Outline

Week 9: complex numbers; complex exponential and polar form

Course Notes: 5.1, 5.2, 5.3, 5.4

Goals:
Fluency with arithmetic on complex numbers
Using matrices with complex entries: finding determinants and inverses, solving systems, etc.
Visualizing complex numbers in coordinate systems

Complex Arithmetic

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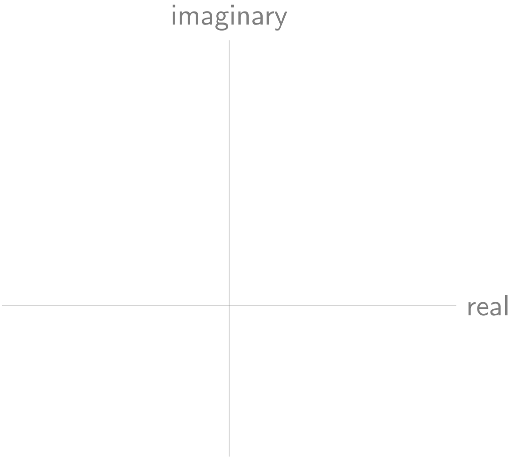
When we talk about "complex numbers," we allow numbers to have real parts and imaginary parts:

$$2+3i \qquad -1 \qquad 2i$$

Complex Arithmetic

$2 + 3i$ $\quad -1 \quad$ $2i$



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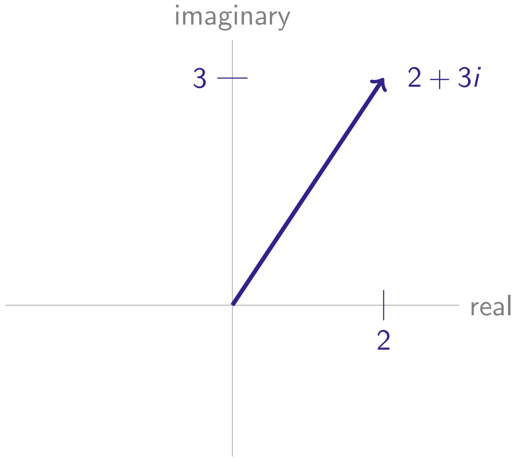

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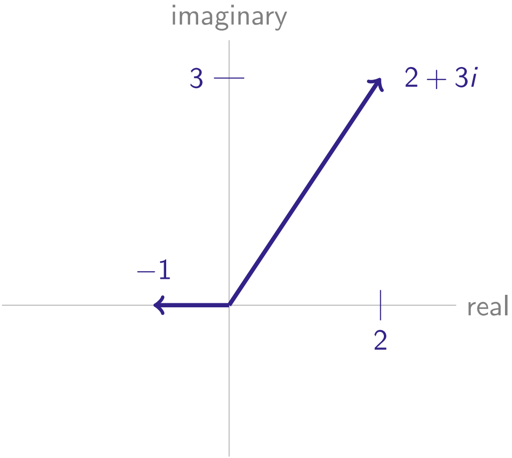

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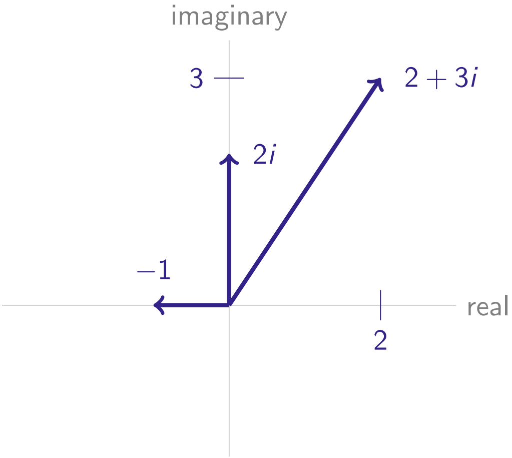

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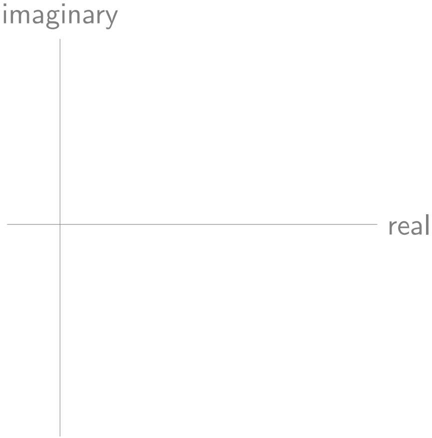
Addition happens component-wise, just like with vectors or polynomials.



Complex Arithmetic

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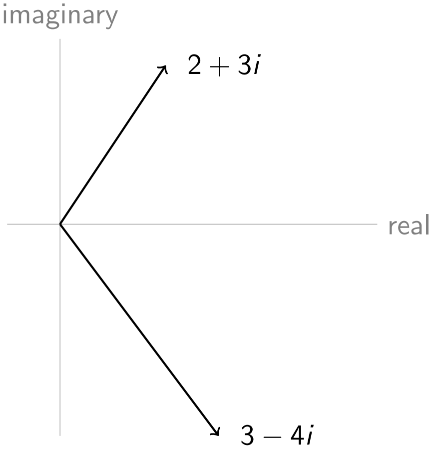
$(2 + 3i) + (3 - 4i) =$



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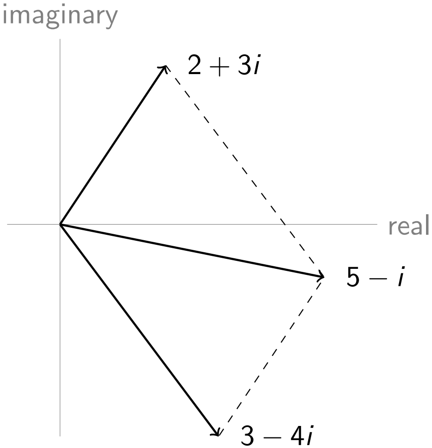
$(2+3i)+(3-4i)=5-i$



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$(2+3i)(3-4i) = 2\cdot 3 + 3i\cdot 3 + (2)(-4i) + (3i)(-4i)$

$= 6 + 9i - 8i + 12$

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A: $(-4+3i)+(1-i)$

B: $i(2+3i)$

C: $(i+1)(i-1)$

D: $(2i+3)(i+4)$

I: 0

II: -1

III: -2

IV: 2i+12

V: -3+2i

VI: 3+2i

VII: 10+11i

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Complex Conjugate

The **complex conjugate** of $(x + yi)$ is:

$$\overline{x + yi} = x - yi$$

the reflection of the vector over the real (x) axis.

Complex Arithmetic

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Suppose $z = x + yi$ and $w = a + bi$. Calculate the following.

- $z - \overline{z}$
- $z + \overline{z}$
- $z\overline{z} - |z|^2$
- $\overline{zw} - (\overline{z})(\overline{w})$

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- $\frac{2+3i}{3+4i}$
- $\frac{1+3i}{1-3i}$
- $\frac{2}{1+i}$
- $\frac{5}{i}$

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- $\frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i$
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- $\frac{2}{1+i} = 1 - i$
- $\frac{5}{i} = -5i$ (dividing by i is the same as multiplying by $-i$)

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Every polynomial can be factored completely over the complex numbers.

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Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

$$\begin{array}{rcrcrcl} ix_1 & + & x_2 & + & 2x_3 & = & 0 \\ & & ix_2 & + & 3x_3 & = & 0 \\ 2ix_1 & + & (2-i)x_2 & + & x_3 & = & 0 \end{array}$$

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Find the inverse of the matrix $\begin{bmatrix} i & 1 \\ 2 & 3i \end{bmatrix}$ $\begin{bmatrix} \frac{-3}{5}i & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5}i \end{bmatrix}$

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Maclaurin (Taylor) Series: (you won't be assessed on this explanation)

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Exponentials

What to do when i is the power of a function?
Maclaurin (Taylor) Series: (you won't be assessed on this explanation)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \end{aligned}$$

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Does that even make sense?

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$e^{x+y} = e^x e^y$;
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$= \cos x \cos y - \sin x \sin y + i[\sin x \cos y + \cos x \sin y]$

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$$= \cos x \cos y - \sin x \sin y + i[\sin x \cos y + \cos x \sin y]$$
$$= (\cos x + i \sin y)(\cos y + i \sin x) = e^{ix}e^{iy}$$

Computation Practice

$$e^{ix} = \cos x + i \sin x$$

Evaluate:

$e^{\frac{\pi i}{2}}$

e^{2+i}

$\sqrt{2}e^{\frac{\pi i}{4}}$

2^i

$e^{\pi i} + 1$

$|e^{xi}|$, where x is any real number.

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$$e^{ix} = \cos x + i \sin x$$

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$e^{\frac{\pi i}{2}} = i$

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$|e^{xi}|$, where x is any real number. $= 1$

Complex exponentiation: $e^{ix} = \cos x + i \sin x$

Let x be a real number.
True or False?

(1) $e^{x} = \cos x$

(2) $e^{ix} = e^{i(x+2\pi)}$

(3) $e^{ix} = -e^{i(x+\pi)}$

(4) $e^{ix} + e^{-ix}$ is a real number

Complex exponentiation: $e^{ix} = \cos x + i \sin x$

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Remember these are real numbers: e^x is unbounded, $\cos x$ stays between -1 and 1.

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For real numbers, a larger exponent gives a larger e^x; complex numbers, not necessarily: $e^{ix} = a + bi$ where $|a|, |b| \leq 1$.

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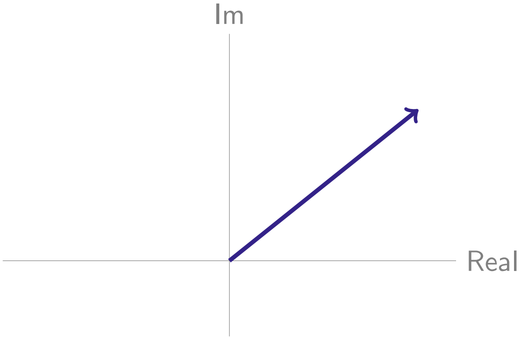
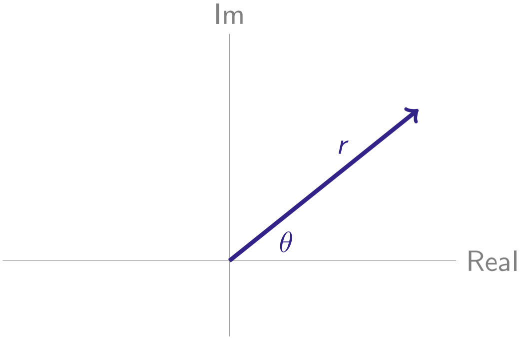
(4) $e^{ix} + e^{-ix}$ is a real number True
using even and odd symmetry of cosine and sine,
$e^{ix} + e^{-ix} = 2\cos x$

Coordinates Revisited



Coordinates Revisited



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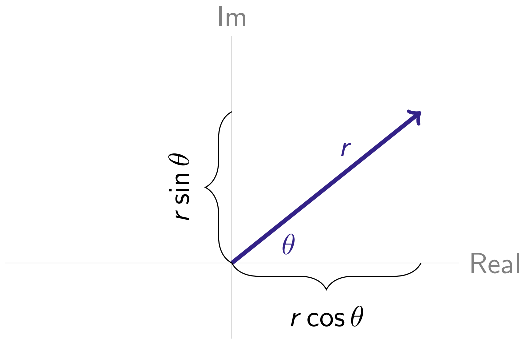

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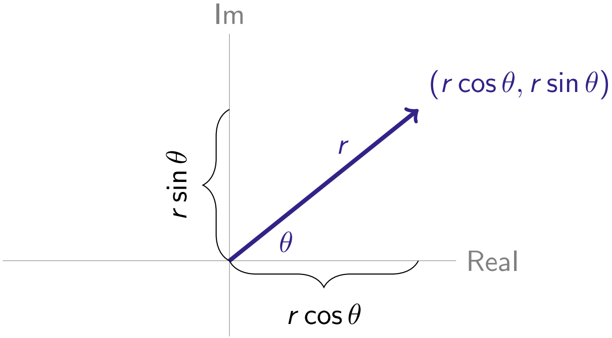

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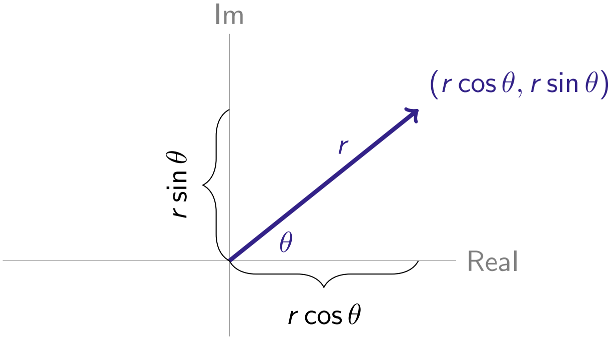
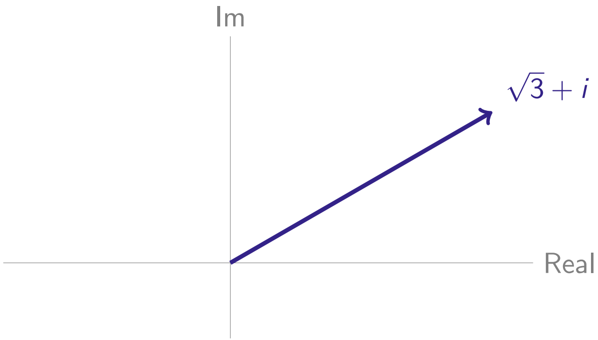


Complex number : $r(\cos\theta + i\sin\theta) = re^{i\theta}$

Coordinates and Exponentials



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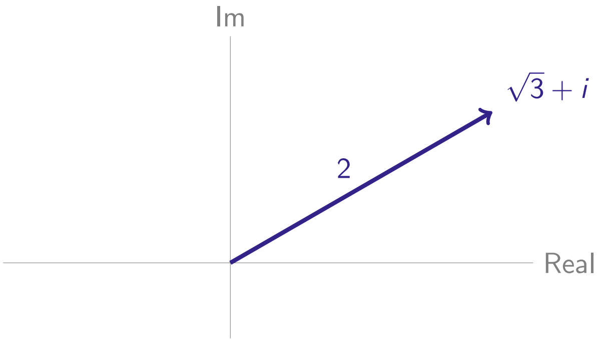

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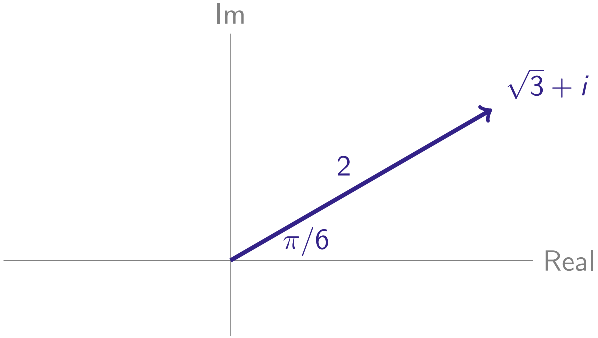

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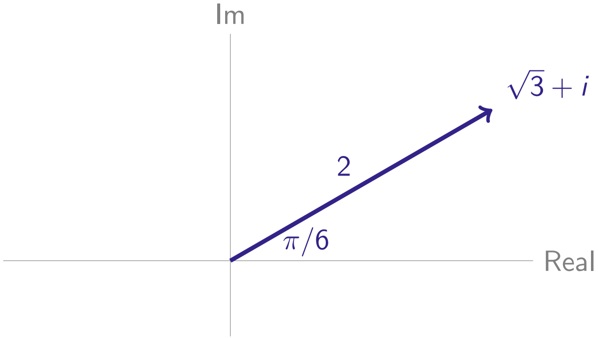
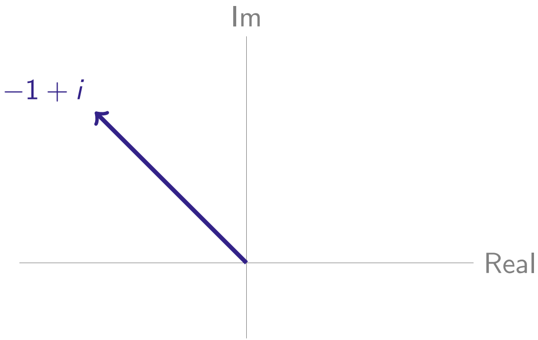


$$\sqrt{3}+i = 2(\cos(\pi/6)+i\sin(\pi/6)) = 2e^{\frac{\pi}{6}i}$$

Coordinates and Exponentials



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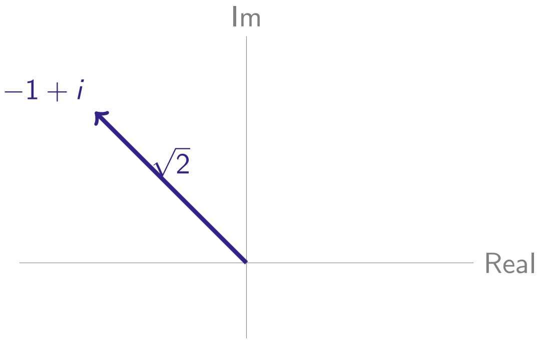

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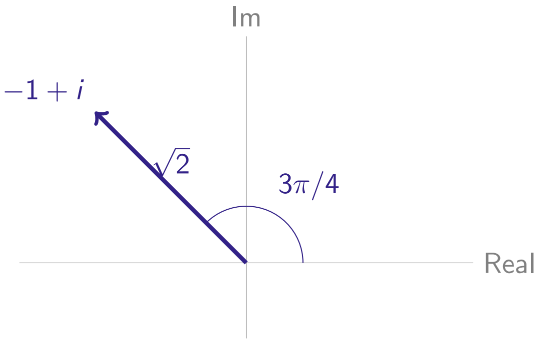

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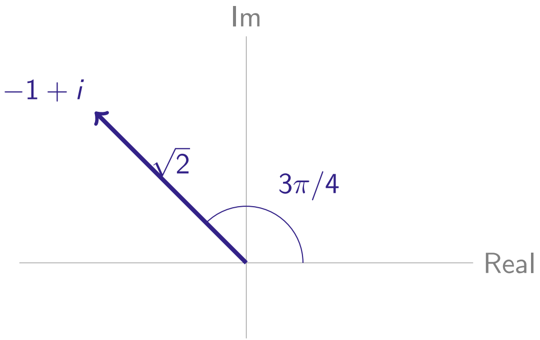
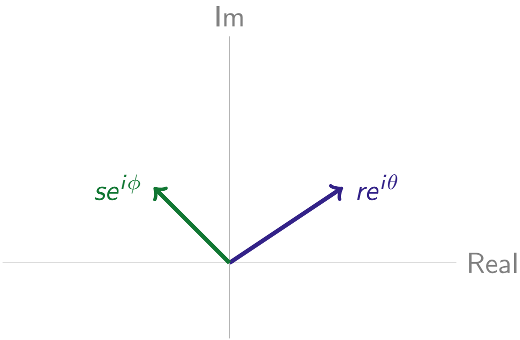


$$-1+i=\sqrt{2}(\cos(3\pi/4)+i\sin(3\pi/4))=\sqrt{2}e^{\frac{3\pi}{4}i}$$

Coordinates and Multiplication



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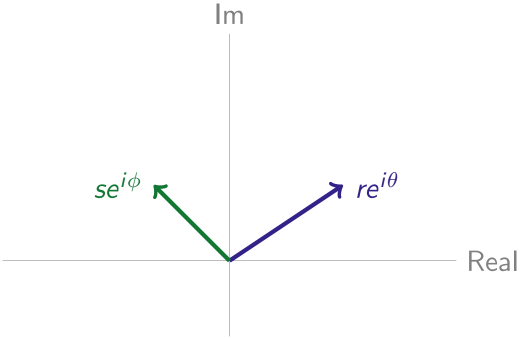
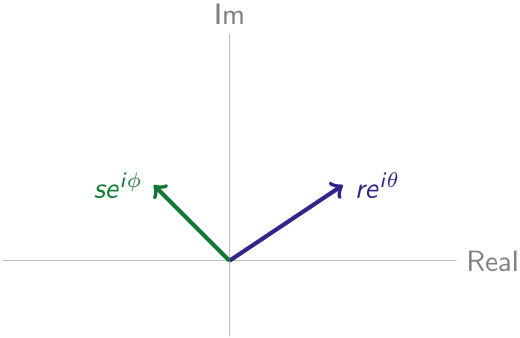


$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

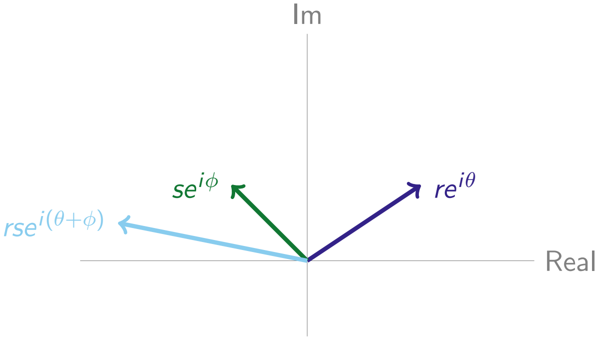
Coordinates and Multiplication



$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

Geometric interpretation of multiplication of two complex numbers: add the angles, multiply the lengths (moduli).

Coordinates and Multiplication



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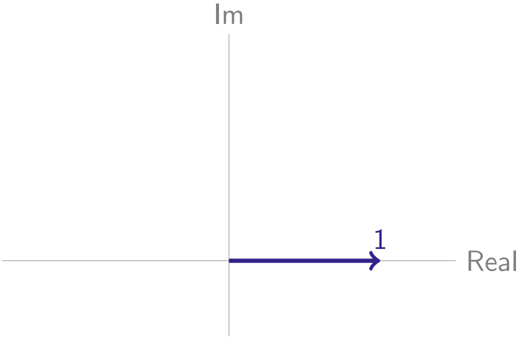
Roots of Unity



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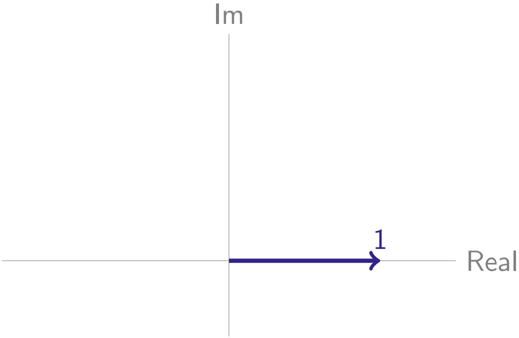
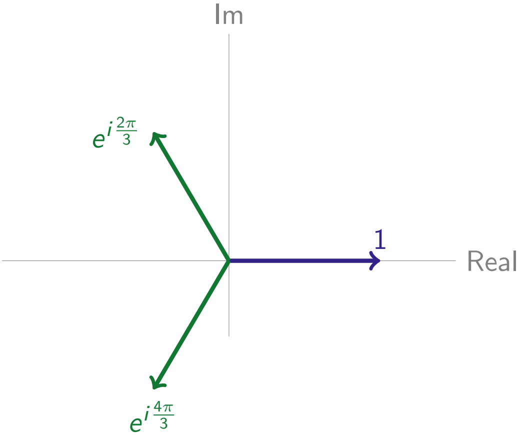


$$\left(re^{i\theta}\right)^3 = 1$$

Roots of Unity



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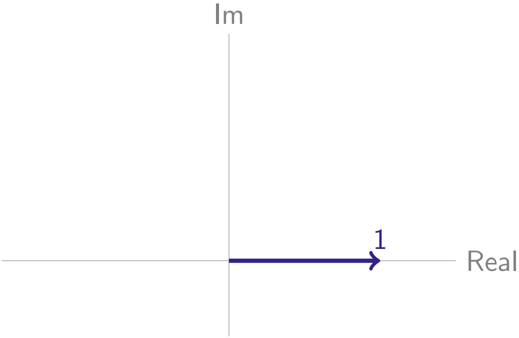
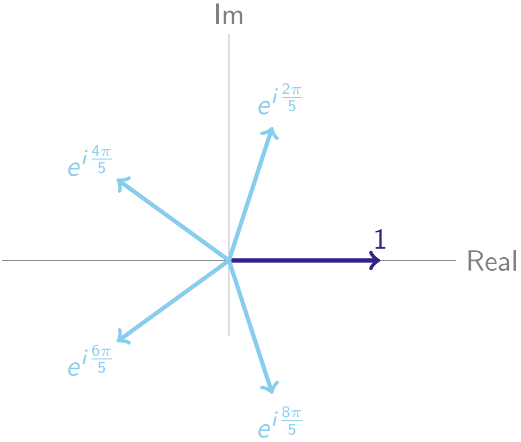


$$\left(re^{i\theta}\right)^5 = 1$$

Roots of Unity



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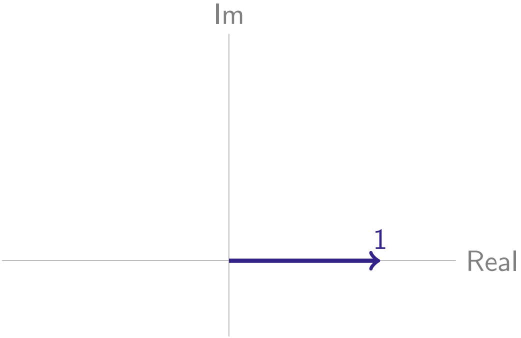
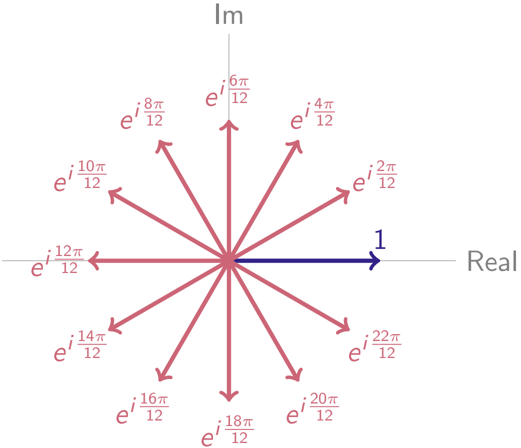


$$\left(re^{i\theta}\right)^{12} = 1$$

Roots of Unity



$$\left(re^{i\theta}\right)^{12} = 1$$

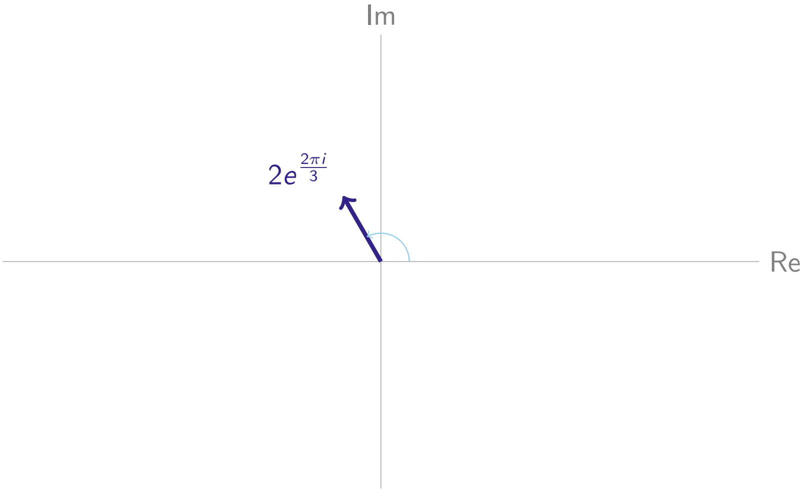
Roots

Find all complex numbers z such that $z^3 = 8$.

Find all complex numbers z such that $z^3 = 27e^{\frac{i\pi}{2}}$.

Find all complex numbers z such that $z^4 = 81e^{2i}$.

$z^3 = 8$



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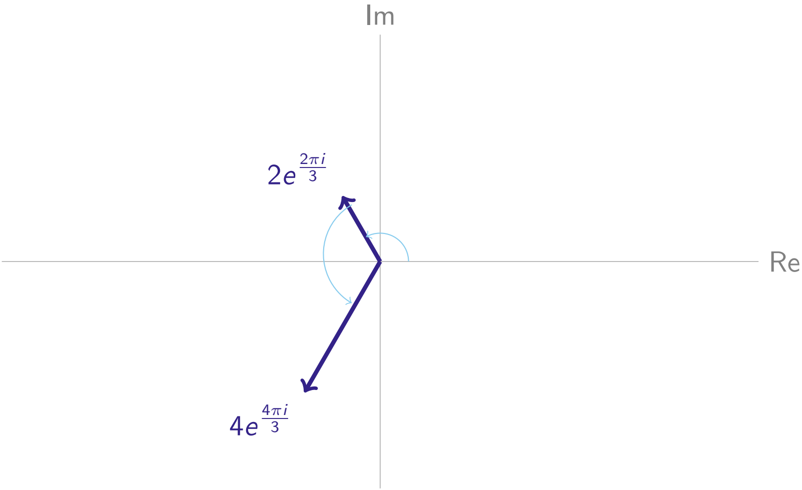

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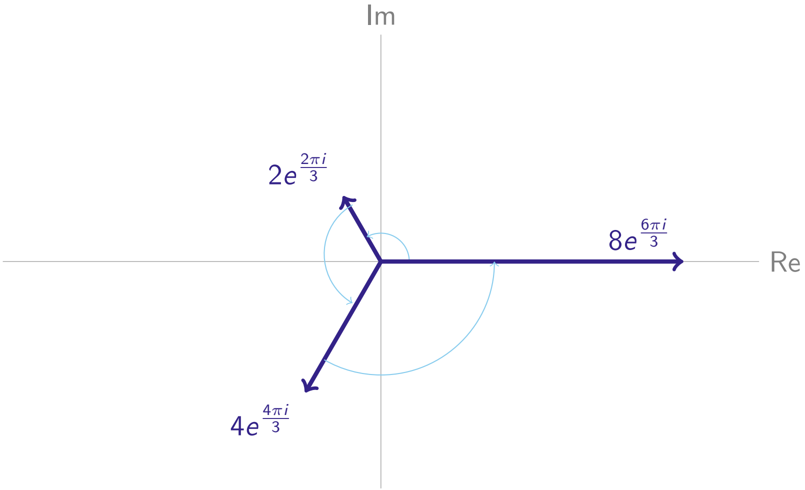

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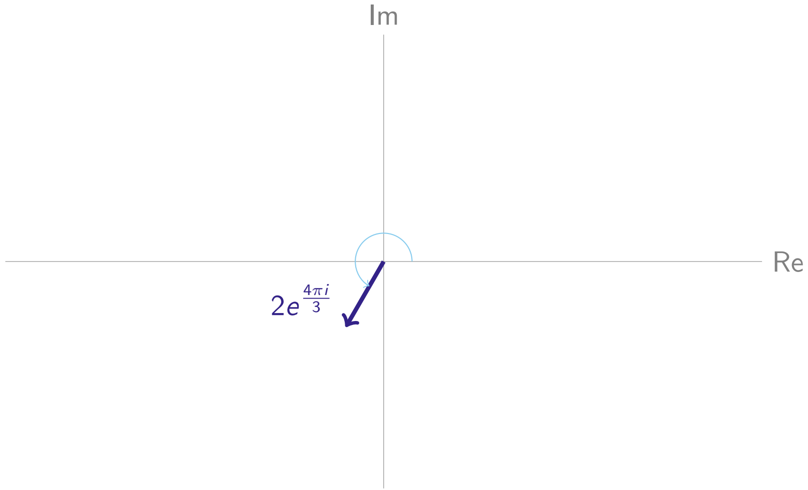

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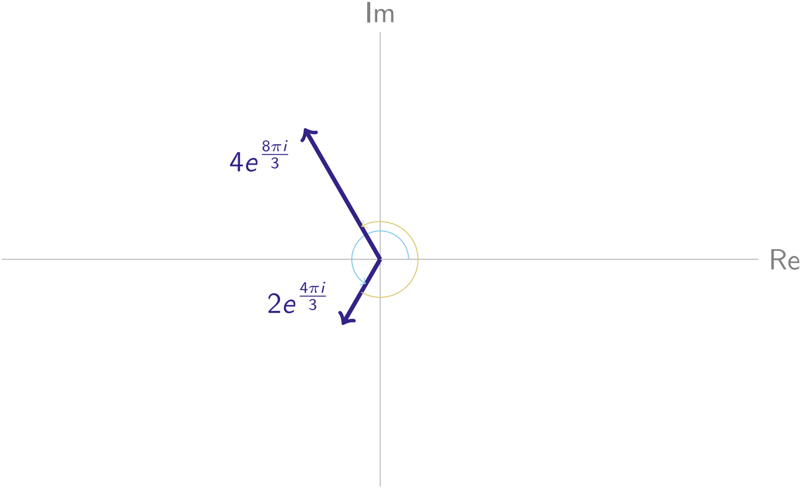

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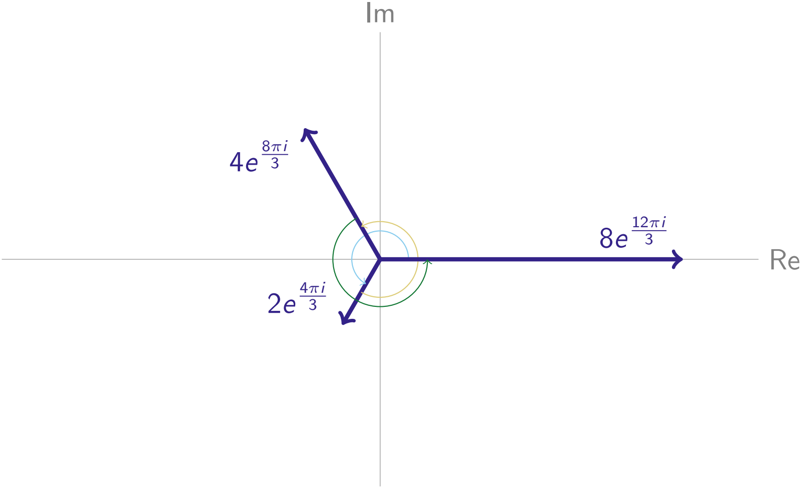

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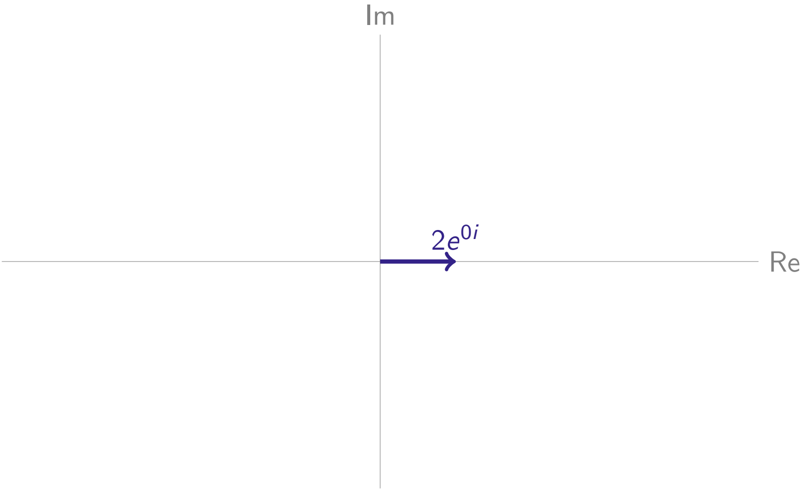

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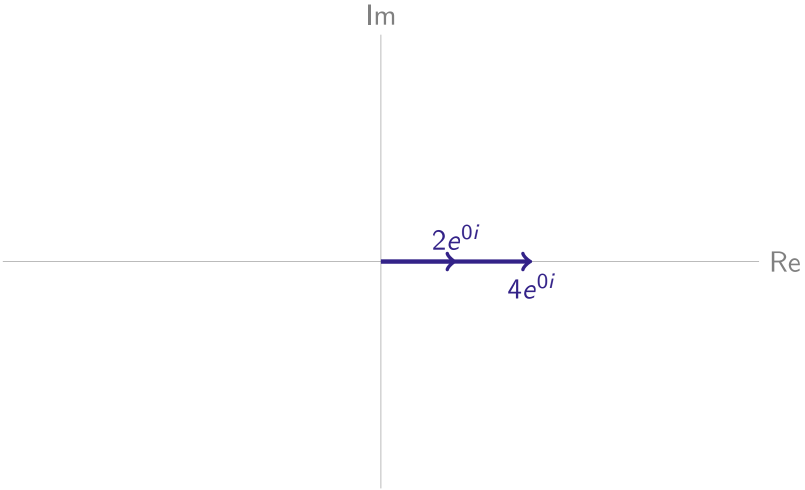

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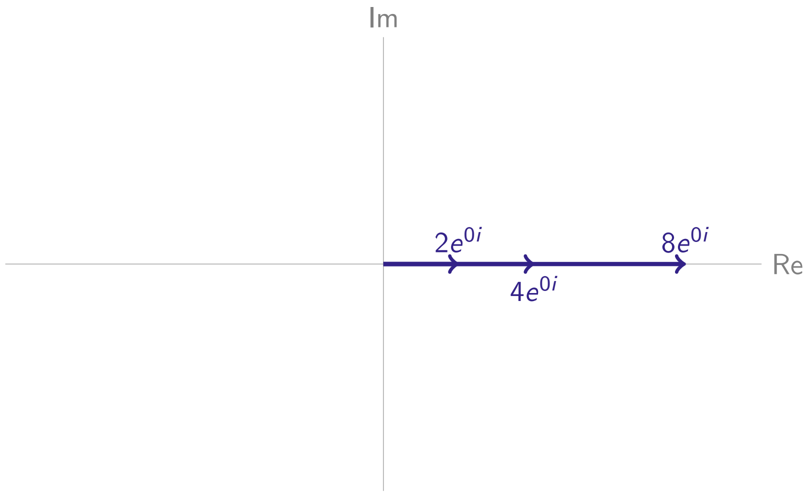

Roots

Find all complex numbers z such that $z^3 = 8$.
$2,\ e^{\frac{2\pi i}{3}},\ 2e^{\frac{4\pi i}{3}}$

Find all complex numbers z such that $z^3 = 27e^{\frac{i\pi}{2}}$.

Find all complex numbers z such that $z^4 = 81e^{2i}$.

$z^3 = 27e^{\frac{i\pi}{2}}$

We solve $\left(re^{i\theta}\right) = 27e^{\frac{i\pi}{2}}$. That is, $r^3e^{i3\theta} = 27e^{\frac{i\pi}{2}}$

- The modulus of our answer is 27; the modulus of $re^{i\theta}$ is r.
- So, we need $r^3 = 27$, so $r = 3$.
- That leaves us with $e^{3i\theta} = e^{\frac{i\pi}{2}}$.
  - There are going to be three distinct answers (since there are three roots of unity)
  - We write $e^{\frac{i\pi}{2}}$ three ways: $e^{\frac{i\pi}{2}} = e^{i\left(\frac{\pi}{2}+2\pi\right)} = e^{i\left(\frac{\pi}{2}+4\pi\right)}$.
    - $e^{3i\theta} = e^{\frac{i\pi}{2}} \implies 3\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{6}$
    - $e^{3i\theta} = e^{i\left(\frac{\pi}{2}+2\pi\right)} \implies 3\theta = \frac{\pi}{2} + 2\pi \implies \theta = \frac{5\pi}{6}$
    - $e^{3i\theta} = e^{i\left(\frac{\pi}{2}+4\pi\right)} \implies 3\theta = \frac{\pi}{2} + 4\pi \implies \theta = \frac{3\pi}{2}$
- So, our solutions are $3e^{\frac{\pi i}{6}}$, $3e^{\frac{5\pi i}{6}}$, $3e^{\frac{3\pi i}{2}}$

Roots

Find all complex numbers z such that $z^3 = 8$.

$2,\ e^{\frac{2\pi i}{3}},\ 2e^{\frac{4\pi i}{3}}$

Find all complex numbers z such that $z^3 = 27e^{\frac{i\pi}{2}}$.

$3e^{\frac{\pi i}{6}},\ 3e^{\frac{5\pi i}{6}},\ 3e^{\frac{3\pi i}{2}}$

Find all complex numbers z such that $z^4 = 81e^{2i}$.

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Find all complex numbers z such that $z^4 = 81e^{2i}$.
$3e^{\frac{i}{2}},\ 3e^{\frac{(1+\pi)i}{2}},\ 3e^{\frac{(1+2\pi)i}{2}},\ 3e^{\frac{(1+3\pi)i}{2}}$