Outline

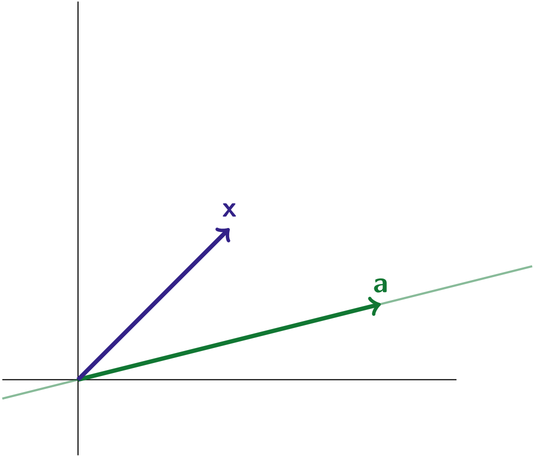
Week 7: Rotations, projections and reflections in 2D; matrix representation and composition of linear transformations; random walks; transpose.

Course Notes: 4.2, 4.3, 4.4

Goals: Understand that a linear transformation of a vector can always be achieved by matrix multiplication; use specific examples of linear transformations.

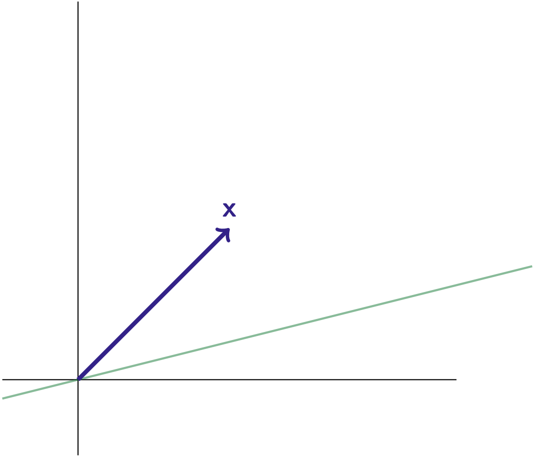
Projections

For a fixed vector $\mathbf{a}$ in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



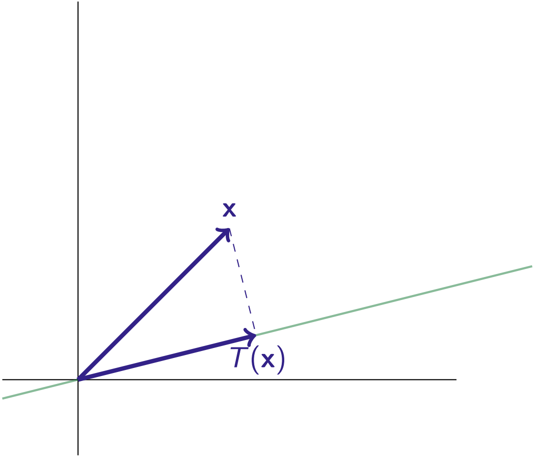
Projections

For a fixed vector $\mathbf{a}$ in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



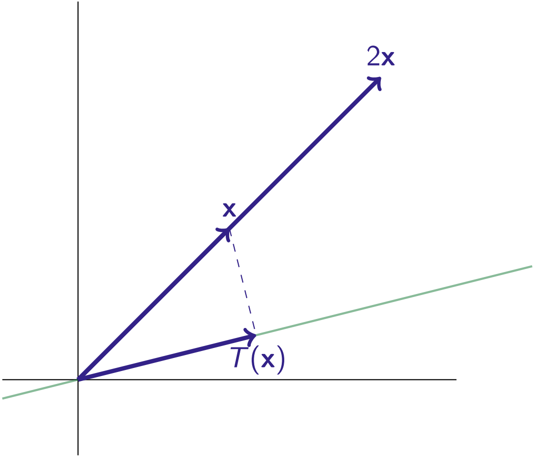
Projections

For a fixed vector $\mathbf{a}$ in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



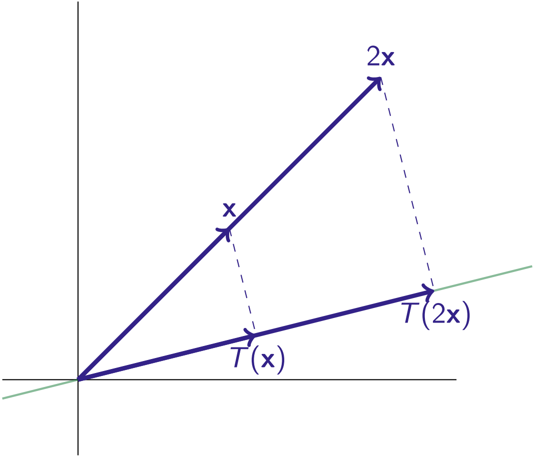
Projections

For a fixed vector **a** in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



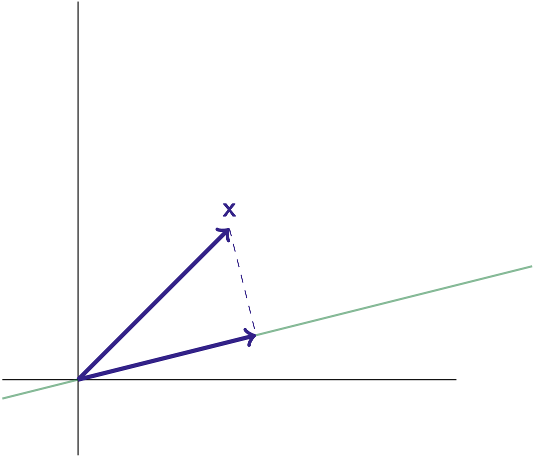
Projections

For a fixed vector **a** in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



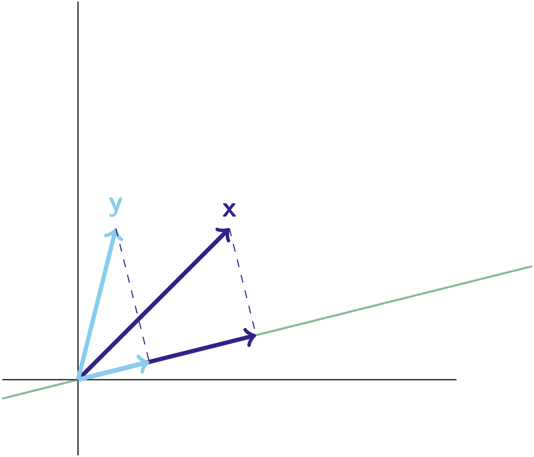
Projections

For a fixed vector $\mathbf{a}$ in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



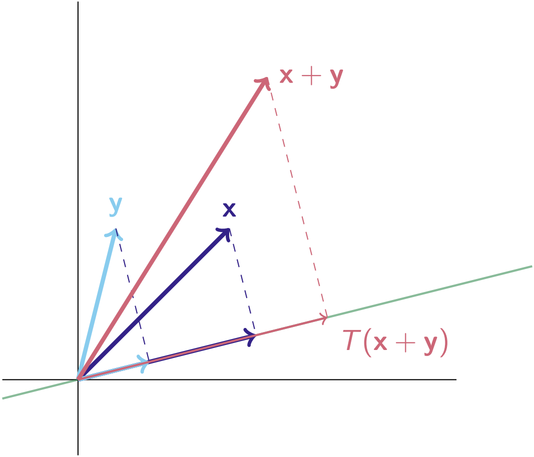
Projections

For a fixed vector $\mathbf{a}$ in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



Projections

For a fixed vector **a** in $\mathbb{R}^2$, let $T(\mathbf{x}) = proj_{\mathbf{a}}\mathbf{x}$



Computing Projections

Let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{x} = [x_1, x_2]$.

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Computing Projections

Let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{x} = [x_1, x_2]$.

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Since $T(\mathbf{x}) = \mathrm{proj}_{\mathbf{a}}\mathbf{x} = A\mathbf{x}$ for a matrix A, then T is a linear transformation.

Computing Projections

Let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{x} = [x_1, x_2]$.

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Since $T(\mathbf{x}) = \text{proj}_{\mathbf{a}}\mathbf{x} = A\mathbf{x}$ for a matrix A, then T is a linear transformation.

Let $\mathbf{a} = [1, 1]$ and $\mathbf{x} = [2, 3]$. Calculate $proj_{\mathbf{a}}\mathbf{x}$ two ways.

Computing Projections

Let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{x} = [x_1, x_2]$.

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Since $T(\mathbf{x}) = \mathrm{proj}_{\mathbf{a}}\mathbf{x} = A\mathbf{x}$ for a matrix A, then T is a linear transformation.

Let $\mathbf{a} = [1, 1]$ and $\mathbf{x} = [2, 3]$. Calculate $proj_{\mathbf{a}}\mathbf{x}$ two ways.

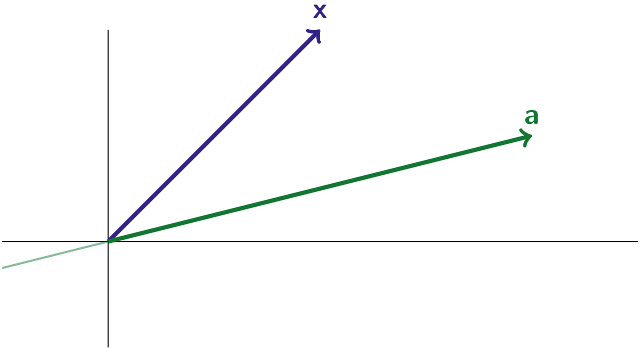
$$T(\mathbf{x}) = proj_{\mathbf{b}}\left(proj_{\mathbf{a}}\mathbf{x}\right)$$

Is the projection of a projection a projection?
(Is there a vector $\mathbf{c}$ so that $T(\mathbf{x}) = proj_{\mathbf{c}}\mathbf{x}$?)

Example: $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

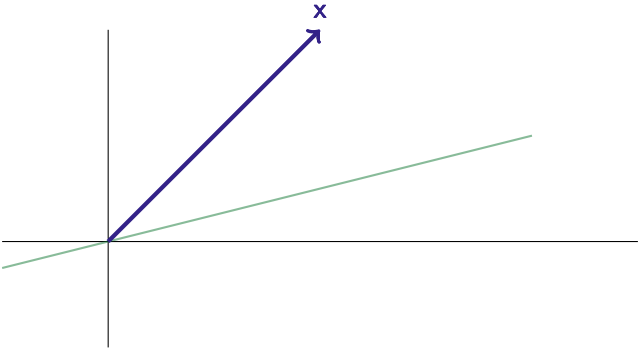
Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



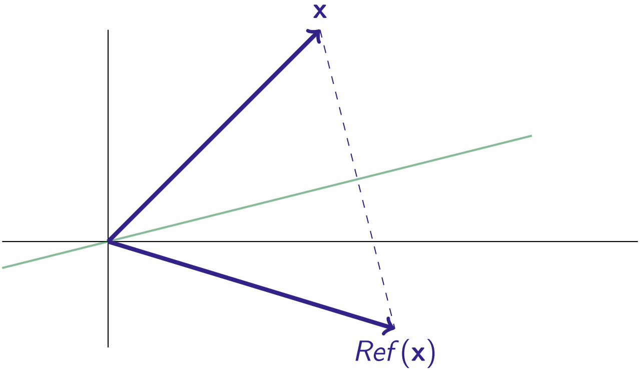
Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



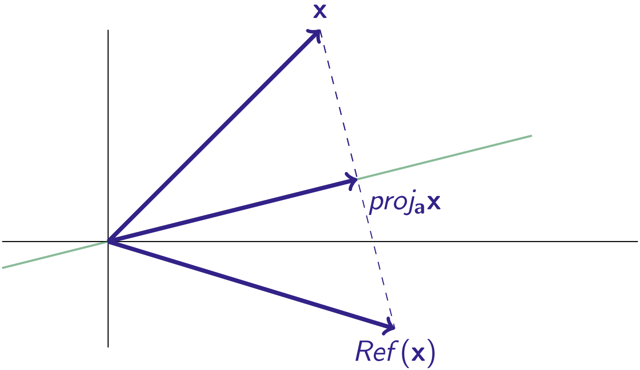
Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



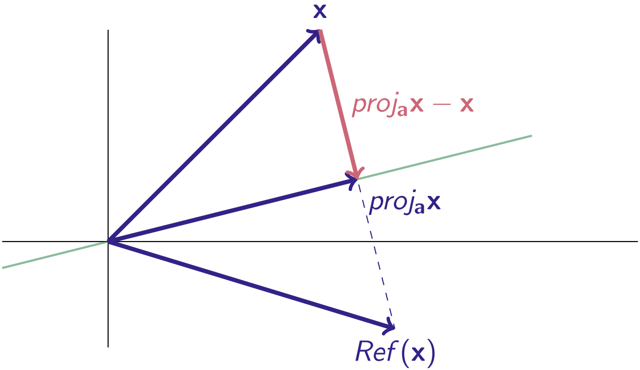
Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



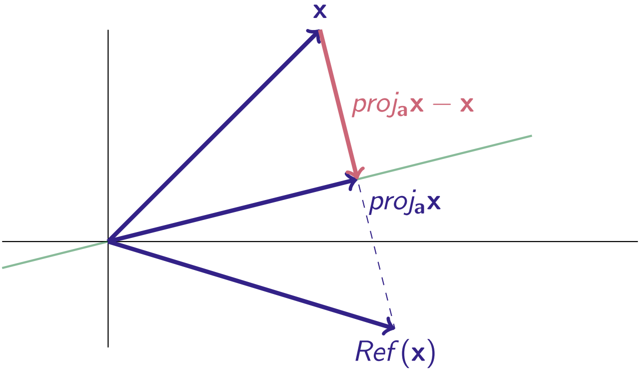
Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



Reflections

For a fixed vector $\mathbf{a}$, let $Ref(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.



$$Ref(\mathbf{x}) = \mathbf{x} + 2(proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}) = 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}$$

Reflections

$$Ref(\mathbf{x}) = 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}$$

Reflections

$$Ref(\mathbf{x}) = 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}$$

Projections:

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reflections

$$Ref\,(\mathbf{x}) = 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x}$$

Projections:

$$proj_{\mathbf{a}}\mathbf{x} = \frac{1}{a_1^2 + a_2^2}\begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} Ref\,(\mathbf{x}) &= 2proj_{\mathbf{a}}\mathbf{x} - \mathbf{x} \\ &= \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1 a_2}{a_1^2+a_2^2} \\ \frac{2a_1 a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Cleanup

$$Ref(\mathbf{x}) = \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1 a_2}{a_1^2+a_2^2} \\ \frac{2a_1 a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Cleanup

$$Ref(\mathbf{x}) = \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\ \frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If **a** is a unit vector, then $a_1^2 + a_2^2 = 1$. Then:

$$Ref(\mathbf{x}) = \begin{bmatrix} 2a_1^2 - 1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Cleanup

$$Ref(\mathbf{x}) = \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\ \frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If **a** is a unit vector, then $a_1^2 + a_2^2 = 1$. Then:

$$Ref(\mathbf{x}) = \begin{bmatrix} 2a_1^2 - 1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And if **a** makes angle θ with the x-axis, then $a_1 = \cos\theta$ and $a_2 = \sin\theta$, so:

$$Ref_\theta(\mathbf{x}) =$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

Cleanup

$$Ref(\mathbf{x}) = \begin{bmatrix} \frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\ \frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If **a** is a unit vector, then $a_1^2 + a_2^2 = 1$. Then:

$$Ref(\mathbf{x}) = \begin{bmatrix} 2a_1^2 - 1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And if **a** makes angle θ with the x-axis, then $a_1 = \cos\theta$ and $a_2 = \sin\theta$, so:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

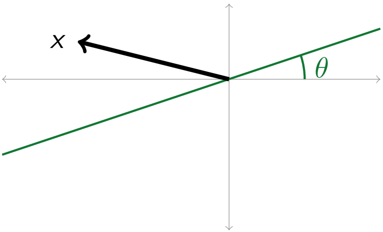
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

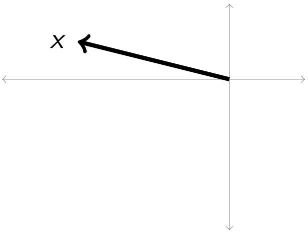
Reflections and Rotations

Compare:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{Rot}_\phi(\mathbf{x}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



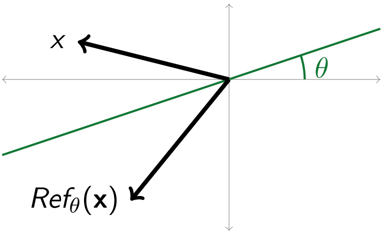


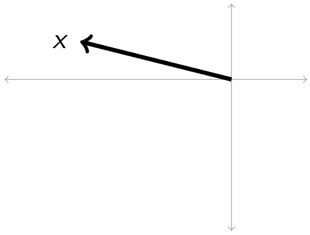
Reflections and Rotations

Compare:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathrm{Rot}_\phi(\mathbf{x}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



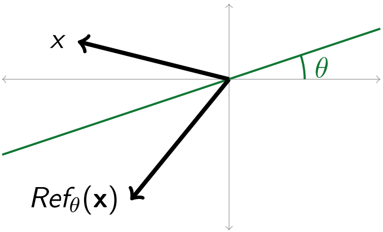


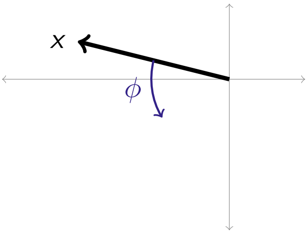
Reflections and Rotations

Compare:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathrm{Rot}_\phi(\mathbf{x}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



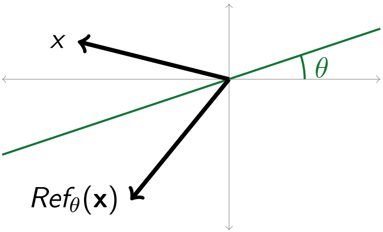


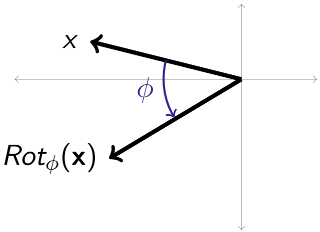
Reflections and Rotations

Compare:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathrm{Rot}_\phi(\mathbf{x}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example: find the reflection of the vector $[2, 4]$ across the line through the origin that makes an angle of 15 degrees ($\pi/12$ radians) with the x-axis.

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example: find the reflection of the vector $[2,4]$ across the line through the origin that makes an angle of 15 degrees ($\pi/12$ radians) with the x-axis.

$$\begin{bmatrix} \cos(2(\pi/12)) & \sin(2(\pi/12)) \\ \sin(2(\pi/12)) & -\cos(2(\pi/12)) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & \sin(\pi/6) \\ \sin(\pi/6) & -\cos(\pi/6)) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{3}+2 \\ 1-2\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 3.7 \\ -2.5 \end{bmatrix}$$

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} =$$

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

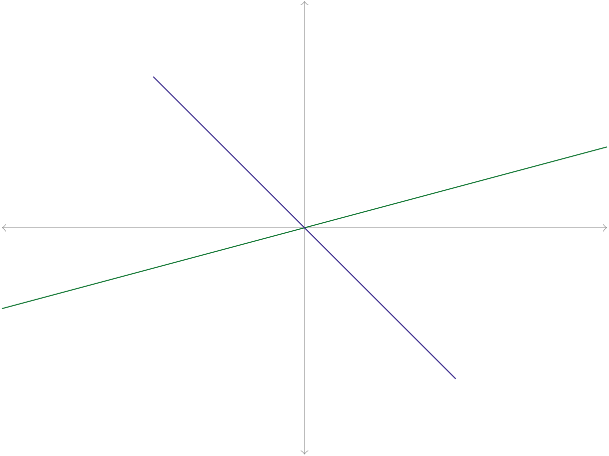
$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = Rot_{2(\theta-\phi)}$$

Two Reflections gives a Rotation

Consider:

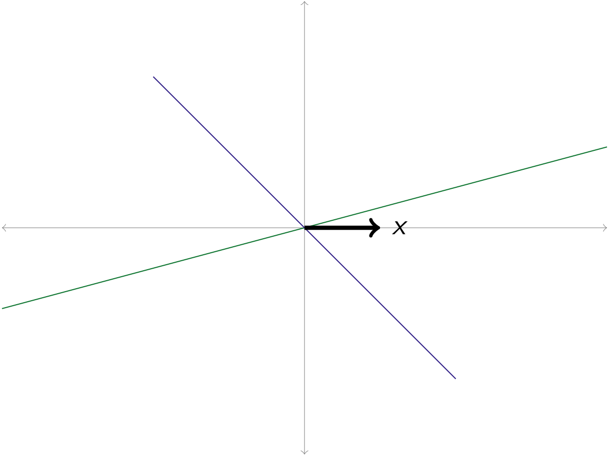
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

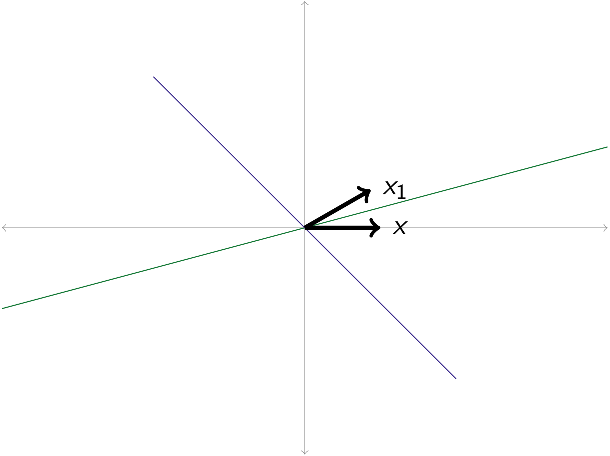
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

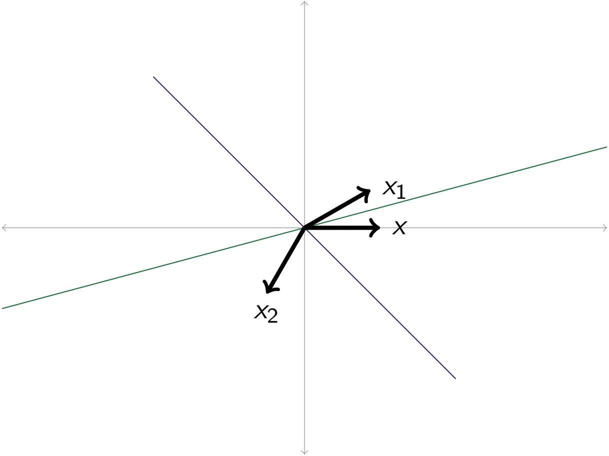
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

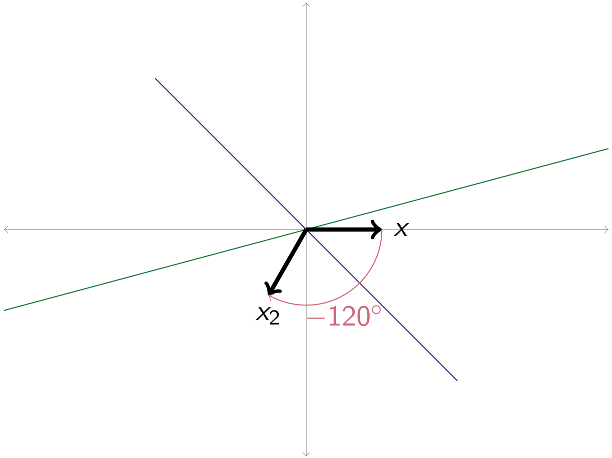
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

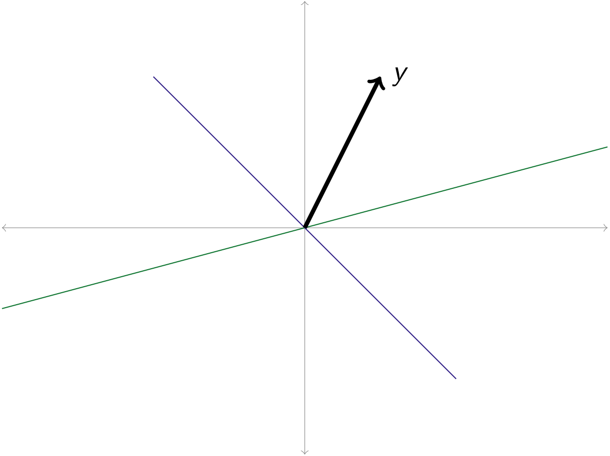
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

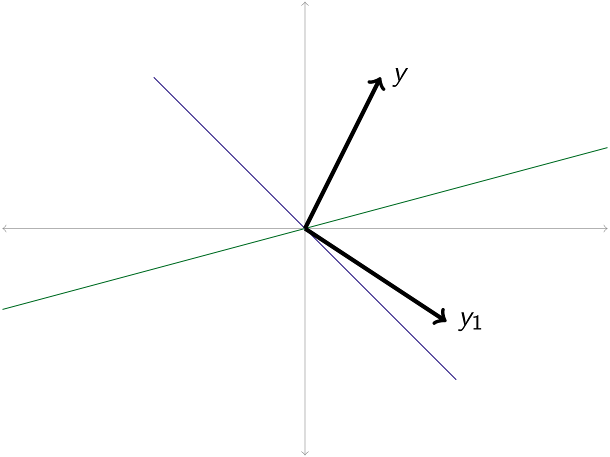
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

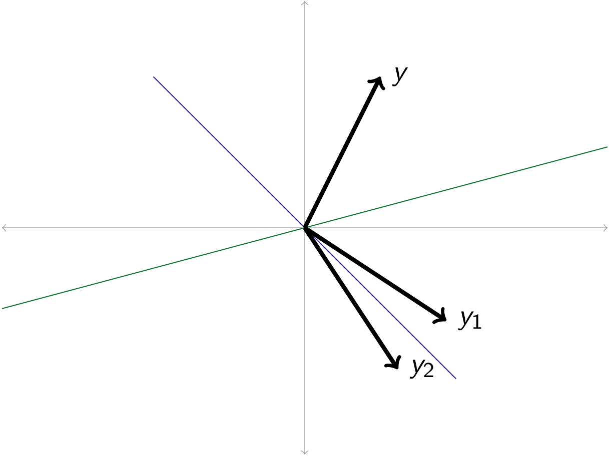
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

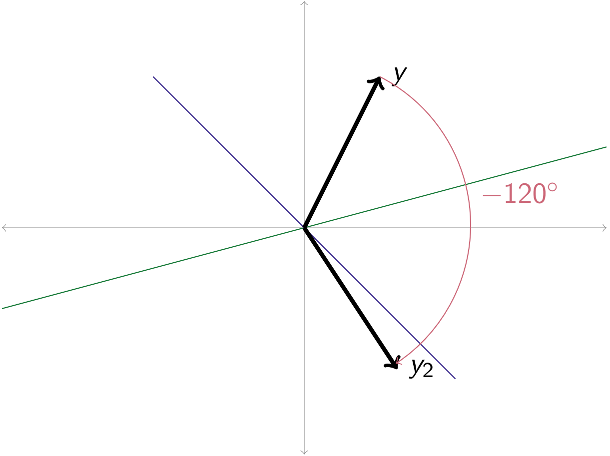
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

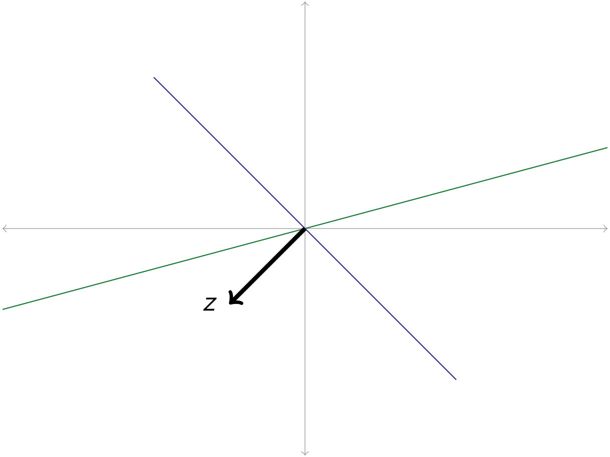
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

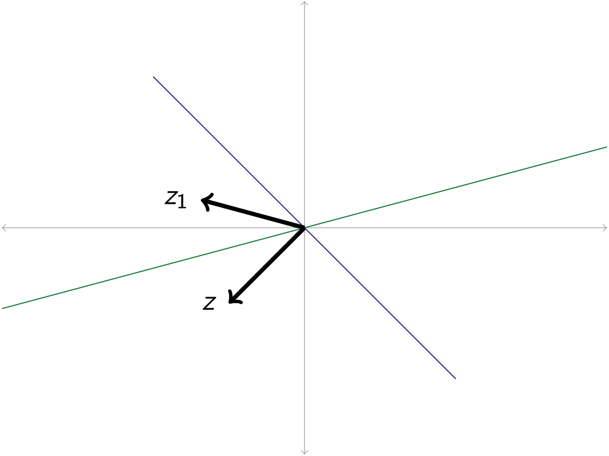
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

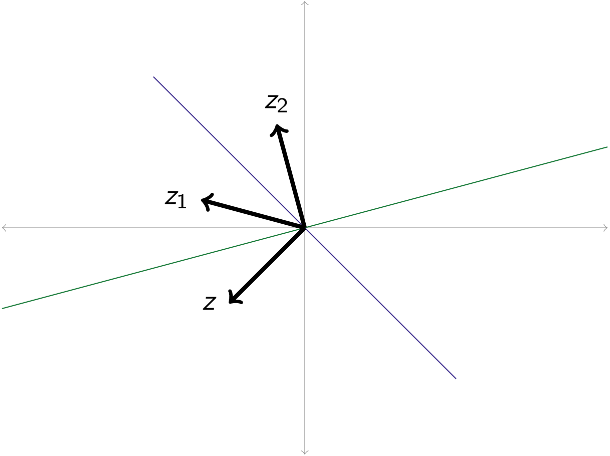
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

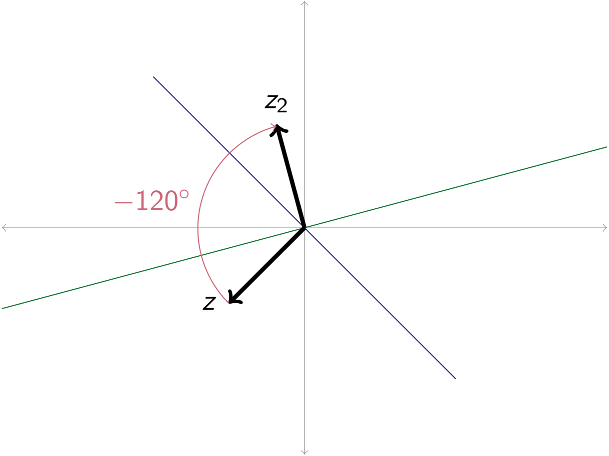
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.



Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = Rot_{2(\theta-\phi)}$$

Are reflections commutative?

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = Rot_{2(\theta-\phi)}$$

Are reflections commutative? No (but almost)

Reflections

To reflect **x** across the line through the origin that makes angle θ with the x-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = Rot_{2(\theta-\phi)}$$

Are reflections commutative? No (but almost)

Are reflections commutative with rotations?

Reflections and Rotations

Are reflections commutative with rotations?

Try the following with a cell phone or book:
1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:
1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise

Reflections and Rotations

Are reflections commutative with rotations?

Try the following with a cell phone or book:
1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:
1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise

Nope.

Reflections and Rotations

Are reflections commutative with rotations?

Try the following with a cell phone or book:
1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:
1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise

Nope.
To prove an operation IS commutative, we have to prove it is commutative ALWAYS.
To prove an operation IS NOT commutative, it suffices to fine ONE EXAMPLE where it doesn't commute.

Summary: Examples of Linear Transformations

To compute the rotation of the vector $\mathbf{x}$ by θ, multiply $\mathbf{x}$ by the matrix

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Summary: Examples of Linear Transformations

To compute the rotation of the vector **x** by θ, multiply **x** by the matrix

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To compute the projection of the vector **x** onto the vector $[a_1, a_2]$, multiply **x** by the matrix

$$proj_{[a_1,a_2]} = \begin{bmatrix} \frac{a_1^2}{a_1^2+a_2^2} & \frac{a_1 a_2}{a_1^2+a_2^2} \\ \frac{a_1 a_2}{a_1^2+a_2^2} & \frac{a_2^2}{a_1^2+a_2^2} \end{bmatrix}$$

Summary: Examples of Linear Transformations

To compute the rotation of the vector **x** by θ, multiply **x** by the matrix

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To compute the projection of the vector **x** onto the vector $[a_1, a_2]$, multiply **x** by the matrix

$$proj_{[a_1,a_2]} = \begin{bmatrix} \frac{a_1^2}{a_1^2+a_2^2} & \frac{a_1 a_2}{a_1^2+a_2^2} \\ \frac{a_1 a_2}{a_1^2+a_2^2} & \frac{a_2^2}{a_1^2+a_2^2} \end{bmatrix}$$

To compute the reflection of the vector **x** across the line through the origin that makes an angle of ϕ with the x-axis, multiply **x** by the matrix

$$Ref_\theta = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

Summary: Examples of Linear Transformations

To compute the rotation of the vector **x** by θ, multiply **x** by the matrix

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To compute the projection of the vector **x** onto the vector $[a_1, a_2]$, multiply **x** by the matrix

$$proj_{[a_1,a_2]} = \begin{bmatrix} \frac{a_1^2}{a_1^2+a_2^2} & \frac{a_1 a_2}{a_1^2+a_2^2} \\ \frac{a_1 a_2}{a_1^2+a_2^2} & \frac{a_2^2}{a_1^2+a_2^2} \end{bmatrix}$$

To compute the reflection of the vector **x** across the line through the origin that makes an angle of ϕ with the x-axis, multiply **x** by the matrix

$$Ref_\theta = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

Which transformations are equivalent to matrix multiplication?

Which transformations are equivalent to matrix multiplication?

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix} = 9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix} = 9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}$
- Since T is linear,
  $T\left(\begin{bmatrix}9\\14\end{bmatrix}\right)$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix} = 9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}$
- Since T is linear,
  $T\left(\begin{bmatrix}9\\14\end{bmatrix}\right) = T\left(9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}\right)$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix} = 9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}$
- Since T is linear,
  $$T\left(\begin{bmatrix}9\\14\end{bmatrix}\right) = T\left(9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}\right) = 9T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 14T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix}=9\begin{bmatrix}1\\0\end{bmatrix}+14\begin{bmatrix}0\\1\end{bmatrix}$
- Since T is linear,
  $T\left(\begin{bmatrix}9\\14\end{bmatrix}\right)=T\left(9\begin{bmatrix}1\\0\end{bmatrix}+14\begin{bmatrix}0\\1\end{bmatrix}\right)=9T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+14T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- In general,
  $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix} = 9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}$
- Since T is linear,
  $T\left(\begin{bmatrix}9\\14\end{bmatrix}\right) = T\left(9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}\right) = 9T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 14T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- In general,
  $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right)$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$
- Since $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.
  For example, $\begin{bmatrix}9\\14\end{bmatrix} = 9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}$
- Since T is linear,
  $$T\left(\begin{bmatrix}9\\14\end{bmatrix}\right) = T\left(9\begin{bmatrix}1\\0\end{bmatrix} + 14\begin{bmatrix}0\\1\end{bmatrix}\right) = 9T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 14T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
- In general,
  $$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) =$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) =$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}5\\5\\5\\5\end{bmatrix}$
-

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$
- $$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$
- $$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= x\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + y\begin{bmatrix}5\\5\\5\\5\end{bmatrix}$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$
-

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= x\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + y\begin{bmatrix}5\\5\\5\\5\end{bmatrix} = \begin{bmatrix}1x+5y\\2x+5y\\3x+5y\\4x+5y\end{bmatrix}$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$
- $$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= x\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + y\begin{bmatrix}5\\5\\5\\5\end{bmatrix} = \begin{bmatrix}1x+5y\\2x+5y\\3x+5y\\4x+5y\end{bmatrix} = \begin{bmatrix}1 & 5\\2 & 5\\3 & 5\\4 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Which transformations are equivalent to matrix multiplication?

Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\5\\5\\5\end{bmatrix}$
-

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= x\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + y\begin{bmatrix}5\\5\\5\\5\end{bmatrix} = \begin{bmatrix}1x+5y\\2x+5y\\3x+5y\\4x+5y\end{bmatrix} = \begin{bmatrix}1 & 5\\2 & 5\\3 & 5\\4 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

- So: $T(x)$ can be computed as a matrix multiplication,

$$T(\mathbf{x}) = \begin{bmatrix} | & | \\ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ | & | \end{bmatrix}\mathbf{x}$$

Suppose a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ satisfies the following:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-2\end{bmatrix}$$

Then $T(\mathbf{x}) = A\mathbf{x}$ for the matrix $A =$

Suppose a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ satisfies the following:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-2\end{bmatrix}$$

Then $T(\mathbf{x})=A\mathbf{x}$ for the matrix $A=\begin{bmatrix}2&0&3\\5&1&-2\end{bmatrix}$

Which transformations are equivalent to matrix multiplication?

Theorem

Every linear transformation T that takes a vector as an input, and gives a vector as an output, is equivalent to a matrix multiplication.

Which transformations are equivalent to matrix multiplication?

Theorem

Every linear transformation T that takes a vector as an input, and gives a vector as an output, is equivalent to a matrix multiplication.

Extended Theorem

Suppose T is a linear transformation that transforms vectors of $\mathbb{R}^n$ into vectors of $\mathbb{R}^m$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, then:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

That is: $e_1 = [1, 0, \ldots, 0]$, $e_2 = [0, 1, 0, \ldots, 0]$, etc.

Geometric interpretation of an n-by-m matrix:
linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.
Every matrix can be viewed as a linear transformation, and every linear transformation between $\mathbb{R}^n$ and $\mathbb{R}^m$ can be viewed as a matrix.

A matrix can be viewed as a particular kind of function.

General Linear Transformations

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear

General Linear Transformations

$$T : \mathbb{R}^n \to \mathbb{R}^m \qquad \text{linear}$$

Standard basis of $\mathbb{R}^n$:

$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

General Linear Transformations

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear

Standard basis of $\mathbb{R}^n$:

$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}7\\7\end{bmatrix}$$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}7\\7\end{bmatrix}$$

Give a matrix A so that $T(x)=Ax$ for every vector x in $\mathbb{R}^2$.

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}7\\7\end{bmatrix}$$

Give a matrix A so that $T(x)=Ax$ for every vector x in $\mathbb{R}^2$.

Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ has the following properties:

$$T\left(\begin{bmatrix}5\\7\end{bmatrix}\right) = \begin{bmatrix}7\\5\\12\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) = \begin{bmatrix}6\\4\\10\end{bmatrix}$$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

Examples

Suppose a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ has the following properties:

$$T\left(\begin{bmatrix}5\\7\end{bmatrix}\right)=\begin{bmatrix}7\\5\\12\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\6\end{bmatrix}\right)=\begin{bmatrix}6\\4\\10\end{bmatrix}$$

Give a matrix A so that $T(x) = Ax$ for every vector x in $\mathbb{R}^2$.

$$A=\begin{bmatrix}0&1\\1&0\\1&1\end{bmatrix}$$

Examples

Suppose T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, where $T(x) = Ax$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Which vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has $T(x) = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$?

Which vector $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ has $T(y) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$?

Examples

Suppose T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, where $T(x) = Ax$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Which vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has $T(x) = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$?

Which vector $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ has $T(y) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$?

Characterize vectors that can come out of T.

Random Walks: Another Use of Matrix Multiplication

- n states
- Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

Random Walks: Another Use of Matrix Multiplication

- n states
- Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

Examples:

`https://en.wikipedia.org/wiki/Random_walk`

model Brownian Motion (Wiener process)

Random Walks: Another Use of Matrix Multiplication

- n states
- Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

Examples:
https://en.wikipedia.org/wiki/Random_walk
model Brownian Motion (Wiener process)
genetic drift

Random Walks: Another Use of Matrix Multiplication

- n states
- Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

Examples:
https://en.wikipedia.org/wiki/Random_walk
model Brownian Motion (Wiener process)
genetic drift
stock markets

Random Walks: Another Use of Matrix Multiplication

•n states

•Fixed probability $p_{i,j}$ of moving to state i if you are in state j.

Examples:

`https://en.wikipedia.org/wiki/Random_walk`

model Brownian Motion (Wiener process)

genetic drift

stock markets

use sampling to estimate properties of a large system

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

Sleeping: `https://pixabay.com/en/penguin-linux-sleeping-animal-159784/`
Fishing: By Mimooh (Own work), via Wikimedia Commons
Playing: By Silvermoonlight217
`http://silvermoonlight217.deviantart.com/art/Penguin-Sledding-262107547`

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

Let x_n be the vector describing the probability that the penguin is sleeping/fishing/playing after n hours.

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

Let x_n be the vector describing the probability that the penguin is sleeping/fishing/playing after n hours.

x_0: initial state of penguin. For example: $[1, 0, 0]$ if we know the penguin is sleeping.

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

Let x_n be the vector describing the probability that the penguin is sleeping/fishing/playing after n hours.

x_0: initial state of penguin. For example: $[1, 0, 0]$ if we know the penguin is sleeping.

x_1:

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

Let x_n be the vector describing the probability that the penguin is sleeping/fishing/playing after n hours.

x_0: initial state of penguin. For example: $[1, 0, 0]$ if we know the penguin is sleeping.

x_1: $\begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

$$x_1: \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$$

$$x_2: \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = Px_1 =$$

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

$$x_1: \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} \qquad x_1 = \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = Px_0$$

$$x_2: \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = Px_1 =$$

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

$$x_1: \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} \qquad x_1 = \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = Px_0$$

$$x_2: \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = Px_1 = P\left(Px_0\right)$$

Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

from / *to*	*sleeping*	*fishing*	*playing*
sleeping	.5	.7	.4
fishing	.25	0	.3
playing	.25	.3	.3

$$x_1: \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} \qquad x_1 = \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = Px_0$$

$$x_2: \begin{bmatrix} .5 & .7 & .4 \\ .25 & 0 & .3 \\ .25 & .3 & .3 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = Px_1 = P\left(Px_0\right) = P^2 x_0$$

Random Walks

In general:

• n states

• $p_{i,j}$ probability of moving to state i if you are in state j; $P = [p_{i,j}]$

Random Walks

In general:

•n states

•$p_{i,j}$ probability of moving to state i if you are in state j; $P = [p_{i,j}]$

Given x_n:

$x_{n+1} = Px_n = P^{n+1}x_0$

Random Walks

In general:

•n states

•$p_{i,j}$ probability of moving to state i if you are in state j; $P = [p_{i,j}]$

Given x_n:

$x_{n+1} = Px_n = P^{n+1}x_0$

P: "transition matrix"

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

Rob, https://www.flickr.com/photos/rh1985/22218233156

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground			
Rope			
Right ground			

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1		
Rope			
Right ground			

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1		
Rope	0		
Right ground			

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1		
Rope	0		
Right ground	0		

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\text{from}}{\text{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	
Rope	0		
Right ground	0		

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\text{from}}{\text{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	
Rope	0	0.94	
Right ground	0		

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	
Rope	0	0.94	
Right ground	0	0.01	

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

from / *to*	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	
Right ground	0	0.01	

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

$\frac{\textit{from}}{\textit{to}}$	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	0
Right ground	0	0.01	

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

from / *to*	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	0
Right ground	0	0.01	1

Notice: columns add to 1; rows don't have to

Random Walk Example: Falling Down

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

from / *to*	Left ground	Rope	Right ground
Left ground	1	0.05	0
Rope	0	0.94	0
Right ground	0	0.01	1

Where are you after 100 steps?

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

After 100 steps:

x_{100}

$$\begin{pmatrix} \text{left} \\ \text{rope} \\ \text{right} \end{pmatrix}$$

We approximate this using software FOR NOW; later we'll learn analytical ways.

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

After 100 steps:

$$x_{100} = P^{100} x_0 \qquad \begin{pmatrix} \text{left} \\ \text{rope} \\ \text{right} \end{pmatrix}$$

We approximate this using software FOR NOW; later we'll learn analytical ways.

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

After 100 steps:

$$x_{100} = P^{100} x_0 = \begin{bmatrix} 1 & 0.05 & 0 \\ 0 & 0.94 & 0 \\ 0 & 0.01 & 1 \end{bmatrix}^{100} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{pmatrix} \text{left} \\ \text{rope} \\ \text{right} \end{pmatrix}$$

We approximate this using software FOR NOW; later we'll learn analytical ways.

You are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you're scared. When you fall, you stay on the ground where you landed.

After 100 steps:

$$x_{100} = P^{100}x_0 = \begin{bmatrix} 1 & 0.05 & 0 \\ 0 & 0.94 & 0 \\ 0 & 0.01 & 1 \end{bmatrix}^{100} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.8316209 \\ 0.0020549 \\ 0.1663242 \end{bmatrix} \qquad \begin{pmatrix} \text{left} \\ \text{rope} \\ \text{right} \end{pmatrix}$$

We approximate this using software FOR NOW; later we'll learn analytical ways.

Random Walk Example: Error Messages

Suppose you are using a buggy program. You start up without a problem.

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, your next click is to uninstall the program, and never use it again.

Random Walk Example: Error Messages

Suppose you are using a buggy program. You start up without a problem.

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, your next click is to uninstall the program, and never use it again.

Possible states: no errors; one error; two errors; three errors; uninstalled.

Random Walk Example

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.
- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.
- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.
- After the third error message, you uninstall the program.

Possible states: no errors; one error; two errors; three errors; uninstalled.

from / *to*	0	1	2	3	*u*
0	.99	0	0	0	0
1	.01	.95	0	0	0
2	0	.05	.9	0	0
3	0	0	.1	0	0
u	0	0	0	1	1

Again, notice: columns sum to 1, rows don't have to

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$,

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$

- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx \begin{bmatrix} 0.366 \\ 0.090 \\ 0.049 \\ 0.005 \\ 0.490 \end{bmatrix}$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx \begin{bmatrix} 0.366 \\ 0.090 \\ 0.049 \\ 0.005 \\ 0.490 \end{bmatrix}$
- $\mathbf{x}_{200} = P^{200}\mathbf{x}_0 \approx$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx \begin{bmatrix} 0.366 \\ 0.090 \\ 0.049 \\ 0.005 \\ 0.490 \end{bmatrix}$
- $\mathbf{x}_{200} = P^{200}\mathbf{x}_0 \approx \begin{bmatrix} 0.134 \\ 0.033 \\ 0.019 \\ 0.002 \\ 0.812 \end{bmatrix}$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx \begin{bmatrix} 0.366 \\ 0.090 \\ 0.049 \\ 0.005 \\ 0.490 \end{bmatrix}$
- $\mathbf{x}_{200} = P^{200}\mathbf{x}_0 \approx \begin{bmatrix} 0.134 \\ 0.033 \\ 0.019 \\ 0.002 \\ 0.812 \end{bmatrix}$
- $\lim_{n\to\infty} \mathbf{x}_n =$

With the ordering $\begin{bmatrix} \text{0 errors} \\ \text{1 error} \\ \text{2 errors} \\ \text{3 errors} \\ \text{uninstalled} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Using Octave:

- $\mathbf{x}_{10} = P^{10}\mathbf{x}_0 \approx \begin{bmatrix} 0.904 \\ 0.076 \\ 0.015 \\ 0.001 \\ 0.003 \end{bmatrix}$
- $\mathbf{x}_{20} = P^{20}\mathbf{x}_0 \approx \begin{bmatrix} 0.818 \\ 0.115 \\ 0.0037 \\ 0.004 \\ 0.026 \end{bmatrix}$
- $\mathbf{x}_{100} = P^{100}\mathbf{x}_0 \approx \begin{bmatrix} 0.366 \\ 0.090 \\ 0.049 \\ 0.005 \\ 0.490 \end{bmatrix}$
- $\mathbf{x}_{200} = P^{200}\mathbf{x}_0 \approx \begin{bmatrix} 0.134 \\ 0.033 \\ 0.019 \\ 0.002 \\ 0.812 \end{bmatrix}$
- $\lim\limits_{n\to\infty} \mathbf{x}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (we'll do these computations more generally once we learn about eigenvalues!)

Harder Questions involving Random Walks

- For which value of n does x_n have a certain characteristic?

Harder Questions involving Random Walks

- For which value of n does x_n have a certain characteristic?
- What is $\lim_{n\to\infty} x_n$?

Harder Questions involving Random Walks

- For which value of n does x_n have a certain characteristic?
- What is $\lim_{n \to \infty} x_n$?
  Note: $\lim_{n \to \infty} x_n = \lim_{n \to \infty} P^n x_0$.

Harder Questions involving Random Walks

- For which value of n does x_n have a certain characteristic?
- What is $\lim_{n\to\infty} x_n$?
  Note: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} P^n x_0$.
- Does $\lim_{n\to\infty} x_n$ depend on x_0?

Harder Questions involving Random Walks

- For which value of n does x_n have a certain characteristic?
- What is $\lim_{n \to \infty} x_n$?
  Note: $\lim_{n \to \infty} x_n = \lim_{n \to \infty} P^n x_0$.
- Does $\lim_{n \to \infty} x_n$ depend on x_0?

Stay tuned for more Random Walks excitement

Application: Google!

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix} \qquad BA = DNE$$

Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix} \qquad BA = DNE$$

$$B^T A^T = \begin{bmatrix} 6 & 15 \\ 12 & 30 \\ 18 & 45 \end{bmatrix} \qquad AB = (B^T A^T)^T$$

Transpose and Matrix Multiplication

$$AB = P$$

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

$$B^T A^T = Q$$

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

$$B^T A^T = Q$$

(row b of B^T)$\cdot$(column a of A^T)$\rightarrow Q_{ba}$ (row b, column a)

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

$$B^T A^T = Q$$

(row b of B^T)$\cdot$(column a of A^T)$\rightarrow Q_{ba}$ (row b, column a)
(column b of B)$\cdot$(row a of A)$\rightarrow Q_{ba}$

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

$$B^TA^T = Q$$

(row b of B^T)$\cdot$(column a of A^T)$\rightarrow Q_{ba}$ (row b, column a)
(column b of B)$\cdot$(row a of A)$\rightarrow Q_{ba}$
(row a of A)$\cdot$(column b of B)$\rightarrow Q_{ba}$

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

$$B^T A^T = Q$$

(row b of B^T)$\cdot$(column a of A^T)$\rightarrow Q_{ba}$ (row b, column a)
(column b of B)$\cdot$(row a of A)$\rightarrow Q_{ba}$
(row a of A)$\cdot$(column b of B)$\rightarrow Q_{ba}$

Transpose and Matrix Multiplication

$$AB = P$$

(row a of A)$\cdot$(column b of B)$\rightarrow P_{ab}$ (row a, column b)

$$B^T A^T = Q$$

(row b of B^T)$\cdot$(column a of A^T)$\rightarrow Q_{ba}$ (row b, column a)
(column b of B)$\cdot$(row a of A)$\rightarrow Q_{ba}$
(row a of A)$\cdot$(column b of B)$\rightarrow Q_{ba}$

$$P = Q^T$$

Transpose

Previous example of noncommutativity of matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

Transpose

Previous example of noncommutativity of matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 5 & 0 \end{bmatrix}$$

Transpose and Dot Product

$$\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$$

where A is an m-by-n matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Transpose and Dot Product

$$\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$$

where A is an m-by-n matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 9 \\ 1 \end{bmatrix} = 8 + 18 + 3 = 29$$

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 9 \end{bmatrix} = -16 + 45 = 29$$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T \mathbf{y}) \cdot \mathbf{x}$

- $\left(A^T\right)^T = A$
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T \mathbf{y})$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T\mathbf{y}) \cdot \mathbf{x}$

- $(A^T)^T = A$ true
- $\left(\left(\left((A^T)^T\right)^T\right)^T\right)^T = A$
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T\mathbf{y})$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T \mathbf{y}) \cdot \mathbf{x}$

- $\left(A^T\right)^T = A$ true
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$ false
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T \mathbf{y})$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T \mathbf{y}) \cdot \mathbf{x}$

- $\left(A^T\right)^T = A$ true
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$ false
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$ false
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T \mathbf{y})$

True or False?

Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $\mathbf{y} \cdot (A\mathbf{x}) = (A^T \mathbf{y}) \cdot \mathbf{x}$

- $\left(A^T\right)^T = A$ true
- $\left(\left(\left(\left(A^T\right)^T\right)^T\right)^T\right)^T = A$ false
- $(AB)\mathbf{x} = \left(\mathbf{x}^T B^T\right)^T A$ false
- $\mathbf{y} \cdot (A\mathbf{x}) = \mathbf{x} \cdot (A^T \mathbf{y})$ true