Outline

Week 2: Determinants, Cross Products, Lines, and Planes

Course Notes: 2.4-2.5

Goals: Introduce determinants and cross products, computationally and with geometric interpretations. Lines and planes.

Notes

Determinants in Two and Three Dimensions

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1$$

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

Tricky way: ONLY in three dimensions:

Notes

$$\det\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\det\begin{bmatrix} -2 & 8 \\ 3 & 5 \end{bmatrix}$$

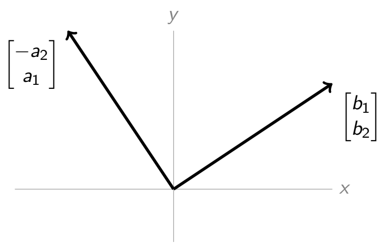
$$\det\begin{bmatrix} 3 & 2 & 5 \\ 5 & 7 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\det\begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 7 \\ 1 & 10 & 5 \end{bmatrix}$$

Notes

Geometric Interpretation: Determinant in Two Dimensions

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1$$
$$= \begin{bmatrix} -a_2 \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



If $\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = 0$,
then

Notes

Zero Determinants

Are the following determinants zero, or nonzero?

$$\det\begin{bmatrix} 1 & 2 \\ -4 & -8 \end{bmatrix}$$

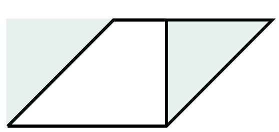
$$\det\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

$$\det\begin{bmatrix} a_1 & a_2 \\ 5a_1 & 5a_2 \end{bmatrix}$$

$$\det\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}$$

Notes

Quick Detour: Parallelograms

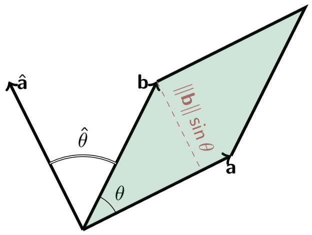


Area: (base)$\times$(height)

Notes

More Geometric Interpretation in Two Dimensions

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1 = \begin{bmatrix} -a_2 \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \hat{\mathbf{a}} \cdot \mathbf{b}$$
$$= \|\hat{\mathbf{a}}\|\|\mathbf{b}\| \cos(\hat{\theta})$$
$$= \|\hat{\mathbf{a}}\|\|\mathbf{b}\| \cos(\pi/2 - \theta)$$
$$= \|\mathbf{a}\|\|\mathbf{b}\| \sin(\theta) = \text{area of parallelogram}$$



Notes

In general:

$$\left|\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right| = \text{area of parallelogram spanned by } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ and } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Example: Find the area of the parallelogram with one side given by $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and the other side $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

Silly Example: Find the area of the rectangle with corners $(0, 0)$, $(x, 0)$, $(0, y)$, and (x, y).

Notes

Cross Product

ONLY defined in three dimensions.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Mnemonic:

$$\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \mathbf{i}\det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \mathbf{j}\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \mathbf{k}\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$
$$= \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$
$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Notes

Practice

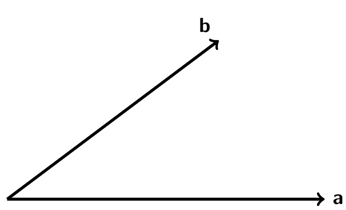
$$\begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}0\\1\\-1\end{bmatrix} \qquad \begin{bmatrix}0\\1\\-1\end{bmatrix} \times \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Notes

Geometric Interpretation

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$.
   Verify:
   $$\begin{bmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{bmatrix} \cdot \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = 0$$
2. $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$,
   where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$.
   Thus, $\sin\theta$ is positive, and $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
3. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ obey the *right hand rule*. That is, if you curl your fingers towards your palm from $\mathbf{a}$ to $\mathbf{b}$, your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Notes



Notes

$\|\mathbf{a}\times\mathbf{b}\|$ = area of parallelogram

$$
\begin{aligned}
A^2 &= \|\mathbf{a}\|^2\|\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}\|^2 \\
&= \|\mathbf{a}\|^2\left\|\mathbf{b} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}\right\|^2 \\
&= \|\mathbf{a}\|^2\left(\mathbf{b}\cdot\mathbf{b} - 2(\mathbf{b}\cdot\mathbf{a})\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right) + \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\right)^2\|\mathbf{a}\|^2\right) \\
&= \|\mathbf{a}\|^2\left(\|\mathbf{b}\|^2 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2}\right) \\
&= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\
&= (a_1^2+a_2^2+a_3^2)^2(b_1^2+b_2^2+b_3^2)^2 - (a_1b_1+a_2b_2+a_3b_3)^2 \\
&= \cdots = (a_2b_3-a_3b_2)^2 + (a_3b_1-a_1b_3)^2 + (a_1b_2-a_2b_1)^2 \\
&= \|\mathbf{a}\times\mathbf{b}\|^2
\end{aligned}
$$

Notes

Find the Area of the Parallelograms

Find the area of the parallelogram spanned by $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\\-2\end{bmatrix}$.

Find the area of the parallelogram spanned by $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}4\\3\end{bmatrix}$.

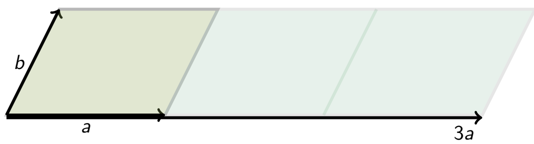
Notes

Suppose a plane contains the points $P_1(3,2,2)$, $P_2(2,2,1)$, and $P_3(1,1,1)$. Find a normal vector to the plane. That is, find a vector that is perpendicular to every line on the plane.

Notes

Properties of Cross Product (p. 31)

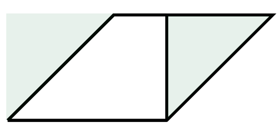
1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$
   `https://proofwiki.org/wiki/Lagrange's_Formula`
3. $s(\mathbf{a} \times \mathbf{b}) = (s\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (s\mathbf{b})$



4. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ "triple product"
   Is it also true that $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$?

Notes

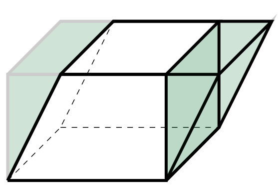
Parallelograms



Area: (base)×(height)

Notes

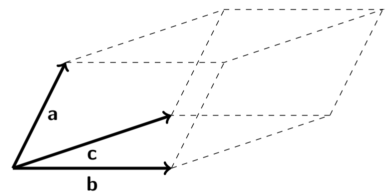
Parallelepipeds



Volume: (area of base)×(height)

Notes

Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$



Notes

Calculating the Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \mathbf{a} \cdot \begin{bmatrix} \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} \\ -\det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} \\ \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \end{bmatrix}$$

$$= a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Notes

Find the volume of the parallelepiped spanned by:
$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

For positive a, b, and c, find the determinant and interpret it as a volume:

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Calculate and explain geometrically:

$$\det \begin{bmatrix} 2 & 0 & 3 \\ 8 & 1 & 7 \\ 20 & 3 & 15 \end{bmatrix}$$

Notes

Right-Hand Rule

Predict the following cross products without using the cross-product calculation. Draw your results. Check using the cross-product calculation.

$$\begin{bmatrix}2\\0\\0\end{bmatrix}\times\begin{bmatrix}0\\0\\7\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\7\end{bmatrix}\times\begin{bmatrix}0\\2\\0\end{bmatrix}$$

$$\begin{bmatrix}-2\\0\\0\end{bmatrix}\times\begin{bmatrix}0\\7\\0\end{bmatrix}$$

Notes

Given any 3-dimensional vector $\mathbf{a}$, is there a simple expression for $\mathbf{a}\times\mathbf{a}$?

What about $(s\mathbf{a})\times\mathbf{a}$ for a scalar s?

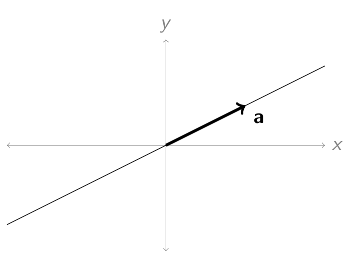
What about $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$?

Consider $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. Will this vector be in the same plane as $\mathbf{b}$ and $\mathbf{c}$, or in an orthogonal plane?

Notice $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{c}\cdot\mathbf{a})\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\mathbf{c}$:

Notes

Parametric Equations of Lines



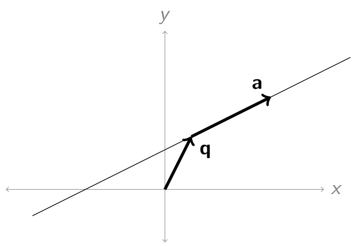
Line passing through the origin:

$$\mathbf{x}=s\mathbf{a}$$

Question: is this the only such equation for the line? Can we use this equation with a line *not* through the origin?

Notes

Parametric Equations of Lines

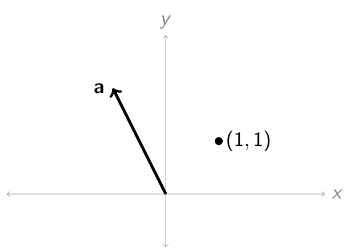


General equation of a line:

$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

Where we mean: the line consists of all points $\mathbf{x}$ that can be written this way for some scalar s.

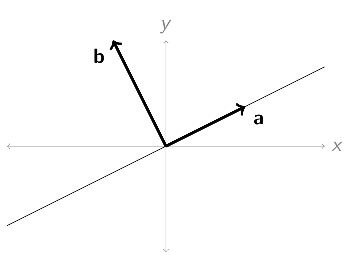
Notes



Find a parametric equation describing the line in the direction of $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, passing through the point $(1, 1)$.

Notes

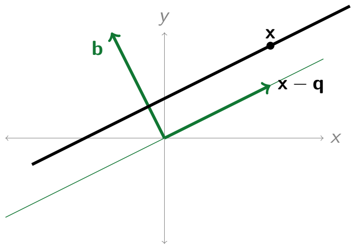
Component Equations of Lines



Line passing through the origin:

Notes

Components Equations of Lines



General Line in $\mathbb{R}^2$

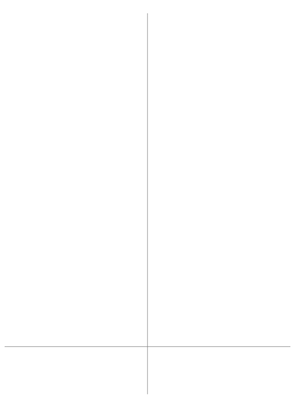
Notes

Suppose the parametric equation of a line is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Convert this to an equation of the form $ax + by = c$.

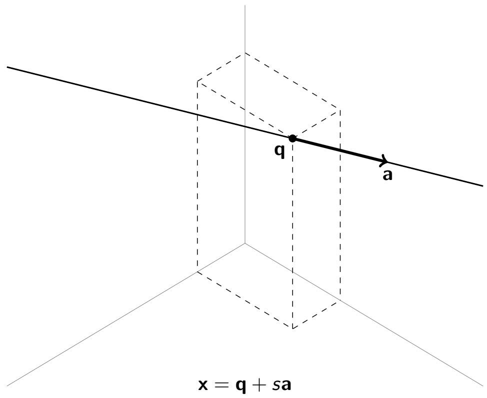
Suppose the parametric equation of a line is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. Convert this to an equation of the form $ax + by = c$.

Notes

Give a parametric equation for the line $y = 3x + 5$.



Notes



$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

Notes

Equation of a Line in $\mathbb{R}^3$

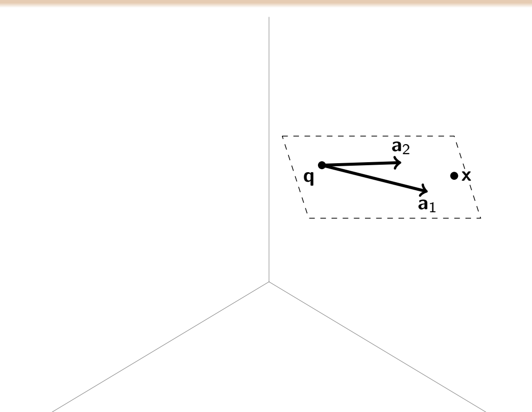
$$(\mathbf{x} - \mathbf{q}) \cdot \mathbf{b} = 0 \text{ and } (\mathbf{x} - \mathbf{q}) \cdot \mathbf{c} = 0$$

$$\mathbf{x} \cdot \mathbf{b} = \mathbf{q} \cdot \mathbf{b} \text{ and } \mathbf{x} \cdot \mathbf{c} = \mathbf{q} \cdot \mathbf{c}$$

To define a line in $\mathbb{R}^3$, we need a *system* of equations:

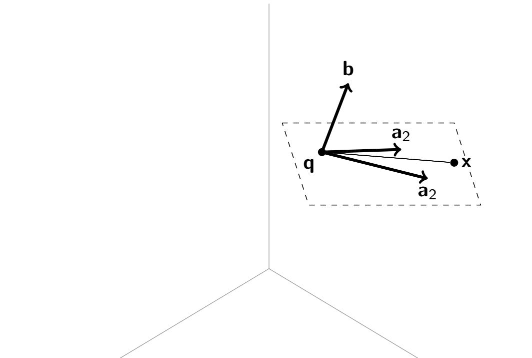
$$\begin{cases} xb_1 + yb_2 + zb_3 & = & s_1 \\ xc_1 + yc_2 + zc_3 & = & s_2 \end{cases}$$

Notes



$$\boxed{\mathbf{x} = \mathbf{q} + s\mathbf{a}_1 + t\mathbf{a}_2}$$

Notes



$$(\mathbf{x}-\mathbf{q})\cdot\mathbf{b}=0$$
$$\mathbf{x}\cdot\mathbf{b}=s$$
$$b_1x+b_2y+b_3z=s$$

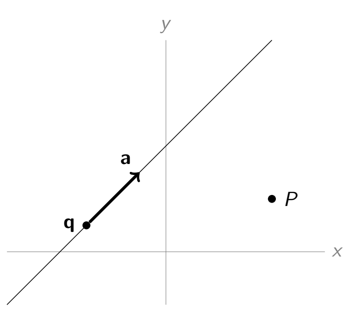
Notes

Equations

	Parametric	Component
Line in $\mathbb{R}^2$	$\mathbf{x}=\mathbf{q}+s\mathbf{a}$	$b_1x+b_2y=s$
Line in $\mathbb{R}^3$	$\mathbf{x}=\mathbf{q}+s\mathbf{a}$	$\begin{cases} b_1x+b_2y+b_3z=s \\ c_1x+c_2y+c_3z=t \end{cases}$
Plane in $\mathbb{R}^3$	$\mathbf{x}=\mathbf{q}+s\mathbf{a}+t\mathbf{b}$	$b_1x+b_2y+b_3z=s$

Suppose $\mathbf{q}$ and $\mathbf{a}$ are vectors in $\mathbb{R}^{18}$. What would you call the geometric object resulting from the equation $\mathbf{x}=\mathbf{q}+s\mathbf{a}$?
Suppose P and Q are planes. What is the intersection of P and Q?
Are there any vectors $\mathbf{q}$ and $\mathbf{a}$ in $\mathbb{R}^3$ for which the equation $\mathbf{x}=\mathbf{q}+s\mathbf{a}$ is **not** a line?
Recall: $\mathbf{b}$ was the normal vector to the plane $b_1x+b_2y+b_3z=s$.
True or False: for a point P on the plane $5x+7y+11z=22$, the vector with head at P and tail at the origin is orthogonal to the vector $[5,7,11]$.

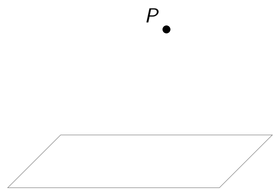
Notes



How can you find the distance from the point P to the line $\mathbf{x}=\mathbf{q}+s\mathbf{a}$?

Notes

Find the distance from the point P to the plane Q.



Find the distance from the point $(3, 5, 1)$ to the plane

$$\mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Notes

Let P be the plane with equation $2x + 2y + 2z = 1$, and let Q be the plane with equation $x + y + z = 1$.

What will their intersection be: a plane, a line, a point, or nothing?

Let P be the plane with equation $2x + y - z = 1$, and let Q be the plane with equation $x + 2y + 3z = 0$.

What will their intersection be: a plane, a line, a point, or nothing?

Find the intersection in parametric form.

Notes

Notes