Outline

Week 3: Introduction to Linear Systems

Course Notes: 2.6, 3.1

Goals: Consider the solution to a system of linear equations as a geometric object; learn basic techniques (back substitution, row reduction) for solving systems of linear equations.

Notes

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$?

A. nothing B. point C. line D. plane E. two points F. two lines

The intersection of the two lines is the set of points (x, y) that are solutions to this system of linear equations:

$$\begin{aligned} a_1x + a_2y &= a_3 \\ b_1x + b_2y &= b_3 \end{aligned}$$

If the intersection is a **point**, what can we say about $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$?

A. zero B. nonzero C. positive D. negative

Notes

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of two planes in $\mathbb{R}^3$?

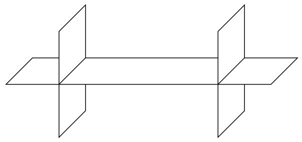
A. nothing B. point C. line D. plane

Notes

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane
E. two points F. two lines G. two planes



$$\begin{array}{ccccccc} a_1x & + & a_2y & + & a_3z & = & a_4 \\ b_1x & + & b_2y & + & b_3z & = & b_4 \\ c_1x & + & c_2y & + & c_3z & = & c_4 \end{array}$$

Suppose $(1, 3, 5)$ and $(2, 6, 10)$ are solutions to the system of equations.

How many solutions total are there?

Give another solution.

Possible solutions:

Notes

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$.
Given a collection of vectors, the set of all their possible linear combinations is the **span** of the vectors.

So, the parametric equation $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ is just the set of all linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

Fact: if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^2$ that are *not parallel*, then every point in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Related Fact: if $\mathbf{a}$, $\mathbf{b}$, an $\mathbf{c}$ are vectors in $\mathbb{R}^3$ that *do not all lie on the same plane*, then every point in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Test for colinearity or coplanarity using determinant.

Notes

Linear (In)dependence

Definition we want:
If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them *linearly independent* if none is a linear combination of the others.

Definition: Linear Independence

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them *linearly independent* if the only solution to the equation

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$$

is $s_1 = s_1 = \cdots = s_n = 0$.
So, the vectors are *linearly dependent* if there exist scalars $s_1, s_2, \ldots, s_n$, at least one of which is nonzero, such that $s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$.

Notes

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are *linearly dependent*.

Example: $2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \end{bmatrix}$, so $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ 14 \end{bmatrix} \right\}$ are *linearly dependent*.

Notes

Basis

Definition: Basis

In $\mathbb{R}^n$, a collection of n linearly independent vectors is called a *basis*.

Any $\mathbf{x}$ in $\mathbf{R}^n$ can be written as a linear combination of basis vectors.

Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ form a basis.

Write $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Notes

Bases

In $\mathbb{R}^3$, what is the *easiest* basis to work with?
That is: find $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ so that it is extremely easy to solve the system

$$s_1\mathbf{a} + s_2\mathbf{b} + s_3\mathbf{c} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Notes

Give one vector in $\mathbb{R}^2$ that can never be in a basis of $\mathbb{R}^2$.

(Remember: a basis in $\mathbb{R}^2$ is a collection of two vectors $\mathbf{a}$ and $\mathbf{b}$ so that the only solution to the equation $s\mathbf{a} + t\mathbf{b} = \mathbf{0}$ is $s = t = 0$.)

Notes

Suppose:

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and also

$$\begin{bmatrix} x \\ y \end{bmatrix} = p \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $s \neq p$.

Is $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$?

Recall: a basis in $\mathbb{R}^2$ is two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $s_1\mathbf{a} + s_2\mathbf{b} = \mathbf{0}$ ONLY when $s_1 = s_2 = 0$.

Notes

Substitution

Give $\begin{bmatrix} 2 \\ 24 \\ 49 \end{bmatrix}$ as a linear combination of the vectors in the basis below.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Notes

Substitution

Suppose a thrown ball at time t has height $h = At^2 + Bt + C$.
At $t = 1$, the ball is at height 0;
at $t = 2$, the ball is at height 1; and
at $t = 3$, the ball is at height 6.

Find A, B, and C.

Notes

General Form

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & c_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & c_2 \\ & \vdots & & & \vdots & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & c_m \end{array}$$

Where $a_{i,j}$ and c_i are known and fixed.

Notes

Goal: easily-solvable system

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 17x_3 & + & 9x_4 & = & 10 \\ -3x_1 & + & -6x_2 & + & 8x_3 & + & 5x_4 & = & 17 \\ \pi x_1 & + & -8x_2 & + & 3x_3 & + & x_4 & = & -2 \\ 8x_1 & + & -8x_2 & + & 5x_3 & + & 2x_4 & = & 2 \end{array}$$

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 17x_3 & + & 9x_4 & = & 10 \\ & & -6x_2 & + & 8x_3 & + & 5x_4 & = & 17 \\ & & & & 3x_3 & + & x_4 & = & -2 \\ & & & & & & 2x_4 & = & 2 \end{array}$$

Upper Triangular

$$\begin{array}{rcrcrcrcr} x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 10 \\ & & -6x_2 & + & 0x_3 & + & 0x_4 & = & 17 \\ & & & & 3x_3 & + & 0x_4 & = & -2 \\ & & & & & & 2x_4 & = & 2 \end{array}$$

Diagonal

Notes

Equivalent Equations

Notice:

$$3x + 5y + 7z = 10$$

and

$$6x + 10y + 14z = 20$$

have the same solutions.

Caution:

$$0x + 0y + 0z = 0$$

has more solutions.

Notes

Elementary Row Operations: Multiplication of a row by a non-zero number

$$\begin{array}{rcrcrcr} 3x & - & 9y & + & 6z & = & 30 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Same solutions as:

$$\begin{array}{rcrcrcr} 1x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Notes

Elementary Row Operations: Adding a Multiple of a Row to Another Row

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Same solutions as:

$$\begin{array}{rcrcrcr} 0x & - & 0y & + & 7z & = & 14 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Notes

Elementary Row Operations: Interchanging Rows

$$\begin{array}{rcrcrcr} & & & & 7z & = & 14 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Same solutions as:

$$\begin{array}{rcrcrcr} -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \\ & & & & 7z & = & 14 \end{array}$$

Notes

Streamlined Notation: Augmented Matrices

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

We'll write this as:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 10 \\ -1 & 3 & 5 & 4 \\ 1 & 1 & 1 & -6 \end{array}\right]$$

Notes

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

Notes

Using Elementary Row Operations

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \leftrightarrow R_2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{c} x = 11 \\ y = -4 \\ z = 0 \end{array}$$

Notes

Solve using Elementary Row Operations

$$\begin{cases} 2x + y + z & = 8 \\ x - y - 3z & = -5 \\ -x - 2y + z & = 2 \end{cases}$$

Notes

Row Operation Calculator (link)

Notes