Outline

Week 8: Inverses and determinants

Course Notes: 4.5, 4.6

Goals: Be able to calculate a matrix's inverse;
understand the relationship between the invertibility of a matrix and the solutions of associated linear systems;
calculate the determinant of a square matrix of any size, and learn some tricks to make the computation more efficient.

Notes

Calculate:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

Calculate:

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Notes

Identity Matrix

$$I=\begin{bmatrix}1&0&0&0&&0&0&0&0\\0&1&0&0&\cdots&0&0&0&0\\0&0&1&0&&0&0&0&0\\0&0&0&1&&0&0&0&0\\&\vdots&&&\ddots&&&\vdots&\\0&0&0&0&&1&0&0&0\\0&0&0&0&&0&1&0&0\\0&0&0&0&\cdots&0&0&1&0\\0&0&0&0&&0&0&0&1\end{bmatrix}$$

The identity matrix, I, is a square matrix with 1s along its main diagonal, and 0s everywhere else.

For any matrix A that can be multiplied with I, $AI = IA = A$.

Notes

What is Division?

$$(a+5)x = 7x$$

Divide both sides by x ***as long as*** $x \neq 0$
There are some numbers we can't divide by.

$$(a+5)\frac{x}{x} = 7\frac{x}{x}$$

To divide by x, we multiply by a special number (in this case, $1/x$) that has the following property: $x(1/x)$ gives the multiplicative identity.

$$(a+5)(1) = 7(1)$$

1 is the *multiplicative identity*. If you multiply it by a number, that number doesn't change.

$$(a+5) = 7$$

Notes

What is Division?

To divide by x, we multiply by a special number (in this case, $1/x$) that has the following property: $x(1/x)$ gives the multiplicative identity.
To replicate "division" in matrices, we want to find a matrix A (called A^{-1}) with the property that $AA^{-1} = I$, the identity matrix.

For example, $4 \times 0.25 = 1$, so dividing by 4 is the same as multiplying by 0.25.

$0.1 \times 10 = 1$, so dividing by 0.1 is the same as multiplying by 10.

We can't divide by 0 because there is NO number x such that $0 \times x = 1$.
There are MANY matrices A such that $AB \neq I$ no matter what matrix B we try.

Notes

Matrix Inverses: The Closest we can Get to Division

Linear System Setup:

$$\begin{cases} x+2y+3z &= 10 \\ 4x+5y+6z &= 20 \\ 7x+8y+9z &= 30 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

Solve for $\mathbf{x}$.

Notes

Definition

A matrix A^{-1} is the **inverse** of a square matrix A if $A^{-1}A = I$, where I is the identity matrix. In this case, also[a] $AA^{-1} = I$.

[a] we won't prove this bit

What do you think the inverse of the following matrix should be?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What do you think the inverse of the following matrix should be?

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Notes

Check:

Check your guesses!

Notes

Existence of Matrix Inverses

Definition

A matrix A^{-1} is the **inverse** of a square matrix A if

$$A^{-1}A = I$$

where I is the identity matrix.

Find the inverses of the following matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

If $A\mathbf{x} = \mathbf{b}$ and A^{-1} exists, then $\mathbf{x} = A^{-1}\mathbf{b}$
If A^{-1} exists, then $A\mathbf{x} = \mathbf{b}$ has a **unique** solution.

Notes

If an Inverse Exists....

Theorem

If an n-by-n matrix A has an inverse A^{-1}, then for any $\mathbf{b}$ in $\mathbb{R}^n$,

$$A\mathbf{x} = \mathbf{b}$$

has precisely one solution, and that solution is

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

So, if $A\mathbf{x} = \mathbf{b}$ has no solutions:

If $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions:

Notes

Solutions to Systems of Equations

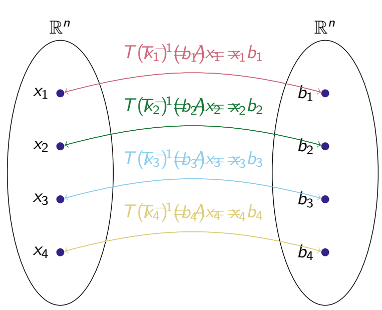
Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.

By previous theorem, if A is invertible, all these statements hold.

Notes

$\boxed{\text{If } A\mathbf{x} = \mathbf{b} \text{ has a unique solution for every } \mathbf{b}, \text{ is } A \text{ invertible?}}$



If T^{-1} is a **linear** transformation, then we can find a matrix B such that

$$T^{-1}(\mathbf{b}) = B\mathbf{b}$$

for every $\mathbf{b}$. Then: $\mathbf{x} = B\mathbf{b} = B(A\mathbf{x}) = (BA)\mathbf{x}$, so $BA = I$

Notes

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, is A invertible?

Need to show: T^{-1} is a **linear** transformation.

- Fix A.
- Given $\mathbf{b}$, we can solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.
- So, given $\mathbf{b}$, we find $\mathbf{x}$.
- This is a transformation: $T^{-1}(\mathbf{b}) = \mathbf{x}$. That is, given input $\mathbf{b}$, the output $\mathbf{x}$ is the vector we multiply A by to get $\mathbf{b}$.
- T^{-1} is linear:
  - Let $T^{-1}(\mathbf{b_1}) = \mathbf{x_1}$ and $T^{-1}(\mathbf{b_2}) = \mathbf{x_2}$.
  - Note $A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \mathbf{b_1} + \mathbf{b_2}$.
    So, $T^{-1}(\mathbf{b_1} + \mathbf{b_2}) = \mathbf{x_1} + \mathbf{x_2} = T^{-1}(\mathbf{b_1}) + T^{-1}(\mathbf{b_2})$.
    So, T^{-1} preserves addition.
  - Note $A(s\mathbf{x_1}) = sA(\mathbf{x_1}) = s\mathbf{b_1}$, so $T^{-1}(s\mathbf{b_1}) = s\mathbf{x_1} = sT^{-1}(\mathbf{b_1})$.
    So, T^{-1} preserves scalar multiplication.
- Since T^{-1} is a linear transformation from one collection of vectors to another, there exists some matrix B such that $T^{-1}(\mathbf{b}) = B\mathbf{b}$.
- Consider $T^{-1}(A\mathbf{x})$. Note $T^{-1}(A\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$, so $B(A\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x}$. Therefore, $BA = I$, so $B = A^{-1}$.

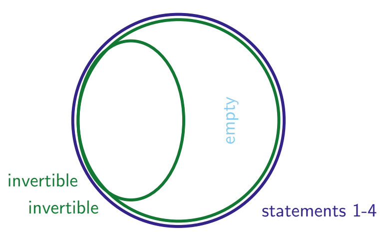
Notes

Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible

By previous theorem, if A is invertible, all these statements hold.
And now we've shown that if the statements hold, then A is invertible



Notes

Solutions to Systems of Equations

Theorem:
A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$.

Suppose A is a matrix with the following reduced form.
Is A invertible?

$$\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$$

Notes

An observation that will help compute inverses

Elementary row operations are equivalent to matrix multiplication.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

Notes

An observation that will help compute inverses

Elementary row operations are equivalent to matrix multiplication.

$$A \to\to\to I$$

Notes

More Conveniently Computing the Inverse (when it exists)

$$[A \mid I] \xrightarrow{reduce} [I \mid A^{-1}]$$

$$\underbrace{\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]}_{[A|I]} \xrightarrow{R1-R2} \left[\begin{array}{cc|cc} 2 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & 1 \end{array}\right]}_{[I|A^{-1}]}$$

Calculate the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 2 & 0 & 7 \end{bmatrix}$

Calculate the inverse of $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Notes

Using Inverses

Suppose $M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Then (as we just found) $M^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix}$.
If $M\mathbf{x} = \begin{bmatrix} 5 \\ 18 \end{bmatrix}$, what is $\mathbf{x}$?

Suppose $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
If $BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, what is B?

Notes

Inverses and Products

Suppose A and B are invertible matrices, with the same dimensions. Simplify:

$$ABB^{-1}A^{-1}$$

What is $(ABC)^{-1}$?

Simplify:

$$\left[(AC)^{-1}A(AB)^{-1}\right]^{-1}$$

Notes

Determinants

Recall:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In general:

$$\det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = a_{1,1}D_{1,1} - a_{1,2}D_{1,2} + a_{1,3}D_{1,3} \cdots \pm a_{1,n}D_{1,n}$$

where $D_{i,j}$ is the determinant of the matrix obtained from A by deleting row i and column j.

Notes

Calculate

$$\det\begin{bmatrix}1&2&3&4\\0&1&0&1\\2&0&1&0\\1&0&2&0\end{bmatrix}=$$

$$\det\begin{bmatrix}0&10&10&0\\1&5&0&2\\2&0&5&1\\0&1&3&1\end{bmatrix}=$$

Notes

Determinants of Triangular Matrices

Calculate, where $*$ is any number:

$$\det\begin{bmatrix}1&0&0&0&0\\ *&2&0&0&0\\ *&*&3&0&0\\ *&*&*&4&0\\ *&*&*&*&5\end{bmatrix}$$

$$\det\begin{bmatrix}1&*&*&*&*\\0&2&*&*&*\\0&0&3&*&*\\0&0&0&4&*\\0&0&0&0&5\end{bmatrix}$$

Fact: for any square matrix A, $\det(A)=\det(A^T)$

Notes

Determinants of Upper Triangular Matrices

Is the determinant of ANY triangular matrix the product of the diagonal entries?

Notes

$$\det \begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Careful: this ONLY works with triangular matrices!

Notes

More Determinant Tricks

Helpful Facts for Calculating the Determinant of a Square Matrix A:

1. If B is obtained from A by multiplying one row of A by the constant c then $\det B = c \det A$.
2. If B is obtained from A by switching two rows of A then $\det B = -\det A$.
3. If B is obtained from A by adding a multiple of one row to another then $\det B = \det A$.
4. $\det(A) = 0$ if and only if A is not invertible
5. For all matrices B of the same size as A, $\det(AB) = \det(A)\det(B)$.
6. $\det(A^T) = \det(A)$

Remark: You should understand how the first three lead to the fourth; otherwise, the proofs are optional, found in the notes.

Notes

If A is invertible, then $\det(A) \neq 0$

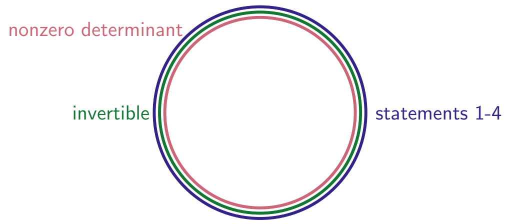
A is invertible
$\Rightarrow$ we can row-reduce A to the identity matrix
$\Rightarrow$ we can row-reduce A to a matrix with determinant 1
• Adding a multiple of one row to another row does not change the determinant
• Swapping two rows multiplies the determinant by -1
• Multiplying a row by a constant a multiplies the determinant by a
$\Rightarrow$ $c\det(A) = 1$, where c is some constant
$\Rightarrow$ $\det(A) \neq 0$

Notes

Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible
6) $\det(A) \neq 0$



Notes

Is A invertible?

$$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 1 & 2 & 0 \\ 10 & 5 & 0 & 1 \\ 10 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} = ?$$

Calculate:

$$\det \begin{bmatrix} 1 & 5 & 10 & 15 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Notes

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Notes

Suppose $\det A = 5$ for an invertible matrix A. What is $\det(A^{-1})$?

Suppose A is an n-by-n matrix with determinant 5. What is the determinant of $3A$?

Suppose A is an n-by-n matrix, and $\mathbf{x}$ and $\mathbf{y}$ are distinct vectors in $\mathbb{R}^n$ with $A\mathbf{x} = A\mathbf{y}$. What is $\det(A)$?

Notes

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{R_2+R_1} \quad \text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right) = \det\left(\begin{bmatrix}1&0&0\\1&2&-3\\1&5&-8\end{bmatrix}\right) = 1\det\left(\begin{bmatrix}2&5\\-3&-8\end{bmatrix}\right)$$

$$= -16 + 15 = -1$$

Is the original 4-by-4 matrix invertible?

Notes

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

$$\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$$

$$\begin{bmatrix}1&2&3\\0&4&5\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Notes

Determinant Expansion across Alternate Lines

"Line" means "row or column"02

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 9 & 8 & 5 & 6 & 10 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 0 & 1 & 1 & 1 \\ 8 & 0 & 1 & 1 & 1 \\ 4 & 3 & 5 & 6 & 7 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 8 & 9 & 5 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 1 & 1 \\ 0 & 8 & 1 & 1 \end{bmatrix}\right)$$

Notes

More practice

$$\det\left(\begin{bmatrix} 2 & 5 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 4 & 4 & 6 & 9 \\ 10 & 5 & 7 & 4 \end{bmatrix}\right)$$

Notes

Notes