Outline

Week 9: complex numbers; complex exponential and polar form

Course Notes: 5.1, 5.2, 5.3, 5.4

Goals:
Fluency with arithmetic on complex numbers
Using matrices with complex entries: finding determinants and inverses, solving systems, etc.
Visualizing complex numbers in coordinate systems

Notes

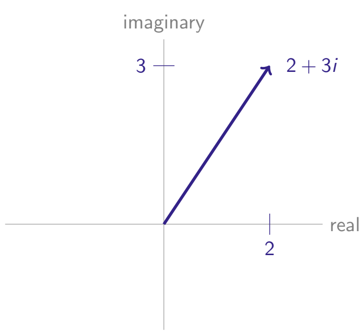
Complex Arithmetic

i

We use i (as in "imaginary") to denote the number whose square is -1.

When we talk about "complex numbers," we allow numbers to have real parts and imaginary parts:

$2+3i$ $\quad -1 \quad$ $2i$



Notes

Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.

Notes

Complex Arithmetic

Multiplication is similar to polynomials.

A: $(-4+3i)+(1-i)$

B: $i(2+3i)$

C: $(i+1)(i-1)$

D: $(2i+3)(i+4)$

I: 0

II: -1

III: -2

IV: 2i+12

V: -3+2i

VI: 3+2i

VII: 10+11i

Notes

Complex Arithmetic

Modulus

The **modulus** of $(x+yi)$ is:

$$|x+yi| = \sqrt{x^2+y^2}$$

like the norm/length/magnitude of a vector.

Complex Conjugate

The **complex conjugate** of $(x+yi)$ is:

$$\overline{x+yi} = x-yi$$

the reflection of the vector over the real (x) axis.

Notes

Complex Arithmetic

$$|x+yi| = \sqrt{x^2+y^2} \qquad \overline{x+yi} = x-yi$$

Notes

Complex Arithmetic

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$$

Compute:

Notes

Polynomial Factorizations

Fundamental Theorem of Algebra

Every polynomial can be factored completely over the complex numbers.

Example: $x^2+1 = x^2-(-1) = x^2-i^2 = (x-i)(x+i)$

$f(x) = x^2+1$ has no real roots, but it has two complex roots.

It is not factorable over $\mathbb{R}$, but it is factorable over $\mathbb{C}$

Notes

Calculating Determinants

We calcuate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

$$\det\begin{bmatrix} 1+i & 1-i \\ 2 & i \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & 2 & 3 \\ i & 4 & 3i \\ 1+i & 2-i & 5 \end{bmatrix} =$$

Notes

Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

$$\begin{array}{rcrcrcl} ix_1 & + & x_2 & + & 2x_3 & = & 0 \\ & & ix_2 & + & 3x_3 & = & 0 \\ 2ix_1 & + & (2-i)x_2 & + & x_3 & = & 0 \end{array}$$

Solve the following system of equations:

$$\begin{array}{rcrcr} ix_1 & + & 2x_2 & = & 9 \\ 3x_1 & + & (1+i)x_2 & = & 5+8i \end{array}$$

Find the inverse of the matrix $\begin{bmatrix} i & 1 \\ 2 & 3i \end{bmatrix}$

Notes

Exponentials

What to do when i is the power of a function?
Maclaurin (Taylor) Series: (you won't be assessed on this explanation)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

Notes

Does that even make sense?

$$e^{ix} = \cos x + i \sin x$$

$\frac{d}{dx}[e^{ax}] = ae^{ax}$;

$e^{x+y} = e^x e^y$;

Notes

Computation Practice

$$e^{ix} = \cos x + i \sin x$$

Evaluate:

$e^{\frac{\pi i}{2}}$

e^{2+i}

$\sqrt{2}e^{\frac{\pi i}{4}}$

2^i

$e^{\pi i} + 1$

$|e^{xi}|$, where x is real.

Notes

Complex exponentiation: $e^{ix} = \cos x + i \sin x$

Let x be a real number.
True or False?

(1) $e^x = \cos x$

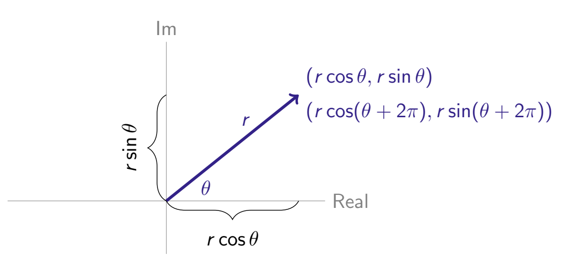
(2) $e^{ix} = e^{i(x+2\pi)}$

(3) $e^{ix} = -e^{i(x+\pi)}$

(4) $e^{ix} + e^{-ix}$ is a real number

Notes

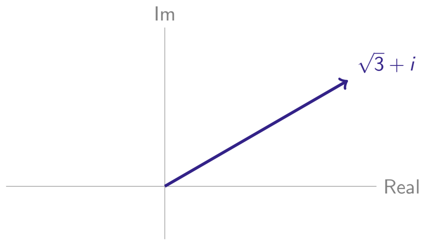
Coordinates Revisited



Complex number : $r(\cos\theta + i\sin\theta) = re^{i\theta} = re^{i(\theta+2\pi)}$

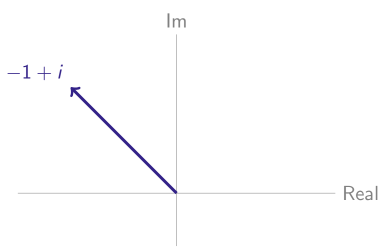
Notes

Coordinates and Exponentials



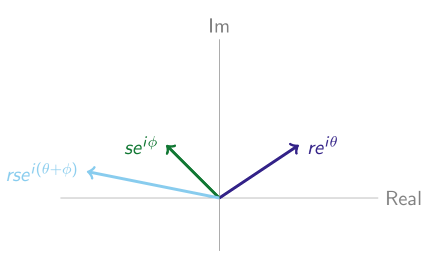
Notes

Coordinates and Exponentials



Notes

Coordinates and Multiplication

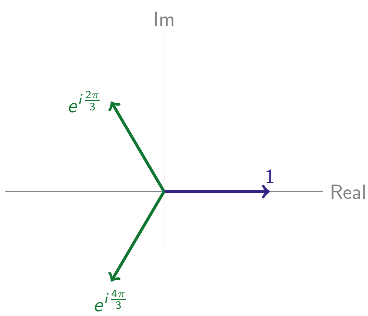


$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

Geometric interpretation of multiplication of two complex numbers: add the angles, multiply the lengths (moduli).

Notes

Roots of Unity



$\left(re^{i\theta}\right)^3 = 1$

Notes

Roots

Find all complex numbers z such that $z^3 = 8$.
$2,\ e^{\frac{2\pi i}{3}},\ 2e^{\frac{4\pi i}{3}}$

Find all complex numbers z such that $z^3 = 27e^{\frac{i\pi}{2}}$.
$3e^{\frac{\pi i}{6}},\ 3e^{\frac{5\pi i}{6}},\ 3e^{\frac{3\pi i}{2}}$

Find all complex numbers z such that $z^4 = 81e^{2i}$.
$3e^{\frac{i}{2}},\ 3e^{\frac{(1+\pi)i}{2}},\ 3e^{\frac{(1+2\pi)i}{2}},\ 3e^{\frac{(1+3\pi)i}{2}}$

Notes

Notes