Outline

Week 7: Rotations, projections and reflections in 2D; matrix representation and composition of linear transformations; random walks; transpose.

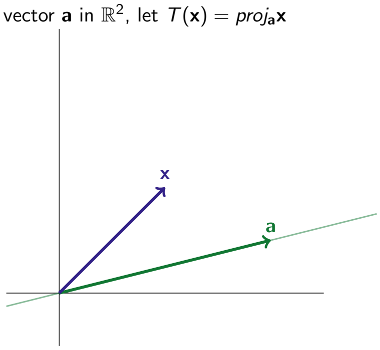
Course Notes: 4.2, 4.3, 4.4

Goals: Understand that a linear transformation of a vector can always be achieved by matrix multiplication; use specific examples of linear transformations.

Notes

Projections

For a fixed vector $\mathbf{a}$ in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



Notes

Computing Projections

Let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{x} = [x_1, x_2]$.

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Since $T(\mathbf{x}) = \text{proj}_{\mathbf{a}}\mathbf{x} = A\mathbf{x}$ for a matrix A, then T is a linear transformation.

Let $\mathbf{a} = [1, 1]$ and $\mathbf{x} = [2, 3]$. Calculate $proj_{\mathbf{a}}\mathbf{x}$ two ways.

$$T(\mathbf{x}) = proj_{\mathbf{b}}\left(proj_{\mathbf{a}}\mathbf{x}\right)$$

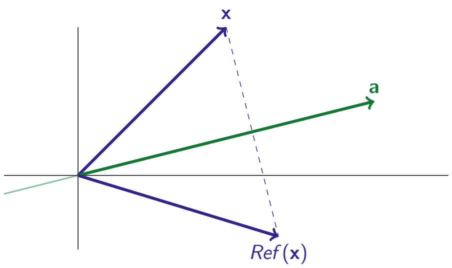
Is the projection of a projection a projection?
(Is there a vector $\mathbf{c}$ so that $T(\mathbf{x}) = proj_{\mathbf{c}}\mathbf{x}$?)

Example: $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Notes

Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



Notes

Reflections

$$Ref(\mathbf{x}) = 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}$$

Projections:

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} Ref(\mathbf{x}) &= 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x} \\ &= \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\ \frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Notes

Cleanup

$$Ref(\mathbf{x}) = \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\ \frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If $\mathbf{a}$ is a unit vector, then $a_1^2 + a_2^2 = 1$. Then:

$$Ref(\mathbf{x}) =$$

And if $\mathbf{a}$ makes angle θ with the x-axis, then $a_1 = \cos\theta$ and $a_2 = \sin\theta$, so:

$$Ref_\theta(\mathbf{x}) =$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

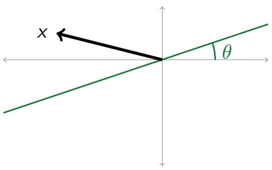
Notes

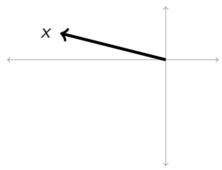
Reflections and Rotations

Compare:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathrm{Rot}_\phi(\mathbf{x}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





Notes

Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

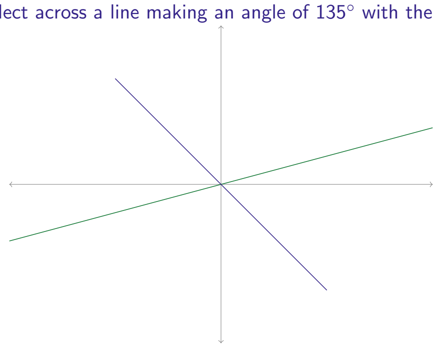
Example: find the reflection of the vector $[2, 4]$ across the line through the origin that makes an angle of 15 degrees ($\pi/12$ radians) with the x-axis.
What happens when we do two reflections?

Notes

Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Notes

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = Rot_{2(\theta-\phi)}$$

Are reflections commutative?
Are reflections commutative with rotations?

Notes

Reflections and Rotations

Are reflections commutative with rotations?

Try the following with a cell phone or book:
1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:
1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise

Notes

Summary: Examples of Linear Transformations

To compute the rotation of the vector **x** by θ, multiply **x** by the matrix

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To compute the projection of the vector **x** onto the vector $[a_1, a_2]$, multiply **x** by the matrix

$$proj_{[a_1,a_2]} = \begin{bmatrix} \frac{a_1^2}{a_1^2+a_2^2} & \frac{a_1a_2}{a_1^2+a_2^2} \\ \frac{a_1a_2}{a_1^2+a_2^2} & \frac{a_2^2}{a_1^2+a_2^2} \end{bmatrix}$$

To compute the reflection of the vector **x** across the line through the origin that makes an angle of ϕ with the x-axis, multiply **x** by the matrix

$$Ref_\theta = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

Notes

Which transformations are equivalent to matrix multiplication?

Notes

Course Notes 4.2: Linear Transformations and Matrices 4.3: Application: Random Walks 4.3: The Transpose

Which transformations are equivalent to matrix multiplication?

Notes

Course Notes 4.2: Linear Transformations and Matrices 4.3: Application: Random Walks 4.3: The Transpose

Suppose a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ satisfies the following:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-2\end{bmatrix}$$

Then $T(\mathbf{x}) = A\mathbf{x}$ for the matrix $A =$

Notes

Which transformations are equivalent to matrix multiplication?

Theorem

Every linear transformation T that takes a vector as an input, and gives a vector as an output, is equivalent to a matrix multiplication.

Extended Theorem

Suppose T is a linear transformation that transforms vectors of $\mathbb{R}^n$ into vectors of $\mathbb{R}^m$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, then:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

That is: $e_1 = [1, 0, \ldots, 0]$, $e_2 = [0, 1, 0, \ldots, 0]$, etc.

Notes

Geometric interpretation of an n-by-m matrix:
linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.
Every matrix can be viewed as a linear transformation, and every linear transformation between $\mathbb{R}^n$ and $\mathbb{R}^m$ can be viewed as a matrix.

A matrix can be viewed as a particular kind of function.

Notes

General Linear Transformations

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear

Standard basis of $\mathbb{R}^n$:

$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Notes

Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}7\\7\end{bmatrix}$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}7\\7\end{bmatrix}$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

Notes

Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ has the following properties:

$T\left(\begin{bmatrix}5\\7\end{bmatrix}\right) = \begin{bmatrix}7\\5\\12\end{bmatrix}$

$T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) = \begin{bmatrix}6\\4\\10\end{bmatrix}$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

Notes

Examples

Suppose T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, where $T(x) = Ax$ for the matrix

$$A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$$

Which vector $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ has $T(x) = \begin{bmatrix}4\\10\\16\end{bmatrix}$?

Which vector $y = \begin{bmatrix}y_1\\y_2\end{bmatrix}$ has $T(y) = \begin{bmatrix}1\\2\\1\end{bmatrix}$?

Characterize vectors that can come out of T.

Notes

Random Walks: Another Use of Matrix Multiplication

•n states
•Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

Examples:
`https://en.wikipedia.org/wiki/Random_walk`
model Brownian Motion (Wiener process)
genetic drift
stock markets
use sampling to estimate properties of a large system

Notes

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3



Sleeping: `https://pixabay.com/en/penguin-linux-sleeping-animal-159784/`
Fishing: By Mimooh (Own work), via Wikimedia Commons
Playing: By Silvermoonlight217
`http://silvermoonlight217.deviantart.com/art/Penguin-Sledding-262107547`

Notes

Random Walks

In general:
•n states
•$p_{i,j}$ probability of moving to state i if you are in state j; $P = [p_{i,j}]$

Given x_n:
$x_{n+1} = Px_n = P^{n+1}x_0$

P: "transition matrix"

Notes

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

Where are you after 100 steps?

Notes

Random Walk Example: Error Messages

Suppose you are using a buggy program. You start up without a problem.

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, your next click is to uninstall the program, and never use it again.

Possible states: no errors; one error; two errors; three errors; uninstalled.

Notes

Random Walk Example

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, you uninstall the program.

Possible states: no errors; one error; two errors; three errors; uninstalled.

from / *to*	0	1	2	3	*u*
0	.99	0	0	0	0
1	.01	.95	0	0	0
2	0	.05	.9	0	0
3	0	0	.1	0	0
u	0	0	0	1	1

Again, notice: columns sum to 1, rows don't have to

Notes

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx \begin{bmatrix} 0.366 \\ 0.090 \\ 0.049 \\ 0.005 \\ 0.490 \end{bmatrix}$
- $\mathbf{x}_{200} = P^{200}\mathbf{x}_0 \approx \begin{bmatrix} 0.134 \\ 0.033 \\ 0.019 \\ 0.002 \\ 0.812 \end{bmatrix}$
- $\lim_{n\to\infty} \mathbf{x}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (we'll do these computations more generally once we learn about eigenvalues!)

Notes

Harder Questions involving Random Walks

- For which value of n does x_n have a certain characteristic?
- What is $\lim_{n\to\infty} x_n$?
  Note: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} P^n x_0$.
- Does $\lim_{n\to\infty} x_n$ depend on x_0?

Stay tuned for more Random Walks excitement

Notes

Application: Google!

Notes

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix} \qquad BA = DNE$$

$$B^T A^T = \begin{bmatrix} 6 & 15 \\ 12 & 30 \\ 18 & 45 \end{bmatrix} \qquad AB = (B^T A^T)^T$$

Notes

Transpose and Matrix Multiplication

$$AB = P$$

$$B^T A^T = Q$$

Notes

Transpose

Previous example of noncommutativity of matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

Notes

Transpose and Dot Product

$$\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$$

where A is an m-by-n matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \left(\begin{bmatrix}1 & 0\\0 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}8\\9\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}8\\9\\1\end{bmatrix} = 8 + 18 + 3 = 29$$

$$\left(\begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right) \cdot \begin{bmatrix}8\\9\end{bmatrix} = \begin{bmatrix}-2\\5\end{bmatrix} \cdot \begin{bmatrix}8\\9\end{bmatrix} = -16 + 45 = 29$$

Notes

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$

Notes

Notes