Course Notes: 2.1-2.3
Course Outline: Week 1

Notes

Vectors

What is a Vector?

Vectors are used to describe quantities with a magnitude (length) and a direction.

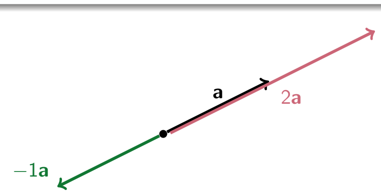
Notice: a vector doesn't intrinsically have a *position*, although we can assign it one in context.

Notes

Vector Operations: Multiply by a Number

Scalar Multiplication

Multiplying a vector $\mathbf{a}$ by a scalar s results in a vector with length $|s|$ times the length of $\mathbf{a}$. The new vector $s\mathbf{a}$ points in the same direction if s is positive, and in the opposite direction if s is negative.



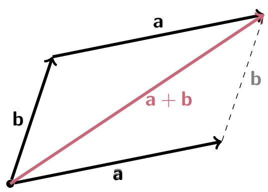
If the length of $\mathbf{a}$ is 1 unit, then the length of $2\mathbf{a}$ is 2. What is the length of $-1\mathbf{a}$: is it 1, or -1?

Notes

Vector Operations: Adding Vectors

Vector Addition

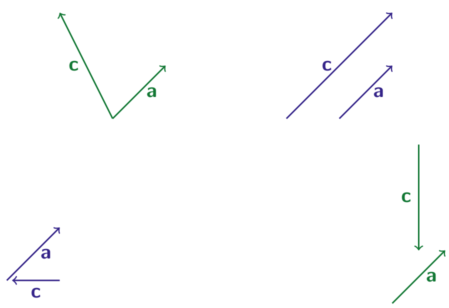
To add vectors **a** and **b** , we can slide the tail of **a** to sit at the head of **b** , and take $\mathbf{a}+\mathbf{b}$ to be the vector with tail where the tail of **b** is, and head where the head of **a** is.
This is equivalent to making a parallelogram out of **a** and **b** (with the same tail) and taking the diagonal (again with the same tail) to be the vector $\mathbf{a}+\mathbf{b}$.



Notes

Vector Addition

In each case, sketch a vector **b** such that $\mathbf{a}+\mathbf{b}=\mathbf{c}$.



Notes

Vector Operations

Example

Suppose we add a vector **a** to the vector $-3\mathbf{a}$. What should be the resulting vector?

Notes

Limits of Sketching

Suppose a ship is sailing in the ocean. The current is pushing the ship at 5 knots per hour due east, while the wind is pushing this ship 3 knots per hour northwest. Rowers onboard are providing a force equal to 2 knots per hour east-southeast. What direction is the ship moving, and how fast?

See also: `https://en.wikipedia.org/wiki/Wind_triangle`

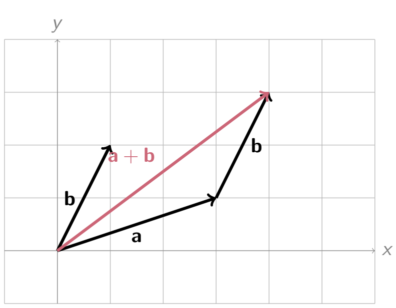
Notes

Coordinates and Vectors

i and **j** are *unit vectors*, and we can write any vector in $\mathbb{R}^2$ as a linear combination of them.
Linear combination: any combination using only addition and scalar multiplication

Notes

Vector Operations on Coordinates



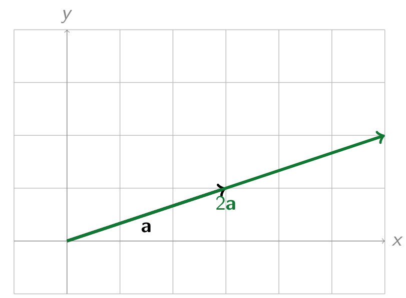
$$\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=$$

$$\begin{bmatrix}3\\1\\7\end{bmatrix}+\begin{bmatrix}1\\2\\0\end{bmatrix}=$$

$$\begin{bmatrix}3\\1\\7\end{bmatrix}+\begin{bmatrix}1\\2\\0\end{bmatrix}=$$

Notes

Vector Operations on Coordinates



$$2\begin{bmatrix}3\\1\end{bmatrix}=$$

$$\frac{1}{3}\begin{bmatrix}3\\1\\6\\9\end{bmatrix}=$$

Notes

Properties of Vector Addition and Scalar Multiplication

(Notes: 2.2.3)
Let $\mathbf{0}$ be the zero vector: this is the vector whose components are all zero. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors, and let s and t be scalars. The following facts about vector addition, and multiplication of vectors by scalars, are true:

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $s(\mathbf{a}+\mathbf{b})=s\mathbf{a}+s\mathbf{b}$
6. $(s+t)\mathbf{a}=s\mathbf{a}+t\mathbf{a}$
7. $(st)\mathbf{a}=s(t\mathbf{a})$
8. $1\mathbf{a}=\mathbf{a}$

Notes

Vectors versus Coordinates

Because we write vectors like coordinates, we will often use them interchangeably with points. You will have to figure this out from context.
Example: Let $\mathbf{a}$ be a fixed, nonzero vector. Describe and sketch the sets of points in two dimensions:

$$\{s\mathbf{a} : s\in\mathbb{R}\}$$

Example: Let $\mathbf{a}$ and $\mathbf{b}$ be fixed, nonzero vectors. Describe and sketch the sets of points in two dimensions:

$$\{s\mathbf{a}+t\mathbf{b} : s,t\in\mathbb{R}\}$$

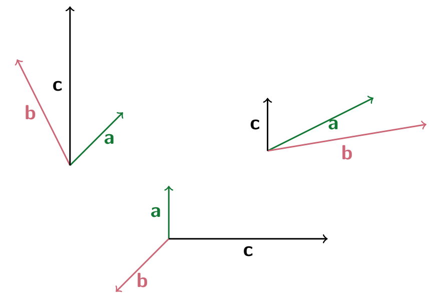
See
`http://thejuniverse.org/PUBLIC/LinearAlgebra/LOLA/spans/two.html`
Example: Let $\mathbf{a}$ and $\mathbf{b}$ be fixed, nonzero vectors. Describe and sketch the sets of points in three dimensions:

$$\{s\mathbf{a}+t\mathbf{b} : s,t\in\mathbb{R}\}$$

Notes

Vectors versus Coordinates

In each case below, show that the vector **c** can be written as $s\mathbf{a} + t\mathbf{b}$ for some $s, t \in \mathbb{R}$.



Notes

Vectors and Coordinates

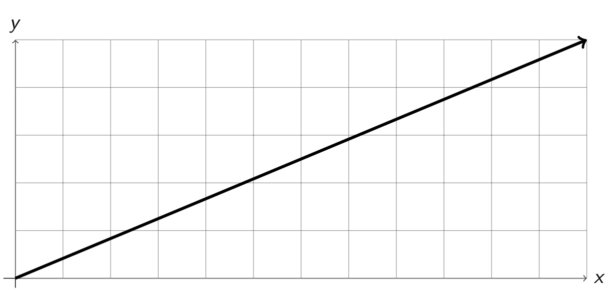
Let **a** and **b** be fixed, nonzero vectors.

- Give an expression for the midpoint of the line segment halfway between **a** and **b** .
- Give an expression for the point that is one-third of the way along the line segment between **a** and **b** .
- What is the geometric interpretation of the following set of points:
$$\{s\mathbf{a} + (1 - s)\mathbf{b} : 0 \leq s \leq 1\}$$
- What is the geometric interpretation of the following set of points:
$$\{(1 - s)\mathbf{a} + s\mathbf{b} : 0 \leq s \leq 1\}$$

Notes

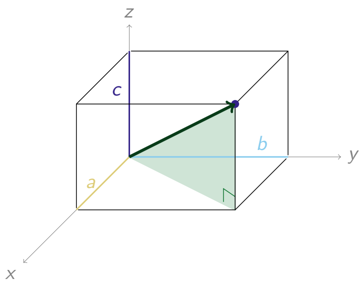
Geometric Aspects of Vectors

How long is the vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$?



The length of $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ is denoted $\left\| \begin{bmatrix} 12 \\ 5 \end{bmatrix} \right\|$, and calculated
We also call this quantity the norm of the vector.
What about vectors with three coordinates?

Notes



$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Notes

Course Notes: Section 2.2, Vectors
Section 2.3, Geometric Aspects of Vectors

Geometric Aspects of Vectors

How long is the vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$?

The length of $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is denoted $\|\mathbf{a}\|$, and calculated

Notes

Course Notes: Section 2.2, Vectors
Section 2.3, Geometric Aspects of Vectors

Quick Concept Test

Let $\mathbf{a}$ be a vector, and let s be a scalar. For each of the following expressions, decide whether it is a vector or a scalar.

A. $\|\mathbf{a}\|$
B. $s\mathbf{a}$
C. $s\|\mathbf{a}\|$
D. $\|s\mathbf{a}\|$
E. $s + \mathbf{a}$
F. $s + \|\mathbf{a}\|$

Notes

Unit Vectors

A **unit vector** is a vector of length one.

What is the unit vector in the direction of the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

Notes

Dot Product

Dot Product

Given vectors $\mathbf{a} = [a_1, \ldots, a_k]$ and $\mathbf{b} = [b_1, \ldots, b_k]$, we define the dot product $\mathbf{a} \cdot \mathbf{b} := a_1 b_1 + \cdots + a_k b_k$. Note $\mathbf{a} \cdot \mathbf{b}$ is a number, not a vector.

Example: $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = -4 + 0 + 15 = 11$

Note: $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} =$$

$$\left\| \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\| =$$

Notes

Properties of the Dot Product

Notes: p. 20

For nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, zero vector $\mathbf{0}$, and scalar s:

1. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $s(\mathbf{a} \cdot \mathbf{b}) = (s\mathbf{a}) \cdot \mathbf{b}$
5. $\mathbf{0} \cdot \mathbf{a} = 0$
6. $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
7. $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a} = 0$, $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular

Example: are $\mathbf{a}$ and $\mathbf{b}$ perpendicular?

- $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Notes

Properties of the Dot Product

$\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a} = 0$, $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Example: are $\mathbf{a}$ and $\mathbf{b}$ perpendicular?

$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ No

$\mathbf{a} = \begin{bmatrix} 2, -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3, 6 \end{bmatrix}$ No

$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ No

$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}$ Yes

Notes

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

Claim 1:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}$$

Claim 2:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

Then:

$$\begin{aligned} \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta && \text{so} \\ -2\mathbf{a} \cdot \mathbf{b} &= -2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta && \text{so} \\ \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta \end{aligned}$$

Notes

Claim 1:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}$$

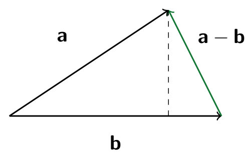
Proof:

$$\begin{aligned} \|\mathbf{a} - \mathbf{b}\|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b} \\ &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b} \end{aligned}$$

Notes

Claim 2:

$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$



Law of Cosines

Notes

Angle between Two Vectors

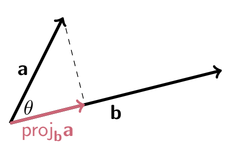
Recall $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

What is the angle between vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}5\\5\end{bmatrix}$?

Notes

Projections

We apply a force to an object in the direction of $\mathbf{a}$, but we're only concerned with the object's movement in the direction of vector $\mathbf{b}$.



- The vector $\text{proj}_{\mathbf{b}}\mathbf{a}$ is in the same direction as $\mathbf{b}$.
- The vector $\text{proj}_{\mathbf{b}}\mathbf{a}$ has length $\|\mathbf{a}\|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}$.

Notes

A man pulls a truck up a hill for some reason. If we take level ground as our coordinate axis, the hill is in the direction of the vector $\begin{bmatrix}10\\2\end{bmatrix}$, and the man applies force represented by the vector $\begin{bmatrix}5\\2\end{bmatrix}$. What vector represents the force acting on the truck in the direction it is moving?

Image credit: stu_spivack, CC, https://www.flickr.com/photos/stuart_spivack/3850975920/in/set-72157622007398607/

Notes

A man pulls a truck up a hill for some reason. He pulls with a force of 1000 pounds, and pulls at an angle of 20 degrees to the hill. What force is exerted in the direction of the hill? That is, what is the magnitude of the component of the force that is in the direction of the truck's motion?

Image credit: stu_spivack, CC, https://www.flickr.com/photos/stuart_spivack/3850975920/in/set-72157622007398607/

Notes

What is the projection of the vector $\begin{bmatrix}0\\2\\5\end{bmatrix}$ onto the vector $\begin{bmatrix}0\\1\\0\end{bmatrix}$?

What is the projection of the vector $\begin{bmatrix}8\\2\\5\end{bmatrix}$ onto the vector $\begin{bmatrix}0\\1\\0\end{bmatrix}$?

What is the projection of the vector $\begin{bmatrix}8\\2\\5\end{bmatrix}$ onto the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$?

What is the projection of the vector **a** onto itself?

Notes