Outline

Week 3: Introduction to Linear Systems

Course Notes: 2.6, 3.1

Goals: Consider the solution to a system of linear equations as a geometric object; learn basic techniques (back substitution, row reduction) for solving systems of linear equations.

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$?

A. nothing B. point C. line D. plane
E. two points F. two lines

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$?

A. nothing
B. point
C. line
D. plane
E. two points
F. two lines

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$?

A. nothing B. point C. line D. plane
E. two points F. two lines

The intersection of the two lines is the set of points (x, y) that are solutions to this system of linear equations:

$$\begin{aligned} a_1x + a_2y &= a_3 \\ b_1x + b_2y &= b_3 \end{aligned}$$

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$?

A. nothing B. point C. line D. plane
E. two points F. two lines

The intersection of the two lines is the set of points (x, y) that are solutions to this system of linear equations:

$$\begin{aligned} a_1x + a_2y &= a_3 \\ b_1x + b_2y &= b_3 \end{aligned}$$

If the intersection is a **point**, what can we say about $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$?

A. zero B. nonzero C. positive D. negative

Intersections: $\mathbb{R}^2$

Which of the following could be the intersection of lines $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$?

A. nothing
B. point
C. line
D. plane
E. two points
F. two lines

The intersection of the two lines is the set of points (x, y) that are solutions to this system of linear equations:

$$\begin{aligned} a_1x + a_2y &= a_3 \\ b_1x + b_2y &= b_3 \end{aligned}$$

If the intersection is a **point**, what can we say about $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$?

A. zero
B. nonzero
C. positive
D. negative

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of two planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

Intersections: $\mathbb{R}^3$

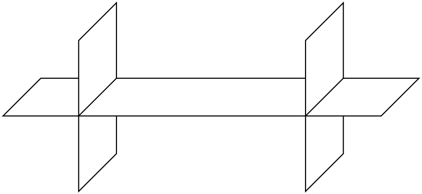
Which of the following could be the intersection of two planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

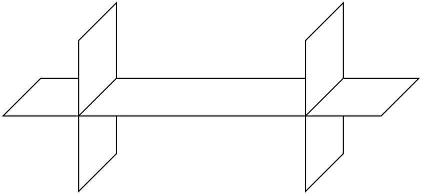
A. nothing B. point C. line D. plane
E. two points F. two lines G. two planes



Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

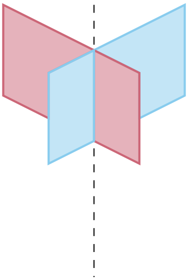
A. nothing B. point C. line D. plane
E. two points F. two lines G. two planes



Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

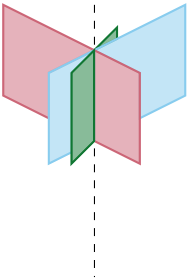
A. nothing B. point C. line D. plane
E. two points F. two lines G. two planes



Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

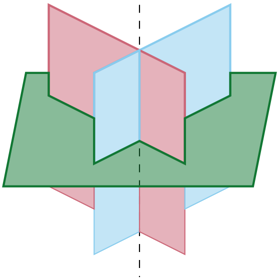
A. nothing
B. point
C. line
D. plane
E. two points
F. two lines
G. two planes



Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

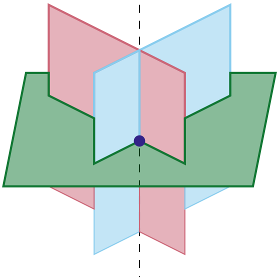
A. nothing B. point C. line D. plane
E. two points F. two lines G. two planes



Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane
E. two points F. two lines G. two planes



Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

E. two points F. two lines G. two planes

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

$$\begin{array}{ccccccc} a_1x & + & a_2y & + & a_3z & = & a_4 \\ b_1x & + & b_2y & + & b_3z & = & b_4 \\ c_1x & + & c_2y & + & c_3z & = & c_4 \end{array}$$

Possible solutions:

$\emptyset$

$\mathbf{x} = \mathbf{q}$

$\mathbf{x} = \mathbf{q} + s\mathbf{a}$

$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

$$\begin{array}{ccccccc} a_1x & + & a_2y & + & a_3z & = & a_4 \\ b_1x & + & b_2y & + & b_3z & = & b_4 \\ c_1x & + & c_2y & + & c_3z & = & c_4 \end{array}$$

Suppose the intersection of the three planes is a point. Then

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \neq 0$$

(Remember the volume of a parallelepiped.)

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

$$\begin{array}{ccccccc} a_1x & + & a_2y & + & a_3z & = & a_4 \\ b_1x & + & b_2y & + & b_3z & = & b_4 \\ c_1x & + & c_2y & + & c_3z & = & c_4 \end{array}$$

Suppose $(1, 3, 5)$ and $(2, 6, 10)$ are solutions to the system of equations.

How many solutions total are there?

Intersections: $\mathbb{R}^3$

Which of the following could be the intersection of **three** planes in $\mathbb{R}^3$?

A. nothing B. point C. line D. plane

$$\begin{array}{ccccccc} a_1x & + & a_2y & + & a_3z & = & a_4 \\ b_1x & + & b_2y & + & b_3z & = & b_4 \\ c_1x & + & c_2y & + & c_3z & = & c_4 \end{array}$$

Suppose $(1,3,5)$ and $(2,6,10)$ are solutions to the system of equations.

How many solutions total are there?

Give another solution.

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$.
Given a collection of vectors, the set of all their possible linear combinations is the **span** of the vectors.

On the other hand, $\mathbf{a}_1 \times \mathbf{a}_2$ and $\mathbf{a}_1 \cdot \mathbf{a}_2$ are **not** linear combinations of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$.
Given a collection of vectors, the set of all their possible linear combinations is the **span** of the vectors.

So, the parametric equation $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ is just the set of all linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$.
Given a collection of vectors, the set of all their possible linear combinations is the **span** of the vectors.

So, the parametric equation $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ is just the set of all linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

Fact: if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^2$ that are *not parallel*, then every point in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$.
Given a collection of vectors, the set of all their possible linear combinations is the **span** of the vectors.

So, the parametric equation $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ is just the set of all linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

Fact: if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^2$ that are *not parallel*, then every point in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Related Fact: if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $\mathbb{R}^3$ that *do not all lie on the same plane*, then every point in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Definition: Linear Combination

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, and $s_1, s_2, \ldots, s_n$ are scalars, then

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n$$

is a *linear combination* of $\mathbf{a}_1$, $\mathbf{a}_2$, ... , $\mathbf{a}_n$.
Given a collection of vectors, the set of all their possible linear combinations is the **span** of the vectors.

So, the parametric equation $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ is just the set of all linear combinations of $\mathbf{a}$ and $\mathbf{b}$.

Fact: if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^2$ that are *not parallel*, then every point in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Related Fact: if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $\mathbb{R}^3$ that *do not all lie on the same plane*, then every point in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Test for colinearity or coplanarity using determinant.

Linear (In)dependence

Definition we want:

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them *linearly independent* if none is a linear combination of the others.

Linear (In)dependence

Definition we want:
If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them *linearly independent* if none is a linear combination of the others.

Definition: Linear Independence

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them **linearly independent** if the only solution to the equation

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$$

is

Linear (In)dependence

Definition we want:
If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them *linearly independent* if none is a linear combination of the others.

Definition: Linear Independence

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them **linearly independent** if the only solution to the equation

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$$

is $s_1 = s_1 = \cdots = s_n = 0$.

Linear (In)dependence

Definition we want:
If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them *linearly independent* if none is a linear combination of the others.

Definition: Linear Independence

If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are a collection of vectors, we call them **linearly independent** if the only solution to the equation

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$$

is $s_1 = s_1 = \cdots = s_n = 0$.
So, the vectors are **linearly dependent** if there exist scalars $s_1, s_2, \ldots, s_n$, at least one of which is nonzero, such that $s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots + s_n\mathbf{a}_n = \mathbf{0}$.

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ (2) Equivalently, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ (2) Equivalently, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ 14 \end{bmatrix} \}$ are **linearly dependent**

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ (2) Equivalently, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ 14 \end{bmatrix}\}$ are **linearly dependent**

(1) $2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \end{bmatrix}$

Example: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix},\begin{bmatrix}2\\3\\3\end{bmatrix}\right\}$ are **linearly dependent**.

(1) $\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}2\\3\\3\end{bmatrix}$ (2) Equivalently, $\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\2\\2\end{bmatrix}-\begin{bmatrix}2\\3\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Example: $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}16\\14\end{smallmatrix}\right]\}$ are **linearly dependent**

(1) $2\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]+7\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]=\left[\begin{smallmatrix}16\\14\end{smallmatrix}\right]$ (2) Equivalently, $2\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]+7\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]-\left[\begin{smallmatrix}16\\14\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ (2) Equivalently, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: $\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right]\}$ are **linearly dependent**

(1) $2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + 7\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right]$ (2) Equivalently, $2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + 7\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \right\}$ are **linearly independent**.

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ (2) Equivalently, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: $\{ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right] \}$ are **linearly dependent**

(1) $2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + 7\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right]$ (2) Equivalently, $2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + 7\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \right\}$ are **linearly independent**.

(1) there's no way to write one as a linear combination of the others;

Example: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ are **linearly dependent**.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ (2) Equivalently, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example: $\{ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right] \}$ are **linearly dependent**

(1) $2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + 7\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right]$ (2) Equivalently, $2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + 7\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 16 \\ 14 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \right\}$ are **linearly independent**.

(1) there's no way to write one as a linear combination of the others;
(2) If we try to solve $s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,
the only solution is $s = t = r = 0$.

Basis

Definition: Basis

In $\mathbb{R}^n$, a collection of n linearly independent vectors is called a *basis*.

Any $\mathbf{x}$ in $\mathbb{R}^n$ can be written as a linear combination of basis vectors.

Basis

Definition: Basis

In $\mathbb{R}^n$, a collection of n linearly independent vectors is called a *basis*.

Any $\mathbf{x}$ in $\mathbb{R}^n$ can be written as a linear combination of basis vectors.

Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ form a basis.

Basis

Definition: Basis

In $\mathbb{R}^n$, a collection of n linearly independent vectors is called a *basis*.

Any $\mathbf{x}$ in $\mathbb{R}^n$ can be written as a linear combination of basis vectors.

Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ form a basis.

Write $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Basis

Definition: Basis

In $\mathbb{R}^n$, a collection of n linearly independent vectors is called a *basis*.

Any $\mathbf{x}$ in $\mathbb{R}^n$ can be written as a linear combination of basis vectors.

Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ form a basis.

Write $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 7 \\ -2 \end{bmatrix} = -11 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Bases

In $\mathbb{R}^3$, what is the *easiest* basis to work with?

That is: find **a** , **b** , and **c** so that it is extremely easy to solve the system

$$s_1\mathbf{a} + s_2\mathbf{b} + s_3\mathbf{c} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Bases

In $\mathbb{R}^3$, what is the *easiest* basis to work with?
That is: find **a** , **b** , and **c** so that it is extremely easy to solve the system

$$s_1\mathbf{a} + s_2\mathbf{b} + s_3\mathbf{c} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$$

"Standard basis"

Give one vector in $\mathbb{R}^2$ that can never be in a basis of $\mathbb{R}^2$.

(Remember: a basis in $\mathbb{R}^2$ is a collection of two vectors **a** and **b** so that the only solution to the equation $s\mathbf{a} + t\mathbf{b} = \mathbf{0}$ is $s = t = 0$.)

Give one vector in $\mathbb{R}^2$ that can never be in a basis of $\mathbb{R}^2$.

(Remember: a basis in $\mathbb{R}^2$ is a collection of two vectors $\mathbf{a}$ and $\mathbf{b}$ so that the only solution to the equation $s\mathbf{a} + t\mathbf{b} = \mathbf{0}$ is $s = t = 0$.)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose:

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and also

$$\begin{bmatrix} x \\ y \end{bmatrix} = p \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $s \neq p$.

Is $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$?

Recall: a basis in $\mathbb{R}^2$ is two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $s_1\mathbf{a} + s_2\mathbf{b} = \mathbf{0}$ ONLY when $s_1 = s_2 = 0$.

Suppose:

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and also

$$\begin{bmatrix} x \\ y \end{bmatrix} = p \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $s \neq p$.

Is $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$?

Recall: a basis in $\mathbb{R}^2$ is two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $s_1\mathbf{a} + s_2\mathbf{b} = \mathbf{0}$ ONLY when $s_1 = s_2 = 0$.
Given a basis, every vector can be represented **uniquely** as a linear combination of basis elements.

Suppose:

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and also

$$\begin{bmatrix} x \\ y \end{bmatrix} = p \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $s \neq p$.

Is $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$?

Find a scalar constant c so that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = c \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Suppose:

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and also

$$\begin{bmatrix} x \\ y \end{bmatrix} = p \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where $s \neq p$.

Is $\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$?

Find a scalar constant c so that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = c \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. $\quad c = \dfrac{q-t}{s-p}$

Substitution

Give $\begin{bmatrix} 2 \\ 24 \\ 49 \end{bmatrix}$ as a linear combination of the vectors in the basis below.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Substitution

Give $\begin{bmatrix} 2 \\ 24 \\ 49 \end{bmatrix}$ as a linear combination of the vectors in the basis below.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 \\ 24 \\ 49 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix} - 8 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Substitution

Suppose a moving object at time t has height $h = At^2 + Bt + C$.
At $t = 1$, the object is at height 0;
at $t = 2$, the object is at height 1; and
at $t = 3$, the object is at height 6.

Find A, B, and C.

Substitution

Suppose a moving object at time t has height $h = At^2 + Bt + C$.
At $t = 1$, the object is at height 0;
at $t = 2$, the object is at height 1; and
at $t = 3$, the object is at height 6.

Find A, B, and C.

$A = 2$, $B = -5$, $C = 3$

General Form

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & c_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & c_2 \\
 & \vdots & & & \vdots & & & \vdots & \\
a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & c_m
\end{array}$$

Where $a_{i,j}$ and c_i are known and fixed.

Goal: easily-solvable system

$$\begin{array}{rcrcrcrcr}
x_1 & + & 3x_2 & + & 17x_3 & + & 9x_4 & = & 10 \\
-3x_1 & + & -6x_2 & + & 8x_3 & + & 5x_4 & = & 17 \\
\pi x_1 & + & -8x_2 & + & 3x_3 & + & x_4 & = & -2 \\
8x_1 & + & -8x_2 & + & 5x_3 & + & 2x_4 & = & 2
\end{array}$$

Goal: easily-solvable system

$$\begin{array}{rcrcrcrcr}
x_1 & + & 3x_2 & + & 17x_3 & + & 9x_4 & = & 10 \\
 & & -6x_2 & + & 8x_3 & + & 5x_4 & = & 17 \\
 & & & & 3x_3 & + & x_4 & = & -2 \\
 & & & & & & 2x_4 & = & 2
\end{array}$$

Upper Triangular

(different from the last system)

Goal: easily-solvable system

$$\begin{array}{rcrcrcrcr}
x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 10 \\
 & & -6x_2 & + & 0x_3 & + & 0x_4 & = & 17 \\
 & & & & 3x_3 & + & 0x_4 & = & -2 \\
 & & & & & & 2x_4 & = & 2
\end{array}$$

Diagonal

(different from the last system)

Equivalent Equations

Notice:

$$3x + 5y + 7z = 10$$

and

$$6x + 10y + 14z = 20$$

have the same solutions.

Equivalent Equations

Notice:

$$3x + 5y + 7z = 10$$

and

$$6x + 10y + 14z = 20$$

have the same solutions.

Caution:

$$0x + 0y + 0z = 0$$

has more solutions.

Equivalent Equations

Notice:

$$3x + 5y + 7z = 10$$

and

$$3x + 5y + 7z + C = 10 + C$$

have the same solutions.

Equivalent Equations

Notice:

$$3x + 5y + 7z = 10$$

and

$$3x + 5y + 7z + C = 10 + C$$

have the same solutions.

By the same logic,

$$\begin{cases} 3x + 5y + 7z & = 10 \\ x + y + z & = 15 \end{cases}$$

and

$$\begin{cases} 3x + 5y + 7z & = 10 \\ x + y + z + 10 & = 15 + 10 \end{cases}$$

have the same solutions.

Equivalent Equations

Notice:

$$3x + 5y + 7z = 10$$

and

$$3x + 5y + 7z + C = 10 + C$$

have the same solutions.

By the same logic,

$$\begin{cases} 3x + 5y + 7z & = 10 \\ x + y + z & = 15 \end{cases}$$

and

$$\begin{cases} 3x + 5y + 7z & = 10 \\ x + y + z + 3x + 5y + 7z & = 15 + 10 \end{cases}$$

have the same solutions.

Equivalent Equations

Useful:

$$\begin{cases} 3x + 5y + 7z & = 10 \\ -3x - 5y + z & = 15 \end{cases}$$

has the same solutions as

Equivalent Equations

Useful:

$$\begin{cases} 3x + 5y + 7z &= 10 \\ -3x - 5y + z &= 15 \end{cases}$$

has the same solutions as

$$\begin{cases} 3x + 5y + 7z &= 10 \\ 8z &= 25 \end{cases}$$

Equivalent Equations

Similarly, the system

$$\begin{cases} 3x + 9y - 6z & = 7 \\ x + 2y - 2z & = 1 \end{cases}$$

has the same solutions as

Equivalent Equations

Similarly, the system

$$\begin{cases} 3x + 9y - 6z &= 7 \\ x + 2y - 2z &= 1 \end{cases}$$

has the same solutions as

$$\begin{cases} 3x + 9y - 6z &= 7 \\ -3x - 6y + 6z &= -3 \end{cases}$$

and also

Equivalent Equations

Similarly, the system

$$\begin{cases} 3x + 9y - 6z &= 7 \\ x + 2y - 2z &= 1 \end{cases}$$

has the same solutions as

$$\begin{cases} 3x + 9y - 6z &= 7 \\ -3x - 6y + 6z &= -3 \end{cases}$$

and also

$$\begin{cases} 3x + 9y - 6z &= 7 \\ 3y &= 4 \end{cases}$$

Elementary Row Operations: Multiplication of a row by a non-zero number

$$\begin{array}{rcrcrcr} 3x & - & 9y & + & 6z & = & 30 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Elementary Row Operations: Multiplication of a row by a non-zero number

$$\begin{array}{rcrcrcr} 3x & - & 9y & + & 6z & = & 30 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Same solutions as:

$$\begin{array}{rcrcrcr} 1x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Elementary Row Operations: Adding a Multiple of a Row to Another Row

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Elementary Row Operations: Adding a Multiple of a Row to Another Row

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Same solutions as:

$$\begin{array}{rcrcrcrl} 0x & - & 0y & + & 7z & = & 14 & \text{(Row1+Row2)} \\ -x & + & 3y & + & 5z & = & 4 & \\ x & + & y & + & z & = & -6 & \end{array}$$

Elementary Row Operations: Interchanging Rows

$$\begin{array}{rcrcrcr} & & & & 7z & = & 14 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Elementary Row Operations: Interchanging Rows

$$\begin{array}{rcrcrcr} & & & & 7z & = & 14 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

Same solutions as:

$$\begin{array}{rcrcrcr} -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \\ & & & & 7z & = & 14 \end{array}$$

Streamlined Notation: Augmented Matrices

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 10 \\ -x & + & 3y & + & 5z & = & 4 \\ x & + & y & + & z & = & -6 \end{array}$$

We'll write this as:

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 10 \\ -1 & 3 & 5 & 4 \\ 1 & 1 & 1 & -6 \end{array}\right]$$

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $x_1 = 1 \quad x_2 = 5 \quad x_3 = -3 \quad x_4 = 2$

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $x_1 = 1 \quad x_2 = 5 \quad x_3 = -3 \quad x_4 = 2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution:

Augmented Matrices

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $x_1 = 1 \quad x_2 = 5 \quad x_3 = -3 \quad x_4 = 2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $x_3 = 2 \quad y = -7 \quad x = 13$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] \; R_1 \to \tfrac{1}{3} R_1$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] \; R_1 \to \tfrac{1}{3} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3} R_3$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3} R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3} R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_3$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3}R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3}R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \leftrightarrow R_2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using Elementary Row Operations (strategy next: Ch 3.2)

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & 9 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_1 \to \tfrac{1}{3} R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 2 & 2 & -1 & 14 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & -3 & 0 \end{array}\right] R_3 \to -\tfrac{1}{3} R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 1 & 1 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -4 \\ 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \leftrightarrow R_2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$x = 11$$
$$y = -4$$
$$z = 0$$

Solve using Elementary Row Operations

$$\begin{cases} 2x + y + z & = 8 \\ x - y - 3z & = -5 \\ -x - 2y + z & = 2 \end{cases}$$

Row Operation Calculator (link)