Outline

Week 8: Inverses and determinants

Course Notes: 4.5, 4.6

Goals: Be able to calculate a matrix's inverse;
understand the relationship between the invertibility of a matrix and the solutions of associated linear systems;
calculate the determinant of a square matrix of any size, and learn some tricks to make the computation more efficient.

Calculate:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$$

Calculate:

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix, I, is a square matrix with 1s along its main diagonal, and 0s everywhere else.

For any matrix A that can be multiplied with I, $AI = IA = A$.

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We can't divide by 0 because there is NO number x such that $0 \times x = 1$.
There are MANY matrices A such that $AB \neq I$ no matter what matrix B we try.

Matrix Inverses: The Closest we can Get to Division

Linear System Setup:

$$\begin{cases} x + 2y + 3z & = & 10 \\ 4x + 5y + 6z & = & 20 \\ 7x + 8y + 9z & = & 30 \end{cases}$$

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Solve for $\mathbf{x}$.

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Wanted: a matrix A^{-1} with the property $A^{-1}A = I$, identity matrix.
Then: $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$, so $\mathbf{x} = A^{-1}\mathbf{b}$

Definition

A matrix A^{-1} is the **inverse** of a square matrix A if $A^{-1}A = I$, where I is the identity matrix. In this case, also[a] $AA^{-1} = I$.

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$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & 0\\ 0 & \sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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If $A\mathbf{x} = \mathbf{b}$ and A^{-1} exists, then $\mathbf{x} = A^{-1}\mathbf{b}$
If A^{-1} exists, then $A\mathbf{x} = \mathbf{b}$ has a **unique** solution.

If an Inverse Exists....

Theorem

If an n-by-n matrix A has an inverse A^{-1}, then for any $\mathbf{b}$ in $\mathbb{R}^n$,

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has precisely one solution, and that solution is

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Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
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3) The rank of A is n.
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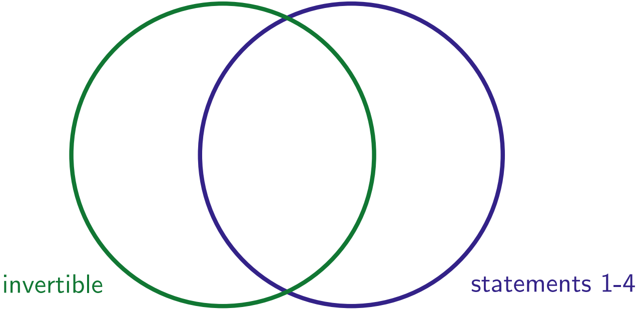

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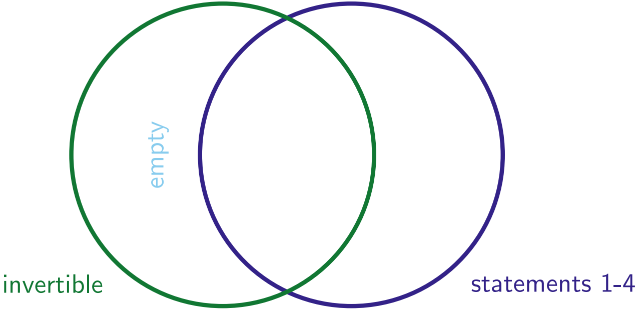

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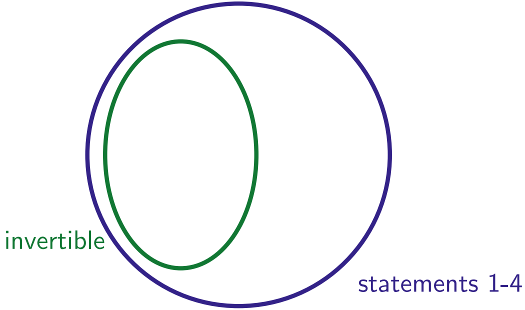

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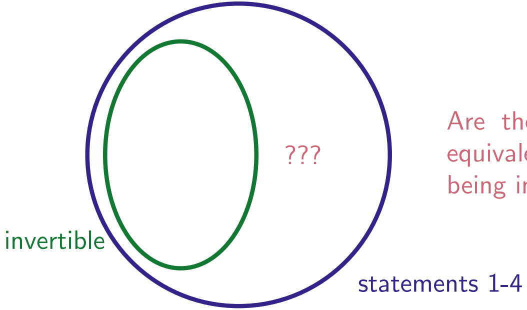
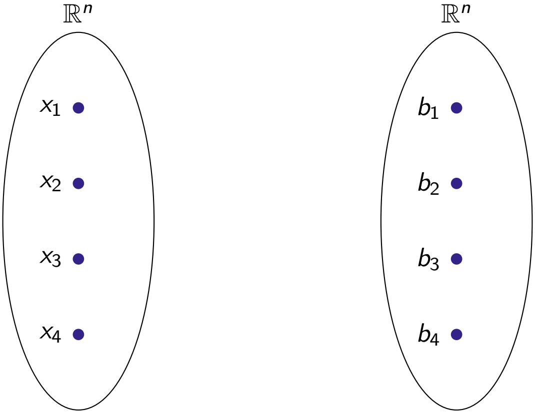


Are the statements equivalent to A being invertible?

$$\boxed{\text{If } A\mathbf{x} = \mathbf{b} \text{ has a unique solution for every } \mathbf{b}, \text{ is } A \text{ invertible?}}$$

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, is A invertible?



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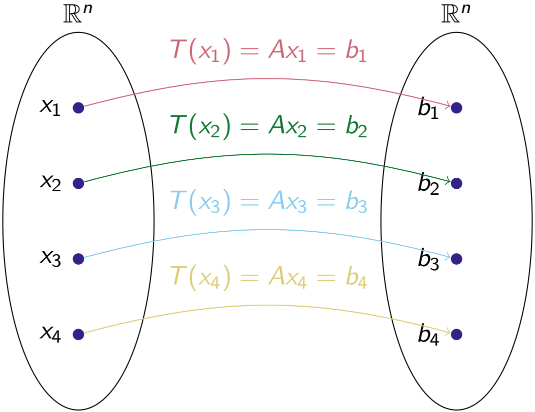

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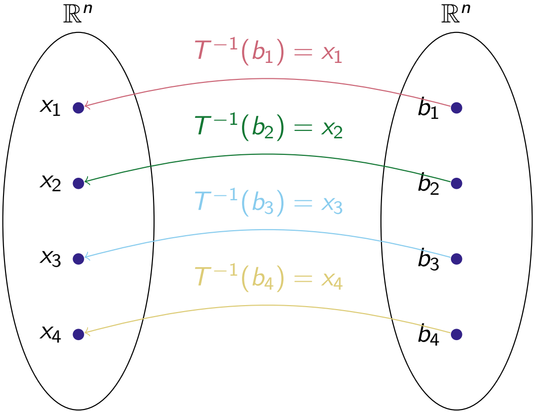

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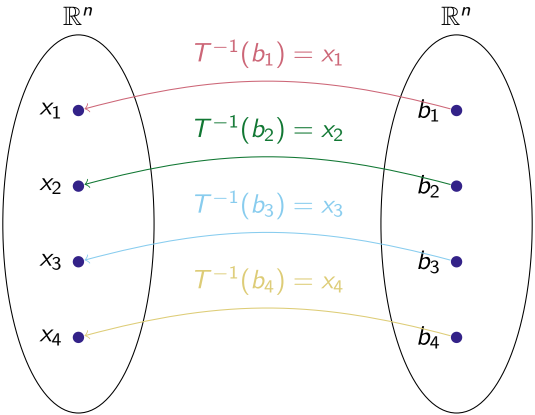

If T^{-1} is a **linear** transformation, then we can find a matrix B such that

$$T^{-1}(\mathbf{b}) = B\mathbf{b}$$

for every $\mathbf{b}$.

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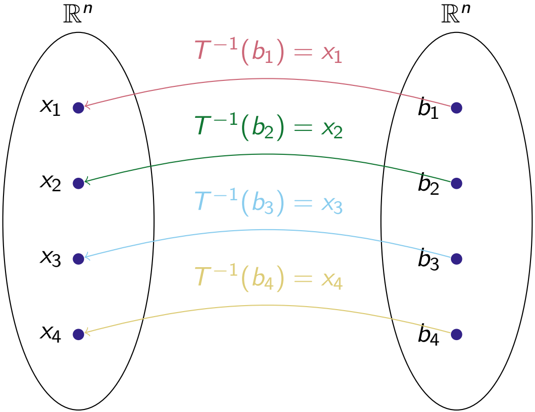

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If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, is A invertible?

Need to show: T^{-1} is a **linear** transformation.

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, is A invertible?

- Fix A.
- Given $\mathbf{b}$, we can solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.
- So, given $\mathbf{b}$, we find $\mathbf{x}$.
- This is a transformation: $T^{-1}(\mathbf{b}) = \mathbf{x}$. That is, given input $\mathbf{b}$, the output $\mathbf{x}$ is the vector we multiply A by to get $\mathbf{b}$.
- T^{-1} is linear:
  - Let $T^{-1}(\mathbf{b_1}) = \mathbf{x_1}$ and $T^{-1}(\mathbf{b_2}) = \mathbf{x_2}$.
  - Note $A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \mathbf{b_1} + \mathbf{b_2}$.
    So, $T^{-1}(\mathbf{b_1} + \mathbf{b_2}) = \mathbf{x_1} + \mathbf{x_2} = T^{-1}(\mathbf{b_1}) + T^{-1}(\mathbf{b_2})$.
    So, T^{-1} preserves addition.
  - Note $A(s\mathbf{x_1}) = sA(\mathbf{x_1}) = s\mathbf{b_1}$, so $T^{-1}(s\mathbf{b_1}) = s\mathbf{x_1} = sT^{-1}(\mathbf{b_1})$.
    So, T^{-1} preserves scalar multiplication.
- Since T^{-1} is a linear transformation from one collection of vectors to another, there exists some matrix B such that $T^{-1}(\mathbf{b}) = B\mathbf{b}$.
- Consider $T^{-1}(A\mathbf{x})$. Note $T^{-1}(A\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$, so $B(A\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x}$. Therefore, $BA = I$, so $B = A^{-1}$.

Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
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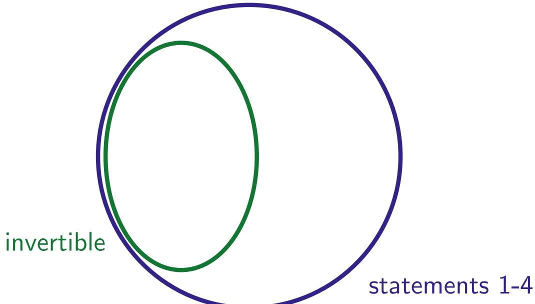
By previous theorem, if A is invertible, all these statements hold.
And now we've shown that if the statements hold, then A is invertible



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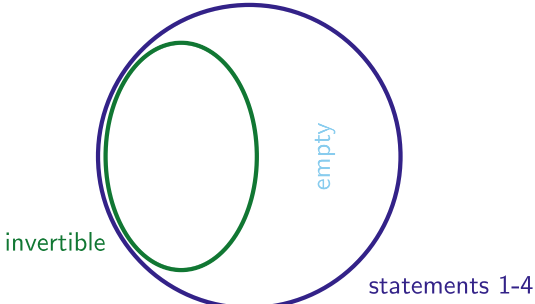

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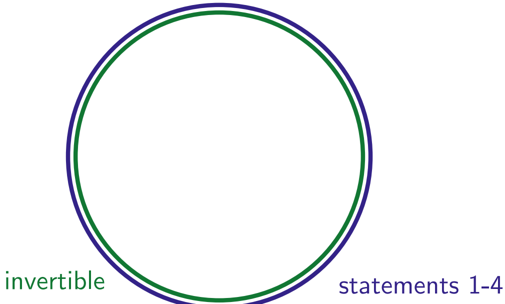

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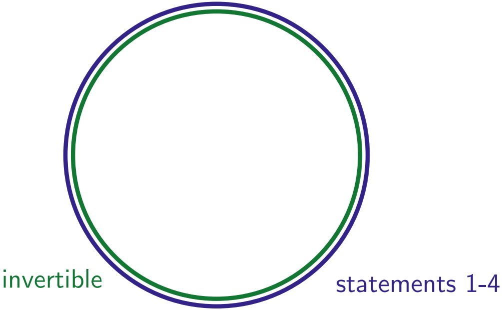

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Row 1 $\leftrightarrow$ Row 3

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$$EI = E$$

so, we reduce I at the same time as A, using the same operations.

$$I \to\to\to [E_3 E_2 E_1]I = E$$

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$$\left[A \quad \middle| \quad I\right] \xrightarrow{reduce} \left[I \quad \middle| \quad A^{-1}\right]$$

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has INFINITELY MANY solutions: $B = \begin{bmatrix} a & (5-a) & -2 \\ a & (5-a) & -2 \\ a & (5-a) & -2 \end{bmatrix}$.

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In general:

$$\det\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = a_{1,1}D_{1,1} - a_{1,2}D_{1,2} + a_{1,3}D_{1,3} \cdots \pm a_{1,n}D_{1,n}$$

where $D_{i,j}$ is the determinant of the matrix obtained from A by deleting row i and column j.

Calculate

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

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Determinants of Triangular Matrices

Calculate, where $*$ is any number:

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 2 & 0 & 0 & 0 \\ * & * & 3 & 0 & 0 \\ * & * & * & 4 & 0 \\ * & * & * & * & 5 \end{bmatrix}$$

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Fact: for any square matrix A, $\det(A) = \det(A^T)$

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For 2-by-2 matrices: yes.

$$\begin{aligned}\det \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} &= a \det \underbrace{\begin{bmatrix} b & * \\ 0 & c \end{bmatrix}}_{\textit{triangular}} - * \det \underbrace{\begin{bmatrix} 0 & * \\ 0 & c \end{bmatrix}}_{\textit{triangular}} + * \det \underbrace{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}_{\textit{triangular}} \\ &= a(bc) - *(0 \cdot c) + *(0 \cdot 0) \\ &= abc\end{aligned}$$

Determinants of Upper Triangular Matrices

Is the determinant of ANY triangular matrix the product of the diagonal entries?

$$\det \begin{bmatrix} a & * \\ 0 & b \end{bmatrix} = ab$$

For 2-by-2 matrices: yes.

$$\begin{aligned} \det \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} &= a \det \underbrace{\begin{bmatrix} b & * \\ 0 & c \end{bmatrix}}_{\textit{triangular}} - * \det \underbrace{\begin{bmatrix} 0 & * \\ 0 & c \end{bmatrix}}_{\textit{triangular}} + * \det \underbrace{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}_{\textit{triangular}} \\ &= a(bc) - *(0 \cdot c) + *(0 \cdot 0) \\ &= abc \end{aligned}$$

The determinant of any triangular matrix (upper or lower) is the product of the diagonal entries.

$$\det \begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\det\begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = (10)(5)\left(\frac{1}{2}\right)(2)(5) = 250$$

$$\det \begin{bmatrix} 10 & 9 & 8 & 4 & 12 \\ 0 & 5 & 9 & 7 & 15 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{7} \\ 0 & 0 & 0 & 2 & 32 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = (10)(5)\left(\frac{1}{2}\right)(2)(5) = 250$$

Careful: this ONLY works with triangular matrices!

More Determinant Tricks

Helpful Facts for Calculating the Determinant of a Square Matrix A:

1. If B is obtained from A by multiplying one row of A by the constant c then $\det B = c \det A$.
2. If B is obtained from A by switching two rows of A then $\det B = -\det A$.
3. If B is obtained from A by adding a multiple of one row to another then $\det B = \det A$.
4. $\det(A) = 0$ if and only if A is not invertible
5. For all matrices B of the same size as A, $\det(AB) = \det(A)\det(B)$.
6. $\det(A^T) = \det(A)$

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Remark: You should understand how the first three lead to the fourth; otherwise, the proofs are optional, found in the notes.

If A is invertible, then $\det(A) \neq 0$

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$$\begin{aligned} & A \text{ is invertible} \\ \Rightarrow \quad & \text{we can row-reduce } A \text{ to the identity matrix} \end{aligned}$$

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$$\begin{array}{ll} & A \text{ is invertible} \\ \Rightarrow & \text{we can row-reduce } A \text{ to the identity matrix} \\ \Rightarrow & \text{we can row-reduce } A \text{ to a matrix with determinant } 1 \end{array}$$

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$\Rightarrow$ we can row-reduce A to the identity matrix

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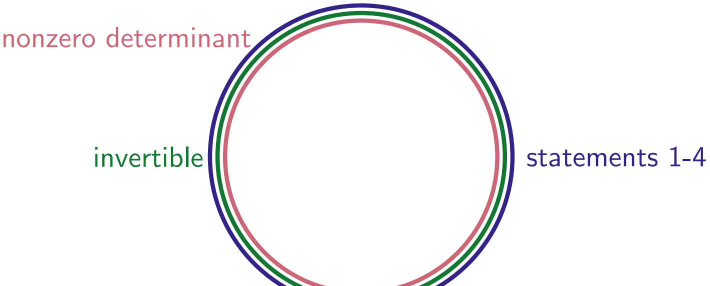
Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible
6) $\det(A) \neq 0$



Is A invertible?

$$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

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$$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 1 & 2 & 0 \\ 10 & 5 & 0 & 1 \\ 10 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} = ?$$

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$$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 1 & 2 & 0 \\ 10 & 5 & 0 & 1 \\ 10 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} = ?$$

Calculate:

$$\det \begin{bmatrix} 1 & 5 & 10 & 15 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\det\begin{bmatrix}0 & 10 & 10 & 0\\1 & 5 & 0 & 2\\2 & 0 & 5 & 1\\0 & 1 & 3 & 1\end{bmatrix} = -210;\qquad \det\begin{bmatrix}0 & 20 & 20 & 0\\1 & 5 & 0 & 2\\2 & 0 & 5 & 1\\0 & 1 & 3 & 1\end{bmatrix} = ?$$

$$\det\begin{bmatrix}2 & 0 & 5 & 1\\1 & 5 & 0 & 2\\0 & 10 & 10 & 0\\0 & 1 & 3 & 1\end{bmatrix} = ?\qquad \det\begin{bmatrix}2 & 0 & 5 & 1\\0 & 10 & 10 & 0\\1 & 5 & 0 & 2\\0 & 1 & 3 & 1\end{bmatrix} = ?$$

$$\det\begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210;$$

$$\det\begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} =?$$

$$\det\begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} =?$$

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$$\det\begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det\begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -420$$

$$\det\begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} = ? \qquad \det\begin{bmatrix} 2 & 0 & 5 & 1 \\ 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix} = ?$$

$$\det\begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det\begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -420$$

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$$\det\begin{bmatrix}0 & 10 & 10 & 0\\ 1 & 5 & 0 & 2\\ 2 & 0 & 5 & 1\\ 0 & 1 & 3 & 1\end{bmatrix} = -210; \qquad \det\begin{bmatrix}0 & 20 & 20 & 0\\ 1 & 5 & 0 & 2\\ 2 & 0 & 5 & 1\\ 0 & 1 & 3 & 1\end{bmatrix} = -420$$

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$$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210; \qquad \det \begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -420$$

$$\det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} = 210 \qquad \det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210$$

Suppose $\det A = 5$ for an invertible matrix A. What is $\det(A^{-1})$?

Suppose A is an n-by-n matrix with determinant 5. What is the determinant of $3A$?

Suppose A is an n-by-n matrix, and $\mathbf{x}$ and $\mathbf{y}$ are distinct vectors in $\mathbb{R}^n$ with $A\mathbf{x} = A\mathbf{y}$. What is $\det(A)$?

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

det : -1

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\xleftarrow{R_3+R_2}$$

det : -1

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \underset{\text{det}\,:\,-1}{\begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix}} \underset{\xleftarrow{R_3+R_2}}{\xrightarrow{R_3-R_2}} \underset{\text{det}\,:\,-1}{\begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\xleftarrow{2R_2} \quad \text{det} : -1 \quad \xleftarrow{R_3+R_2} \quad \text{det} : -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \underset{\xleftarrow{R_2+R_1}}{\xrightarrow{R_2-R_1}} \underset{\text{det}\,:\,-2}{\begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix}} \underset{\xleftarrow{2R_2}}{\xrightarrow{\frac{1}{2}R_2}} \underset{\text{det}\,:\,-1}{\begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix}} \underset{\xleftarrow{R_3+R_2}}{\xrightarrow{R_3-R_2}} \underset{\text{det}\,:\,-1}{\begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}}$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \underset{\xleftarrow{R_2+R_1}}{\xrightarrow{R_2-R_1}} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \underset{\xleftarrow{2R_2}}{\xrightarrow{\frac{1}{2}R_2}} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \underset{\xleftarrow{R_3+R_2}}{\xrightarrow{R_3-R_2}} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$\det : -2$ $\qquad$ $\det : -2$ $\qquad$ $\det : -1$ $\qquad$ $\det : -1$

Using Row Reduction to Calculate a Determinant

$$\begin{array}{ccccccc}
\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} & \xrightarrow{R_2-R_1} & \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} & \xrightarrow{\frac{1}{2}R_2} & \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} & \xrightarrow{R_3-R_2} & \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix} \\
\text{det} : -2 & \xleftarrow{R_2+R_1} & \text{det} : -2 & \xleftarrow{2R_2} & \text{det} : -1 & \xleftarrow{R_3+R_2} & \text{det} : -1
\end{array}$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{array}{ccccccc}
\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} & \xrightarrow{R_2-R_1} & \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} & \xrightarrow{\frac{1}{2}R_2} & \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} & \xrightarrow{R_3-R_2} & \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix} \\
\det : -2 & \xleftarrow{R_2+R_1} & \det : -2 & \xleftarrow{2R_2} & \det : -1 & \xleftarrow{R_3+R_2} & \det : -1
\end{array}$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{R_2+R_1} \quad \text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{array}{ccccccc}
\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} & \overset{R_2-R_1}{\longrightarrow} & \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} & \overset{\frac{1}{2}R_2}{\longrightarrow} & \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} & \overset{R_3-R_2}{\longrightarrow} & \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix} \\
\text{det}:-2 & \overset{R_2+R_1}{\longleftarrow} & \text{det}:-2 & \overset{2R_2}{\longleftarrow} & \text{det}:-1 & \overset{R_3+R_2}{\longleftarrow} & \text{det}:-1
\end{array}$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)=\det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)=-(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$=\det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{R_2+R_1} \quad \text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{R_2+R_1} \quad \text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right) = \det\left(\begin{bmatrix}1&0&0\\1&2&-3\\1&5&-8\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix}$$

$$\text{det}: -2 \quad \xleftarrow{R_2+R_1} \quad \text{det}: -2 \quad \xleftarrow{2R_2} \quad \text{det}: -1 \quad \xleftarrow{R_3+R_2} \quad \text{det}: -1$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right) = \det\left(\begin{bmatrix}1&0&0\\1&2&-3\\1&5&-8\end{bmatrix}\right) = 1\det\left(\begin{bmatrix}2&5\\-3&-8\end{bmatrix}\right)$$

Using Row Reduction to Calculate a Determinant

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{det} : -2 \quad \xleftarrow{R_2+R_1} \quad \text{det} : -2 \quad \xleftarrow{2R_2} \quad \text{det} : -1 \quad \xleftarrow{R_3+R_2} \quad \text{det} : -1$$

$$\det\left(\begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 8 & 7 \\ 9 & 6 & 1 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 8 & 7 \\ 9 & 6 & 1 & 4 \end{bmatrix}\right) = -(-1)\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 8 \\ 9 & 6 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 0 & -3 & -8 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & 5 & -8 \end{bmatrix}\right) = 1\det\left(\begin{bmatrix} 2 & 5 \\ -3 & -8 \end{bmatrix}\right)$$

$$= -16 + 15 = -1$$

Using Row Reduction to Calculate a Determinant

$$\begin{array}{ccccccc}
\begin{bmatrix}1&0&4\\1&2&8\\0&1&1\end{bmatrix} & \xrightarrow{R_2-R_1} & \begin{bmatrix}1&0&4\\0&2&4\\0&1&1\end{bmatrix} & \xrightarrow{\frac{1}{2}R_2} & \begin{bmatrix}1&0&4\\0&1&2\\0&1&1\end{bmatrix} & \xrightarrow{R_3-R_2} & \begin{bmatrix}1&0&4\\0&1&2\\0&0&-1\end{bmatrix} \\
\text{det}:-2 & \xleftarrow{R_2+R_1} & \text{det}:-2 & \xleftarrow{2R_2} & \text{det}:-1 & \xleftarrow{R_3+R_2} & \text{det}:-1
\end{array}$$

$$\det\left(\begin{bmatrix}2&2&2&1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)=\det\left(\begin{bmatrix}0&0&0&-1\\1&1&1&1\\3&5&8&7\\9&6&1&4\end{bmatrix}\right)=-(-1)\det\left(\begin{bmatrix}1&1&1\\3&5&8\\9&6&1\end{bmatrix}\right)$$

$$=\det\left(\begin{bmatrix}1&1&1\\0&2&5\\0&-3&-8\end{bmatrix}\right)=\det\left(\begin{bmatrix}1&0&0\\1&2&-3\\1&5&-8\end{bmatrix}\right)=1\det\left(\begin{bmatrix}2&5\\-3&-8\end{bmatrix}\right)$$

$$=-16+15=-1$$

Is the original 4-by-4 matrix invertible?

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Invertible; determinant unknowable but nonzero

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

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Not invertible; determinant 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose a matrix has the following **reduced** form. Is the matrix invertible? What is its determinant?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Invertible; determinant unknowable but nonzero

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Not invertible; determinant 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Invertible; determinant unknowable but nonzero

Determinant Expansion across Alternate Lines

"Line" means "row or column"02

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Determinant Expansion across Alternate Lines

"Line" means "row or column"02

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 9 & 8 & 5 & 6 & 10 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 0 & 1 & 1 & 1 \\ 8 & 0 & 1 & 1 & 1 \\ 4 & 3 & 5 & 6 & 7 \end{bmatrix}\right)$$

Determinant Expansion across Alternate Lines

"Line" means "row or column"02

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 9 & 8 & 5 & 6 & 10 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 0 & 1 & 1 & 1 \\ 8 & 0 & 1 & 1 & 1 \\ 4 & 3 & 5 & 6 & 7 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 8 & 9 & 5 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 1 & 1 \\ 0 & 8 & 1 & 1 \end{bmatrix}\right)$$

More practice

$$\det\left(\begin{bmatrix} 2 & 5 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 4 & 4 & 6 & 9 \\ 10 & 5 & 7 & 4 \end{bmatrix}\right)$$