## Outline

Week 4: Solving Linear Systems

Course Notes: 3.2, 3.3, 3.4

Goals: Learn the method of Gaussian elimination to efficiently solve linear systems; describe infinite families of solutions as parametric equations; use properties of associated homogeneous systems.

Notes



## Row Operations

- Multiply one row by a non-zero scalar
- Add a multiple of one row to another
- Interchange rows

These operations will change the system of equations, but will not change the *solutions* of the system.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 34 \\ 1 & 1 & 0 & 13 \\ 0 & 1 & 2 & 11 \end{array}\right]$$

Notes



## Quick Notation

Row Echelon Form: the position of the first non-zero entry in a row strictly increases from one row to the row below it.

$$\left[\begin{array}{cccccc|c} 2 & 3 & 5 & 2 & 5 & 9 & 13 \\ 0 & 5 & 4 & 3 & 4 & 0 & 11 \\ 0 & 0 & 0 & 3 & 2 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row Echelon Form: the first non-zero entry in every row is a 1, and is the only non-zero entry in its column; the position of the first non-zero entry in a row strictly increases with every row.

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & 0 & 13 \\ 0 & 1 & 4 & 5 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notes

## Gaussian Elimination

Use row operations to change this system to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 2 & 2 & -6 & -16 \\ 3 & -2 & 1 & 11 \\ -1 & 3 & 1 & -2 \end{array}\right]$$

Notes



## Gaussian Elimination

Use row operations to change this system to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 8 \\ 5 & 3 & 6 & 10 \end{array}\right]$$

(Try to do it using at most 6 row operations!)

Notes



## Gaussian Elimination

Use row operations to change this system to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 15 \\ 4 & 4 & 0 & 36 \\ 3 & -1 & 6 & -31 \end{array}\right]$$

(Try to do it using at most 8 row operations!)

Notes

## Use row operations to solve this system

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 6 \\ 5 & 10 & 3 & 3 & 28 \\ 2 & 4 & 1 & 2 & 15 \\ 3 & 6 & 2 & 3 & 21 \end{array}\right]$$

Notes



Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 17 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notes



Give a parametric equation for the solutions of this system.

$$\left[\begin{array}{cccccc|c} 1 & 5 & 7 & 0 & 2 & 0 & 17 \\ 0 & 0 & 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notes

Give a parametric equation for the solutions of this system.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notes



Give a parametric equation for the solutions of this augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 5 & 5 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

Notes



## Fill in the Table

| | more variables $\left[\begin{array}{ccc\|c} * & * & * & * \\ * & * & * & * \end{array}\right]$ | square $\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \end{array}\right]$ | more equations $\left[\begin{array}{cc\|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$ |
|---|---|---|---|
| No Solutions | | | |
| One Solution | | | |
| Infinitely Many | | | |

Notes

## Rank and Solutions

How many solutions?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 1 & * \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 1 & * \\ 0 & 0 & 0 & * \end{array}\right]$$

The *rank* of a matrix (without its augmented part) is the number of non-zero rows in the matrix obtained after reducing it. If a matrix has rank $r$ and $n$ columns, then the solution will require parameters (provided a solution exists at all).

Notes



## Homogeneous Systems

System of equations:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Associated homogeneous system of equations:

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Notes



$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Give a solution to this equation.

Suppose $\mathbf{a}$ and $\mathbf{b}$ are solutions to a homogeneous system of equations.

Then $\mathbf{a} + \mathbf{b}$ is also a solution.

Also, $c\mathbf{a}$ is a solution for any scalar $c$.

Notes

## Connection to Non-homogeneous Systems

Suppose $A$ is an augmented matrix, and $A_0$ is its associated homogeneous system.
If $\mathbf{a}$ and $\mathbf{b}$ are solutions to $A$, then $\boxed{\phantom{xxxx}}$ is a solution to $A_0$.

$$A = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 13 \\ 11 & 3 & -1 & 13 & 42 \\ -7 & -2 & 1 & -8 & -26 \\ 4 & 1 & 0 & 5 & 16 \end{array}\right]$$

$$A_0 = \left[\begin{array}{cccc|c} 4 & 1 & -1 & 4 & 0 \\ 11 & 3 & -1 & 13 & 0 \\ -7 & -2 & 1 & -8 & 0 \\ 4 & 1 & 0 & 5 & 0 \end{array}\right]$$

Given one solution $\mathbf{q}$ to $A$, every solution to $A$ can be written in the form $\mathbf{q}+\mathbf{a}$ for some solution $\mathbf{a}$ of $A_0$.

Notes



## Rank and Linear Independence

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -1 & -3 & 0 \end{array}\right] \quad R_4 \to R_4 - R_3 + R_2 + 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_4 = R_3 - R_2 - 2R_2$$

If row reduction results in a row of all 0s, then the row vectors of the original matrix were not linearly independent.

Notes



## Rank and Linear Independence

Suppose this matrix has rows that are linearly independent:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

What is its reduced form going to be?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notes

## Rank and Linear Independence

Suppose this augmented matrix has rows that are linearly independent:

$$\left[\begin{array}{ccccc|c} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \end{array}\right]$$

What will its solutions set be? (Point, line, plane, etc.)

Notes



## Geometry in Unimaginable Dimensions

What does the equation $0x_1 + 0x_2 + 0x_3 + x_4 = 0$ describe in $\mathbb{R}^4$?
In general, an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = a_5$ describes a three-dimensional space in $\mathbb{R}^4$.
What will be the intersection of four linearly-independent 3-dimensional spaces of this type?
What will be the intersection of $n$ linearly-independent $(n-1)$-dimensional spaces of this type in $\mathbb{R}^n$?

Notes



Check this collection of vectors for linear independence:
$\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and $\mathbf{c} = [c_1, c_2, c_3]$.

**Recall**

Vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly independent if the equation

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$$

has only one solution, $s_1 = s_2 = \cdots = s_n = 0$.

$$s_1\mathbf{a} + s_2\mathbf{b} + s_3\mathbf{c} = \mathbf{0}$$

$$\begin{bmatrix} s_1a_1 \\ s_1a_2 \\ s_1a_3 \end{bmatrix} + \begin{bmatrix} s_2b_1 \\ s_2b_2 \\ s_2b_3 \end{bmatrix} + \begin{bmatrix} s_3c_1 \\ s_3c_2 \\ s_3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{\begin{array}{ll} a_1s_1 + b_1s_2 + c_1s_3 & = 0 \\ a_2s_1 + b_2s_2 + c_2s_3 & = 0 \\ a_3s_1 + b_3s_2 + c_3s_3 & = 0 \end{array}\right. \rightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{array}\right]$$

Notes

$$\mathbf{a} = [1, 1, 1, 1], \qquad \mathbf{b} = [2, 1, 1, 1]$$

Are **a** and **b** linearly independent?

$$x_1\mathbf{a} + x_2\mathbf{b} = \mathbf{0}:$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

Can we row-reduce this matrix to get a row of all 0s?

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

Notes



## Checking Vectors for Linear Independence

Are $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are linearly independent? (Two ways.)

$$\left[\begin{array}{cccc|c} | & | & & | & 0 \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \vdots \\ | & | & & | & 0 \end{array}\right]$$

- If there is only one solution, the vectors are linearly independent.
- If there is more than one solution, the vectors are linearly dependent.

$$\begin{bmatrix} -- & \mathbf{a}_1 & -- \\ -- & \mathbf{a}_2 & -- \\ & \vdots & \\ -- & \mathbf{a}_n & -- \end{bmatrix}$$

- If the reduced matrix has no rows of all zeroes, the vectors are linearly independent.
- If the reduced matrix has some row of all zeroes, the vectors are linearly dependent.

Notes



Write $\begin{bmatrix} 16 \\ 5 \\ 11 \end{bmatrix}$ as a linear combination of vectors from the set

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \right\}$$

Is the set linearly independent? Is it a basis of $\mathbb{R}^3$?

Notes

Is the set linearly independent? Is it a basis?

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ -13 \end{bmatrix} \right\}$$

Is the set linearly independent? Is it a basis?

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 6 \\ 9 \\ 0 \end{bmatrix} \right\}$$

Notes



Two points determine a line. How many points are needed to determine an $n$-th degree polynomial?

Notes



Remove one vector to make this set linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix} \right\}$$

Suppose a vector **a** can be written as a linear combination of vectors in the set. Can you still write **a** as a linear combination of vectors in the set WITHOUT using the first vector?

Notes