Course Notes: 2.1-2.3
Course Outline: Week 1

Vectors

What is a Vector?

Vectors are used to describe quantities with a magnitude (length) and a direction.

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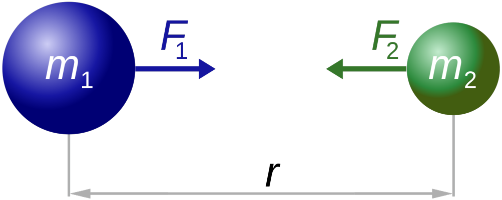

$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

Dennis Nilsson, CC, unedited, `https://en.wikipedia.org/wiki/Kepler_orbit`

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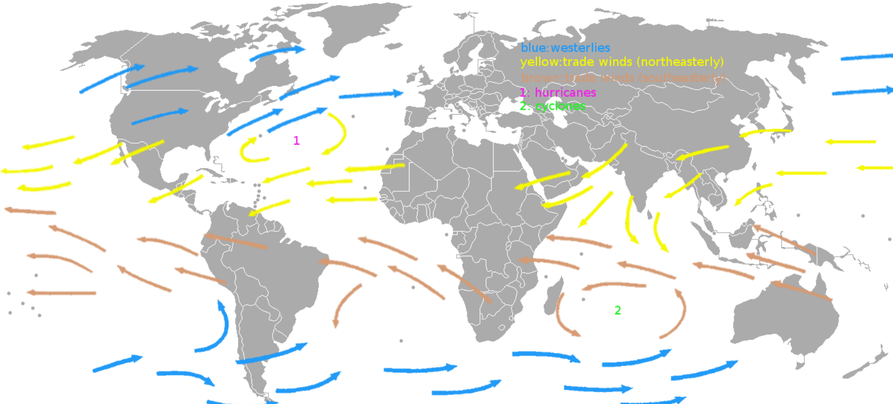

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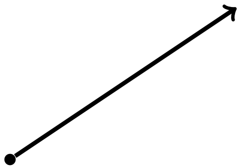

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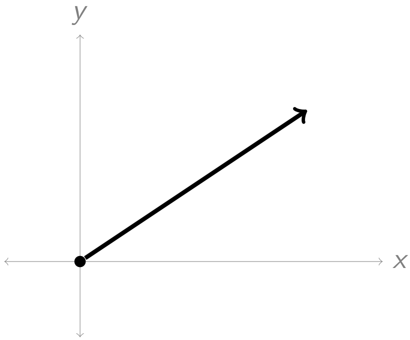

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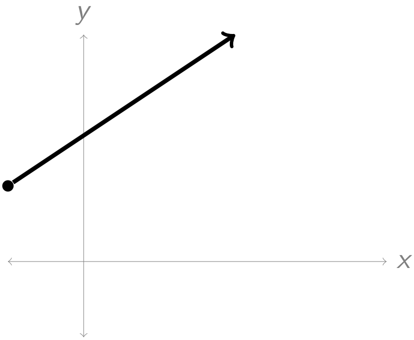

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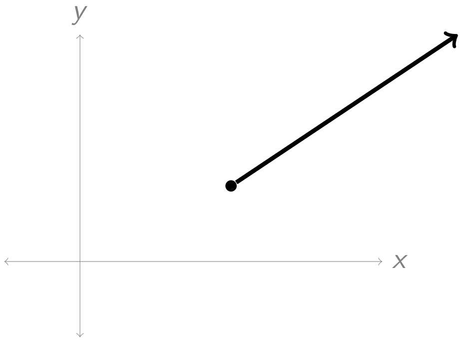

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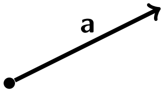
Vector Operations: Multiply by a Number

Scalar Multiplication

Multiplying a vector $\mathbf{a}$ by a scalar s results in a vector with length $|s|$ times the length of $\mathbf{a}$. The new vector $s\mathbf{a}$ points in the same direction if s is positive, and in the opposite direction if s is negative.



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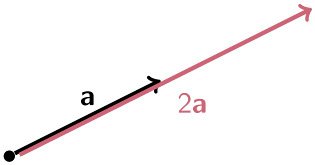

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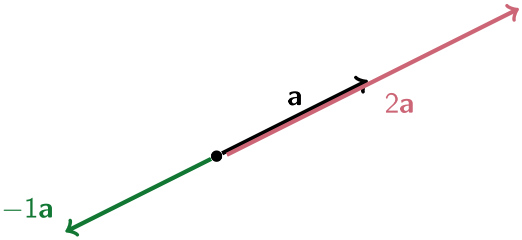

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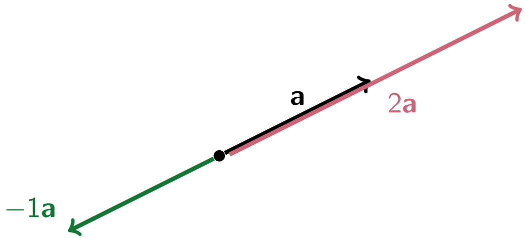
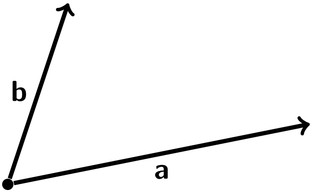


If the length of **a** is 1 unit, then the length of $2\mathbf{a}$ is 2. What is the length of $-1\mathbf{a}$: is it 1, or -1?

Vector Operations: Adding Vectors

Vector Addition

To add vectors $\mathbf{a}$ and $\mathbf{b}$, we can slide the tail of $\mathbf{a}$ to sit at the head of $\mathbf{b}$, and take $\mathbf{a}+\mathbf{b}$ to be the vector with tail where the tail of $\mathbf{b}$ is, and head where the head of $\mathbf{a}$ is.
This is equivalent to making a parallelogram out of $\mathbf{a}$ and $\mathbf{b}$ (with the same tail) and taking the diagonal (again with the same tail) to be the vector $\mathbf{a}+\mathbf{b}$.



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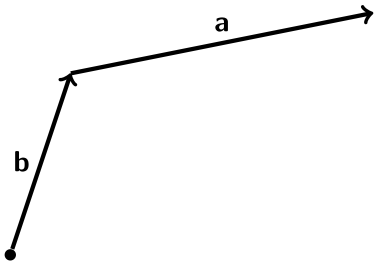

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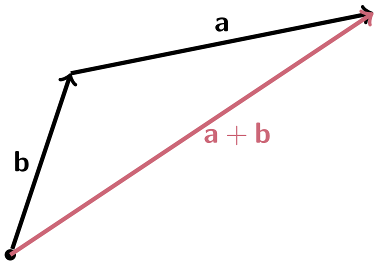

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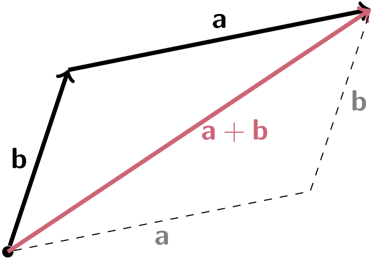

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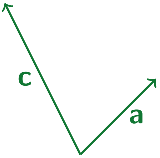

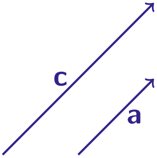

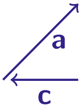

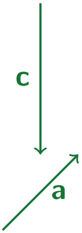

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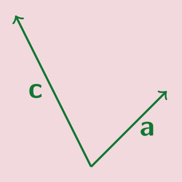

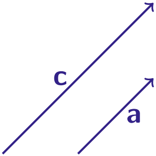

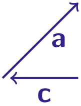

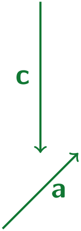

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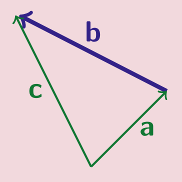

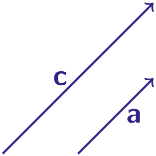

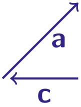

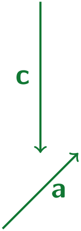

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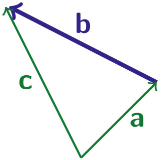

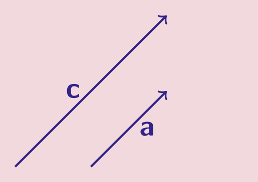

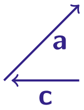

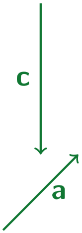

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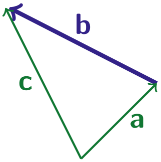

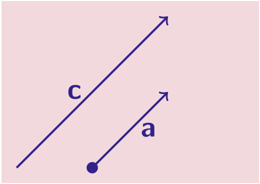

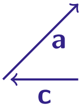

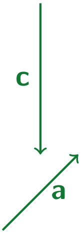

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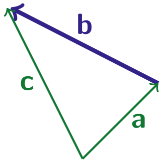

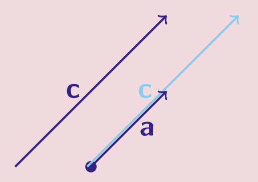

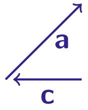

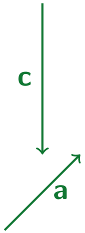

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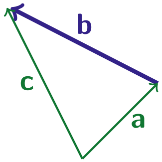

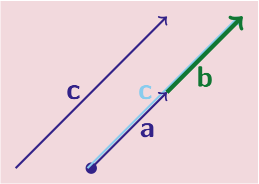

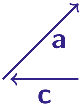

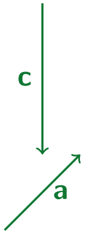

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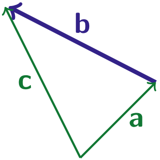

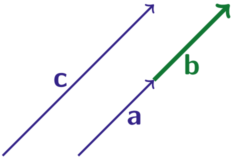

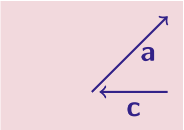

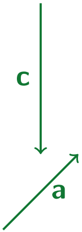

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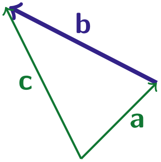

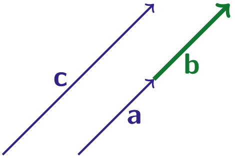

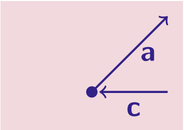

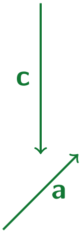

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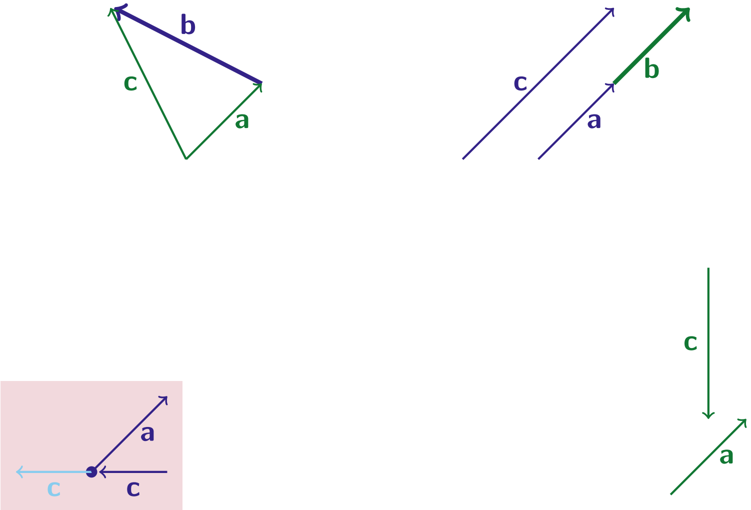

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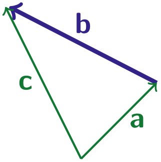

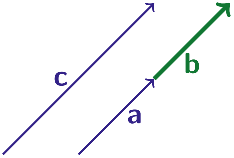

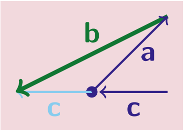

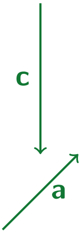

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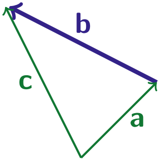

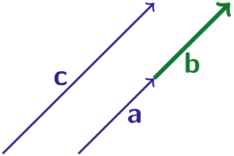

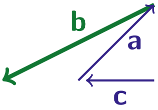

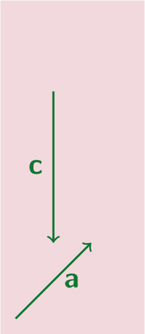

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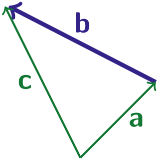

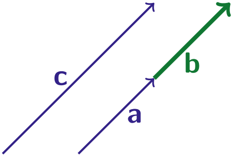

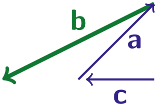

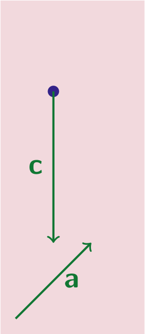

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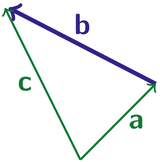

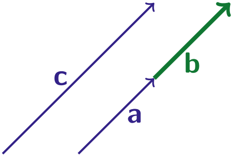

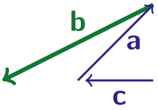

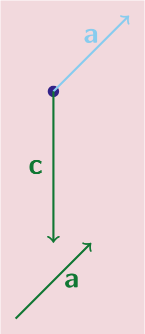

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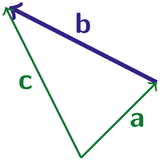

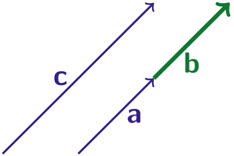

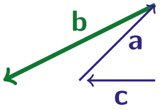

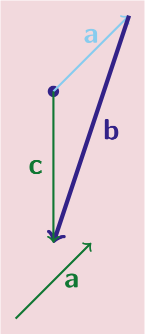

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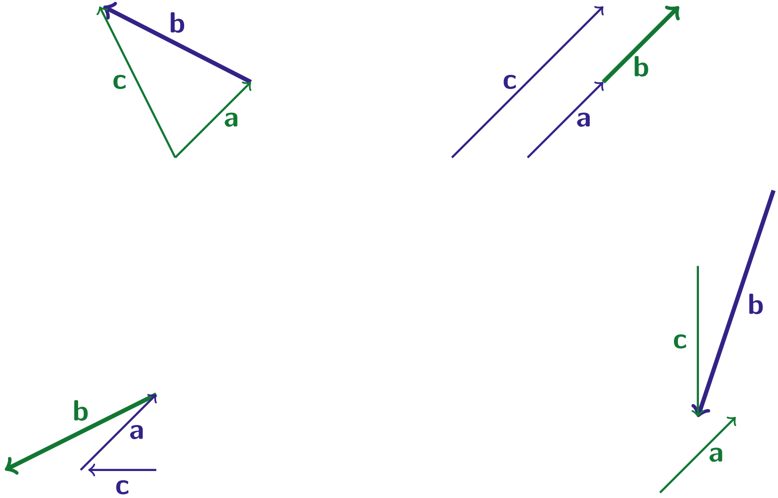

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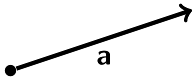

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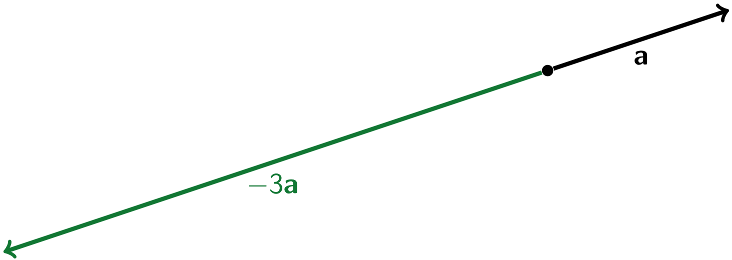

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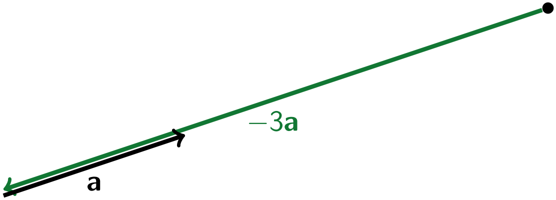

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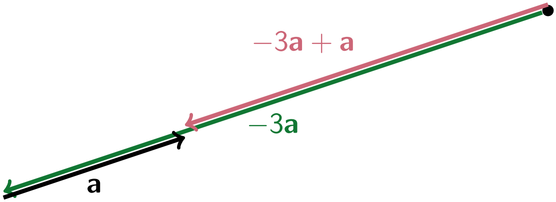

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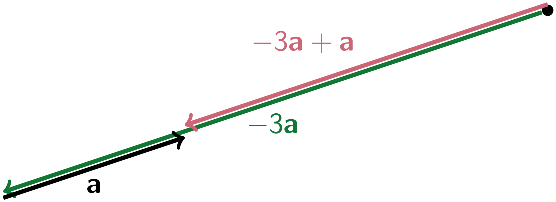

As we might expect, $\mathbf{a} - 3\mathbf{a} = -2\mathbf{a}$.

Limits of Sketching

Suppose a ship is sailing in the ocean. The current is pushing the ship at 5 knots per hour due east, while the wind is pushing this ship 3 knots per hour northwest. Rowers onboard are providing a force equal to 2 knots per hour east-southeast. What direction is the ship moving, and how fast?

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Time for coordinates.

See also: `https://en.wikipedia.org/wiki/Wind_triangle`

Coordinates and Vectors

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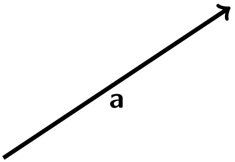

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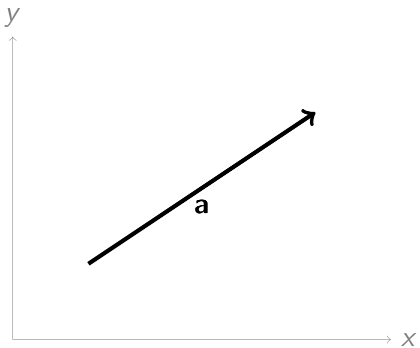

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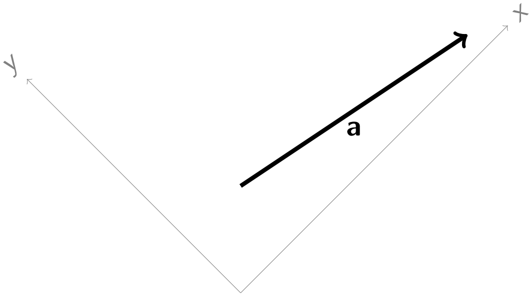

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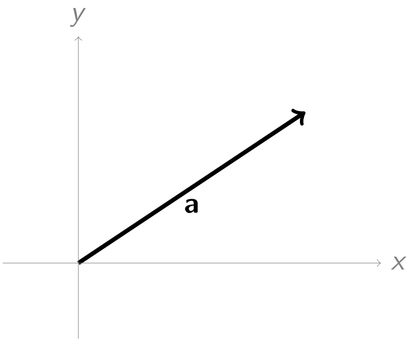

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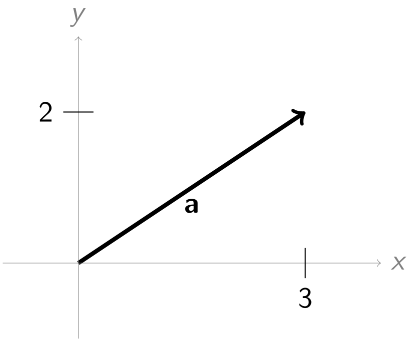

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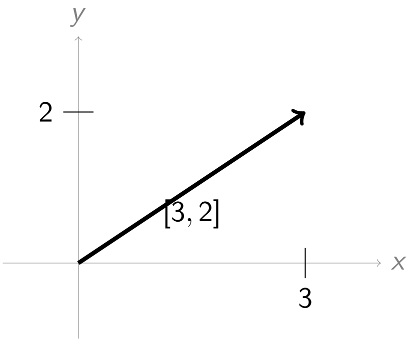

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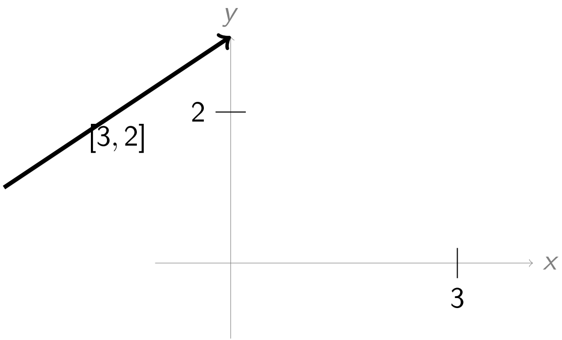

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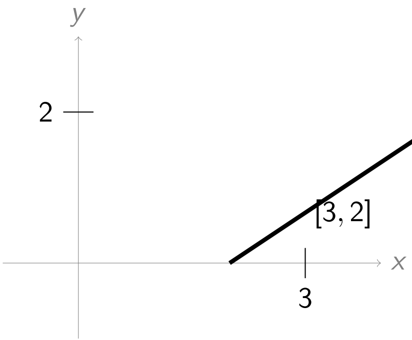

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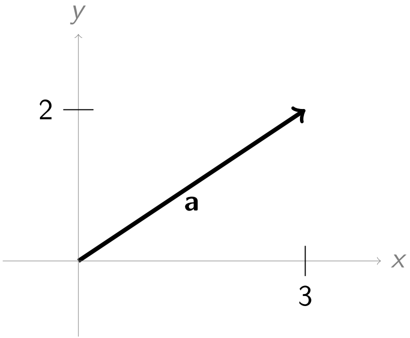

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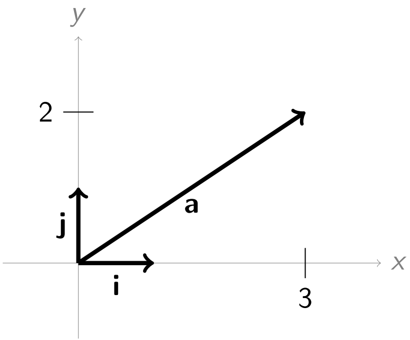

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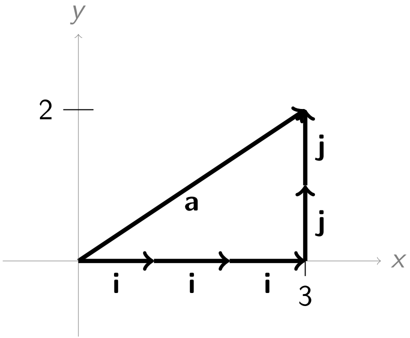

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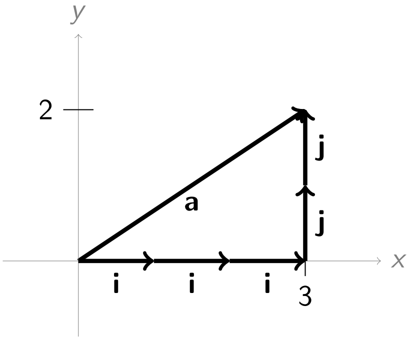

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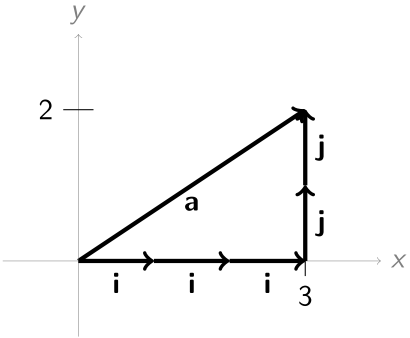

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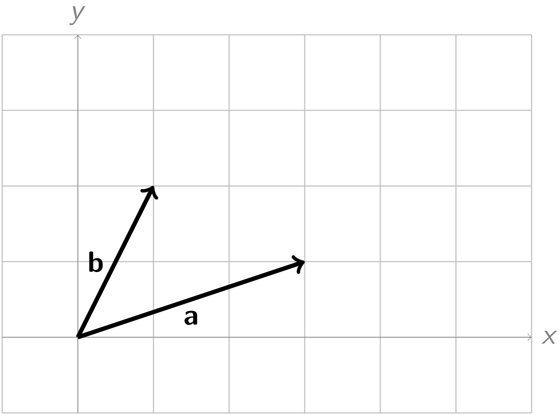
unit vector: length one

linear combination: any combination using only addition and scalar multiplication

Vector Operations on Coordinates



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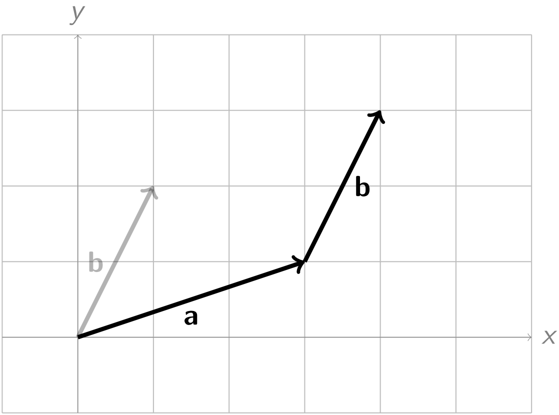

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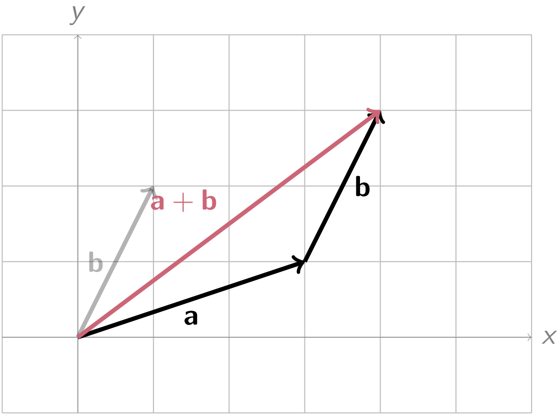

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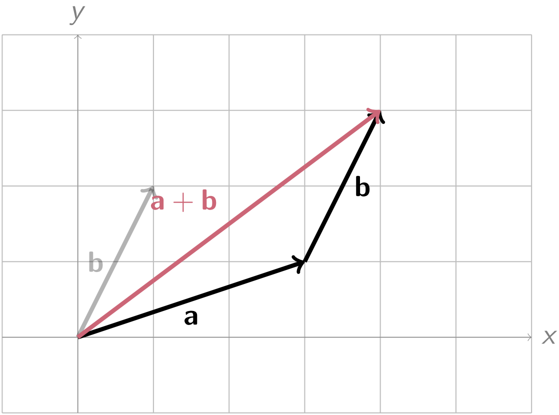

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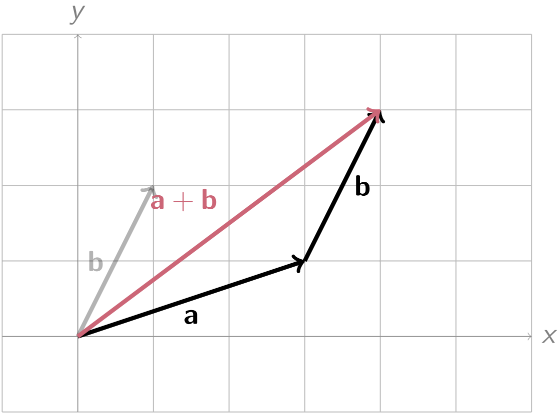

$$\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

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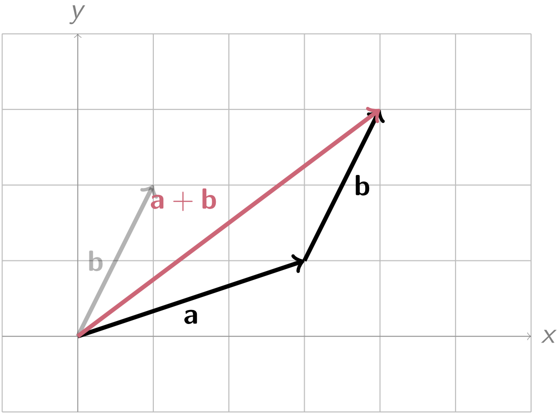

$$\begin{bmatrix} 3 \\ 1 \\ 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 30 \end{bmatrix}$$

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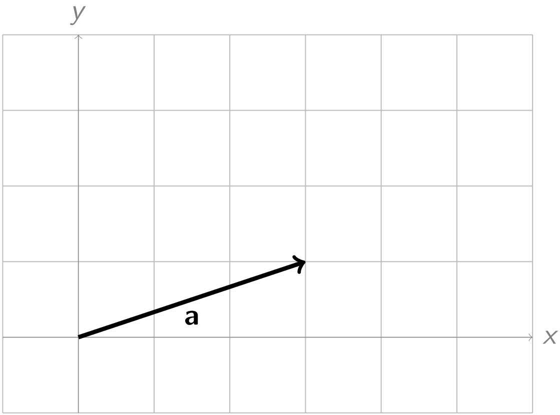

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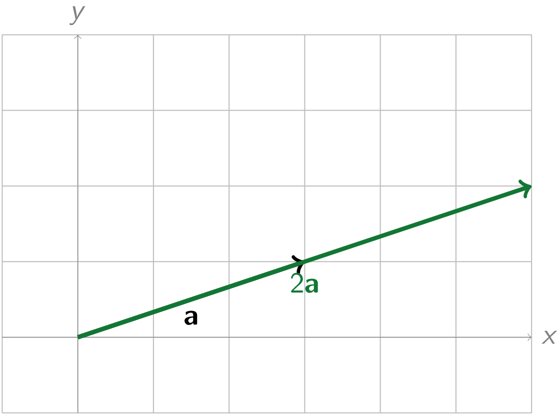

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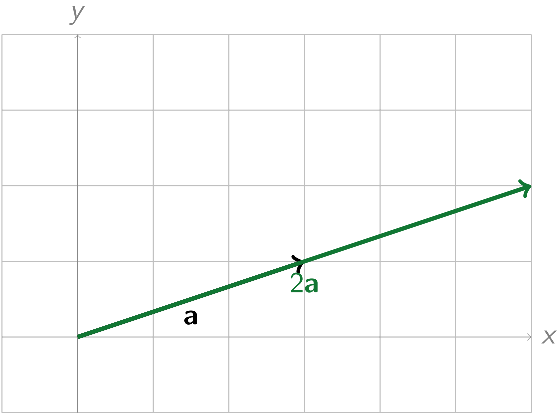

$$2\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}6\\2\end{bmatrix}$$

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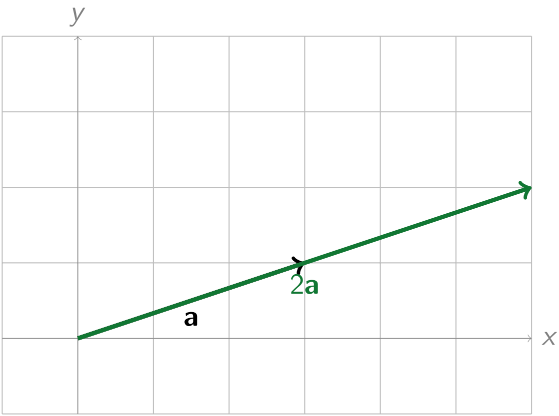

$$\frac{1}{3}\begin{bmatrix}3\\1\\6\\9\end{bmatrix}=\begin{bmatrix}1\\1/3\\2\\3\end{bmatrix}$$

Properties of Vector Addition and Scalar Multiplication

(Notes: 2.2.3)

Let $\mathbf{0}$ be the zero vector: this is the vector whose components are all zero. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors, and let s and t be scalars. The following facts about vector addition, and multiplication of vectors by scalars, are true:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}$
6. $(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}$
7. $(st)\mathbf{a} = s(t\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

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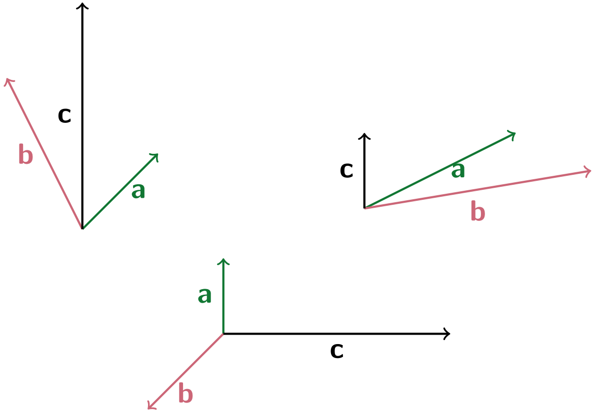
In each case below, show that the vector $\mathbf{c}$ can be written as $s\mathbf{a} + t\mathbf{b}$ for some $s, t \in \mathbb{R}$.



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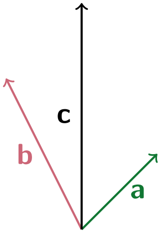

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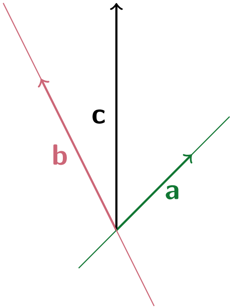

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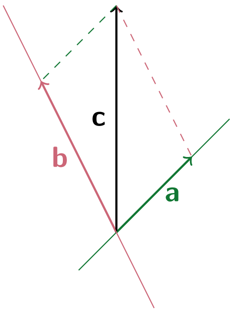

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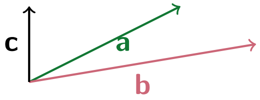

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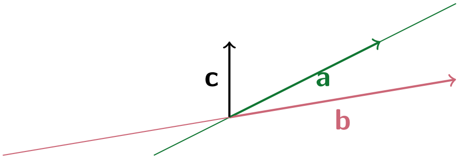

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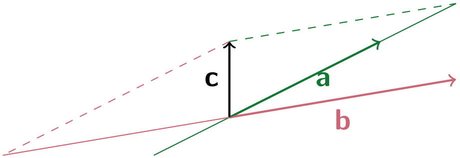

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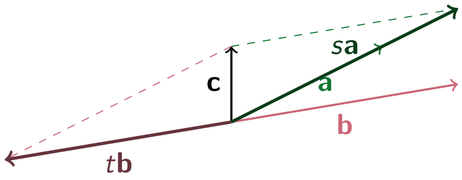

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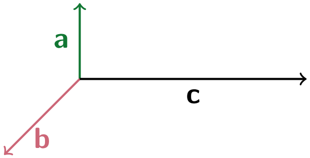

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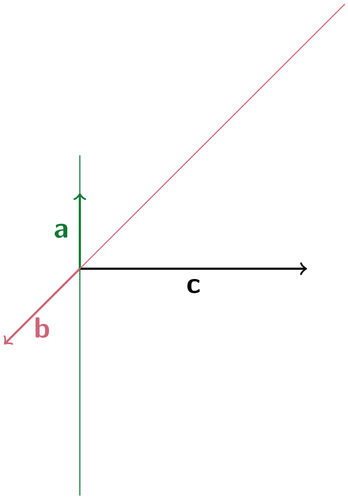

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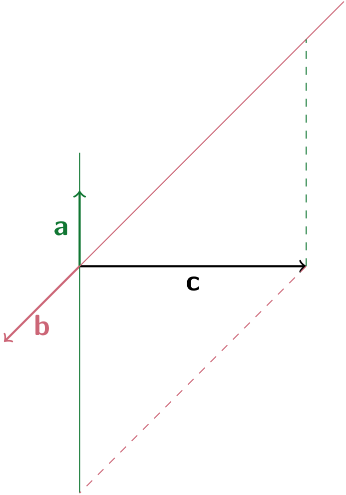
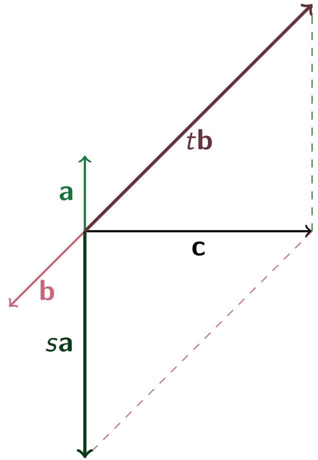


Vectors versus Coordinates

In each case below, show that the vector **c** can be written as $s\mathbf{a} + t\mathbf{b}$ for some $s, t \in \mathbb{R}$.



Vectors and Coordinates

Let $\mathbf{a}$ and $\mathbf{b}$ be fixed, nonzero vectors.

- Give an expression for the midpoint of the line segment halfway between $\mathbf{a}$ and $\mathbf{b}$.

Vectors and Coordinates

Let **a** and **b** be fixed, nonzero vectors.

- Give an expression for the midpoint of the line segment halfway between **a** and **b** .

- Give an expression for the point that is one-third of the way along the line segment between **a** and **b** .

- What is the geometric interpretation of the following set of points:

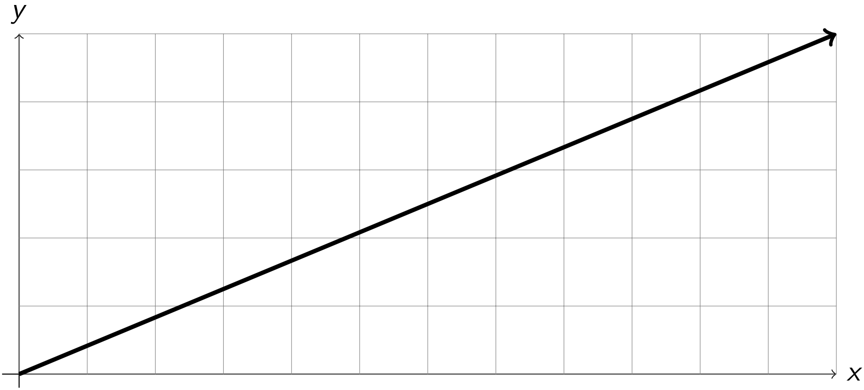
$$\{s\mathbf{a} + (1-s)\mathbf{b} : 0 \leq s \leq 1\}$$

- What is the geometric interpretation of the following set of points:

$$\{(1-s)\mathbf{a} + s\mathbf{b} : 0 \leq s \leq 1\}$$

Geometric Aspects of Vectors

How long is the vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$?



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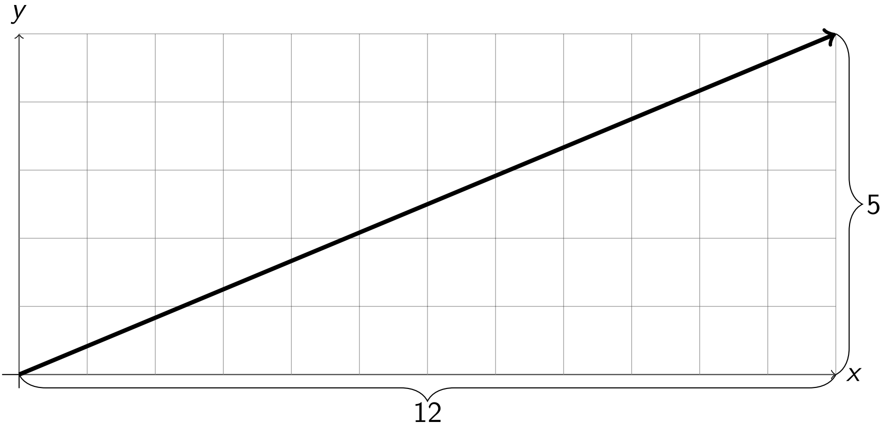

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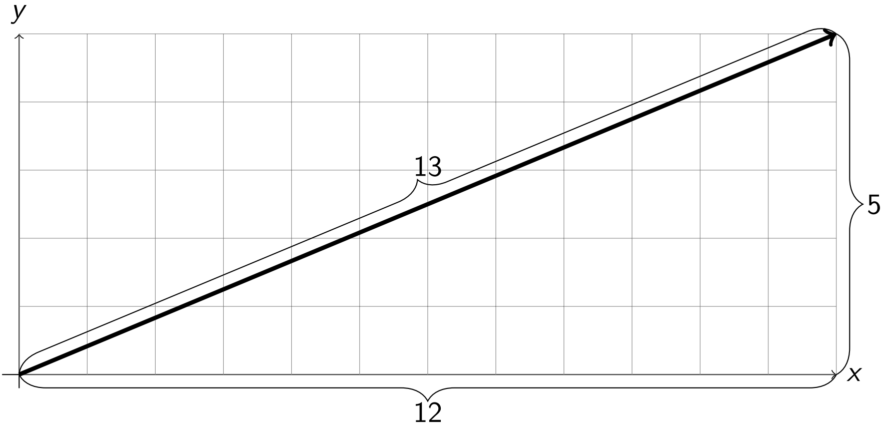

Geometric Aspects of Vectors

How long is the vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$?

The length of $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ is denoted $\left\| \begin{bmatrix} 12 \\ 5 \end{bmatrix} \right\|$, and calculated

$$\left\| \begin{bmatrix} 12 \\ 5 \end{bmatrix} \right\| = \sqrt{12^2 + 5^2} = 13.$$

We also call this quantity the norm of the vector.

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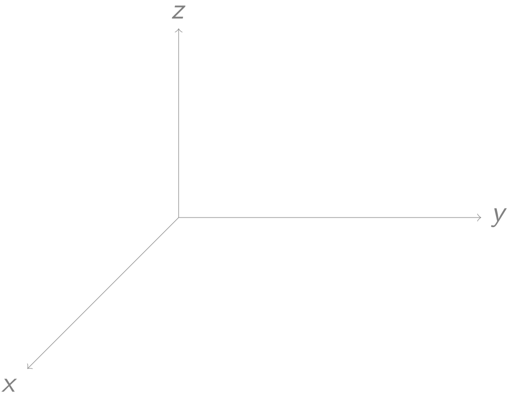
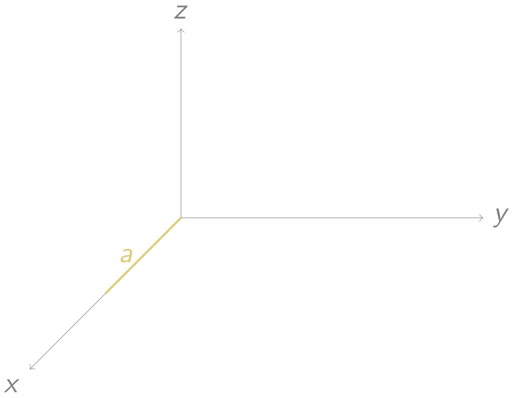
What about vectors with three coordinates?



$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



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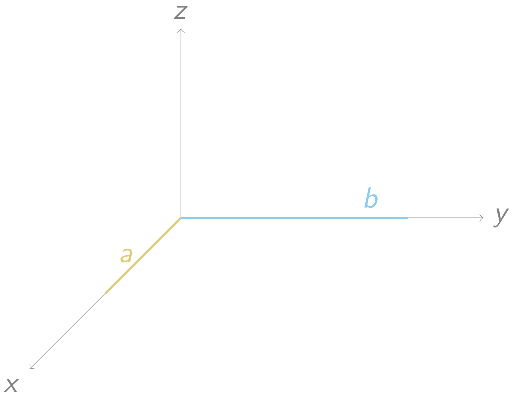

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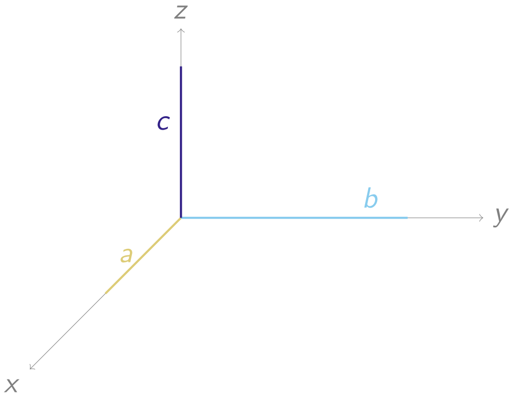

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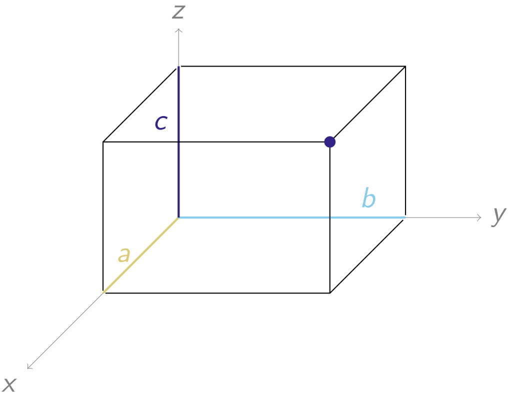

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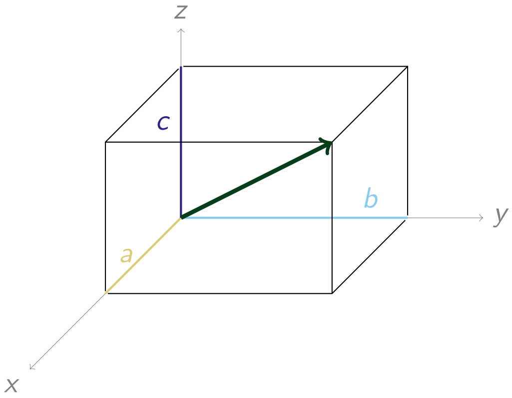

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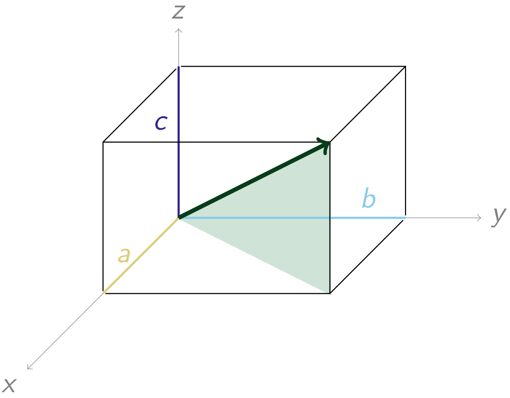

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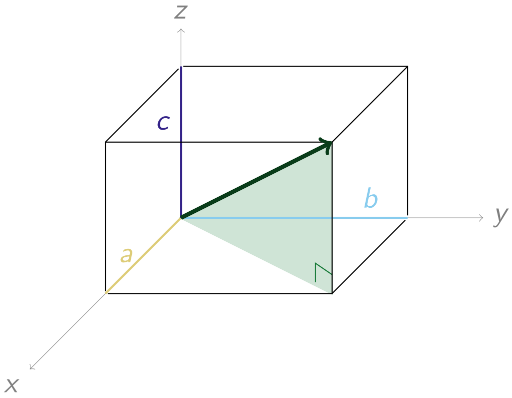

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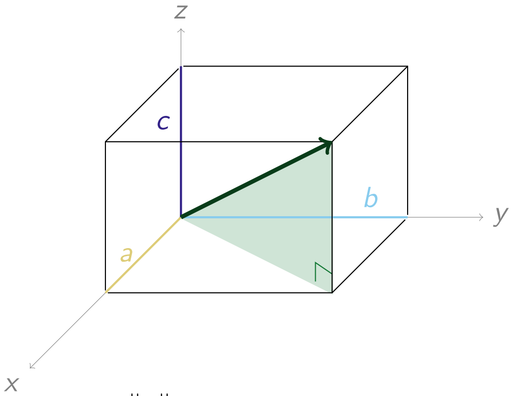
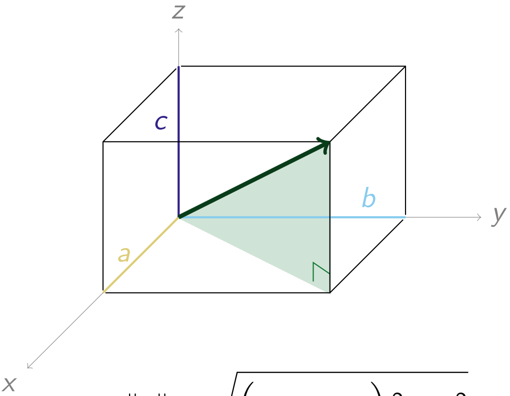


$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\|\mathbf{v}\| =$$



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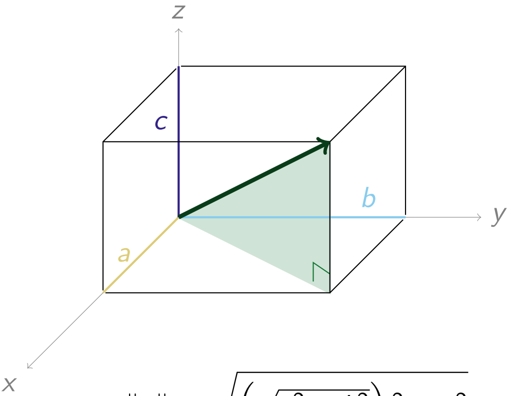

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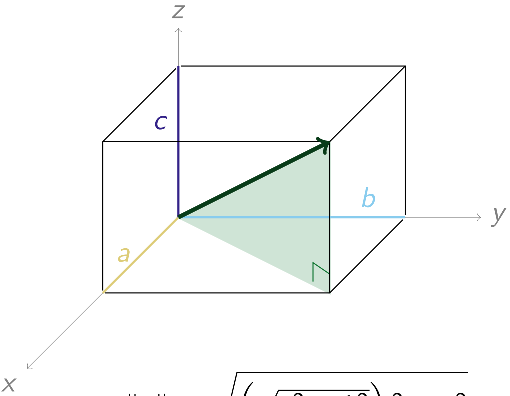

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We also call this quantity the norm of the vector.

The length of $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is denoted $\|\mathbf{a}\|$, and calculated

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Quick Concept Test

Let $\mathbf{a}$ be a vector, and let s be a scalar. For each of the following expressions, decide whether it is a vector or a scalar.

A. $\|\mathbf{a}\|$

B. $s\mathbf{a}$

C. $s\|\mathbf{a}\|$

D. $\|s\mathbf{a}\|$

E. $s + \mathbf{a}$

F. $s + \|\mathbf{a}\|$

Unit Vectors

A **unit vector** is a vector of length one.

What is the unit vector in the direction of the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

Unit Vectors

A **unit vector** is a vector of length one.

What is the unit vector in the direction of the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

If a vector is in the same direction as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then it's a (positive) scalar multiple of that vector. So, we want to divide by the length of our vector.

We compute the length of the vector: $\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2 + 4^2} = 5.$

Then the vector $\frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ is the unit vector in the same direction as the given vector.

Dot Product

Dot Product

Given vectors $\mathbf{a} = [a_1, \ldots, a_k]$ and $\mathbf{b} = [b_1, \ldots, b_k]$, we define the dot product $\mathbf{a} \cdot \mathbf{b} := a_1 b_1 + \cdots + a_k b_k$. Note $\mathbf{a} \cdot \mathbf{b}$ is a number, not a vector.

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$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = 2^2 + 1^2 + 5^2$$

$$\left\| \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\| = \sqrt{2^2 + 1^2 + 5^2}$$

Properties of the Dot Product

Notes: p. 20

For nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, zero vector $\mathbf{0}$, and scalar s:

1. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

Properties of the Dot Product

Notes: p. 20

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4. $s(\mathbf{a} \cdot \mathbf{b}) = (s\mathbf{a}) \cdot \mathbf{b}$
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7. $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a} = 0$, $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular

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For nonzero vectors **a** , **b** , and **c** , zero vector **0** , and scalar s:

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Example: are **a** and **b** perpendicular?

- $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Properties of the Dot Product

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Example: are $\mathbf{a}$ and $\mathbf{b}$ perpendicular?

$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathbf{a} = \begin{bmatrix} 2, -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3, 6 \end{bmatrix}$

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$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ No

$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}$ Yes

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

Claim 1:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}$$

Claim 2:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

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$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

Then:

$$\begin{aligned} \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta && \text{so} \\ -2\mathbf{a} \cdot \mathbf{b} &= -2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta && \text{so} \\ \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta \end{aligned}$$

Claim 1:

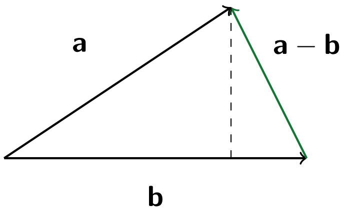
$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}\cdot\mathbf{b}$$

Proof:

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}\\ &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

Claim 2:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$



Law of Cosines

Angle between Two Vectors

Recall $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

What is the angle between vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$?

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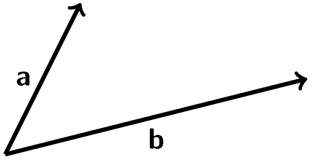
Recall $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

What is the angle between vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$?

$$\begin{aligned}
\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix} &= \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\| \cos\theta \\
10 + 5 &= \sqrt{2^2 + 1^2}\sqrt{5^2 + 5^2} \cos\theta \\
15 &= 5\sqrt{10} \cos\theta \\
\cos\theta &= \frac{3}{\sqrt{10}} \\
\theta &= \arccos\left(\frac{3}{\sqrt{10}}\right)
\end{aligned}$$

Projections

We apply a force to an object in the direction of $\mathbf{a}$, but we're only concerned with the object's movement in the direction of vector $\mathbf{b}$.



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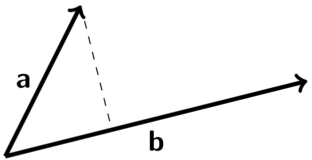

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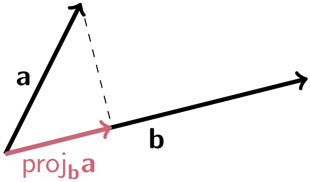

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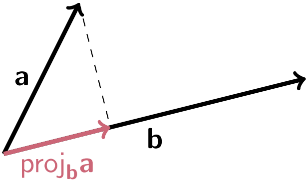

- The vector proj$_{\mathbf{b}}\mathbf{a}$ is in the same or opposite direction as **b**.

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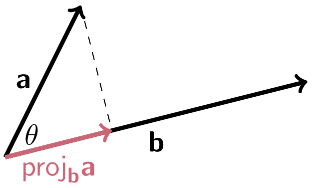

- The vector $\text{proj}_{\mathbf{b}}\mathbf{a}$ is in the same or opposite direction as **b**.
- The vector $\text{proj}_{\mathbf{b}}\mathbf{a}$ has length $\|\mathbf{a}\| \cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}$.

Projections

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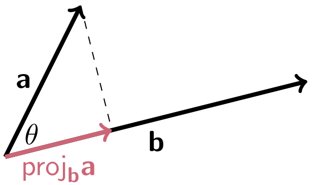

- The vector $\text{proj}_{\mathbf{b}}\mathbf{a}$ is in the same or opposite direction as $\mathbf{b}$.
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$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$$

A man pulls a truck up a hill for some reason. If we take level ground as our coordinate axis, the hill is in the direction of the vector $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$, and the man applies force represented by the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$. What vector represents the force acting on the truck in the direction it is moving?

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$$

$$\mathrm{proj}_{\begin{bmatrix}10\\2\end{bmatrix}}\begin{bmatrix}5\\2\end{bmatrix} = \left(\frac{\begin{bmatrix}5\\2\end{bmatrix}\cdot\begin{bmatrix}10\\2\end{bmatrix}}{\left\|\begin{bmatrix}10\\2\end{bmatrix}\right\|^2}\right)\begin{bmatrix}10\\2\end{bmatrix}$$

$$= \left(\frac{54}{104}\right)\begin{bmatrix}10\\2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{540}{104}\\\frac{108}{104}\end{bmatrix} = \begin{bmatrix}\frac{135}{26}\\\frac{27}{26}\end{bmatrix}$$

A man pulls a truck up a hill for some reason. He pulls with a force of 1000 pounds, and pulls at an angle of 20 degrees to the hill. What force is exerted in the direction of the hill? That is, what is the magnitude of the component of the force that is in the direction of the truck's motion?

Image credit: stu_spivack, CC,
https://www.flickr.com/photos/stuart_spivack/3850975920/in/set-72157622007398607/

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What is the projection of the vector $\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$ onto the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

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A. I solved this by computing $\left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}$

B. I solved this by drawing a picture

C. I solved this by noticing that the y component is precisely the component of the vector in the direction of $\mathbf{j}$

D. I solved this another way

What is the projection of the vector $\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$ onto the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

What is the projection of the vector $\begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$ onto the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

A: still $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

B: probably not $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ any more

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$$\begin{bmatrix} 27/14 \\ 27/7 \\ 81/14 \end{bmatrix}$$

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What is the projection of the vector $\mathbf{a}$ onto itself?

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What is the projection of the vector $\mathbf{a}$ onto itself? $\mathbf{a}$