

## Outline

Week 12: Application of vector differential equations to electrical networks

Course Notes: 6.4

Goals: determine behaviour of LCR (inductor-capacitor-resistor) networks using vector differential equations

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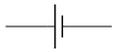
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## Circuit Components



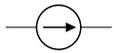
Resistor



Voltage Source



Capacitor  
acts like a changing voltage source



Current Source



Inductor  
acts like a changing current source

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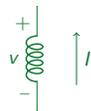
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## Differential Equations



$V$ : Voltage drop  
 $i$ : current  
 $C$ : capacitance (constant)

$$\frac{dV}{dt} = \frac{-i}{C}$$



$v$ : voltage drop  
 $I$ : current  
 $L$ : inductance (constant)

$$\frac{dI}{dt} = \frac{-v}{L}$$

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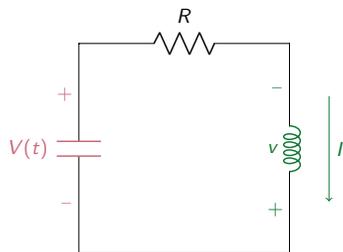
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**Changing:**

$V, v$ : voltage drops  
 $I$ : current

**Constant:**

$R$ : resistance  
 $C$ : capacitance  
 $L$ : inductance



Goal: find equations for  $V(t)$  (voltage across capacitor) and  $I(t)$  (current through inductor).

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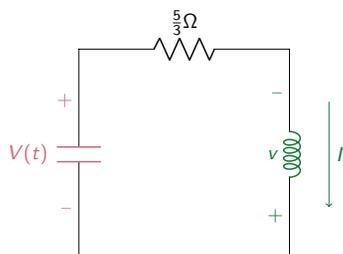
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**Changing:**

$V, v$ : voltage drops  
 $I$ : current

**Constant:**

$R = \frac{5}{3}$ : resistance  
 $C = \frac{1}{2}$ : capacitance  
 $L = \frac{1}{3}$ : inductance



Goal: find equations for  $V(t)$  (voltage across capacitor) and  $I(t)$  (current through inductor).

Kirchoff:  $-v - V + \frac{5}{3}I = 0 \Rightarrow v = \frac{5}{3}I - V$

Differential Equations:

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Differential Equations:  $\frac{dV}{dt} = -2I, \quad \frac{dI}{dt} = 3V - 5I$

Find:  $\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$

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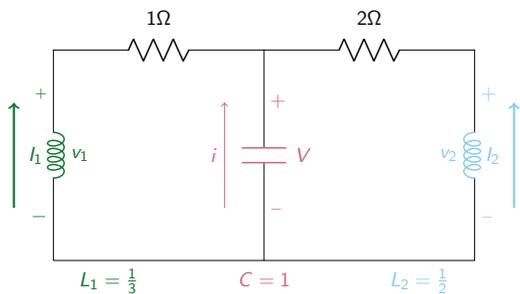
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Want to find:  $i_1(t)$ ,  $i_2(t)$ , and  $V(t)$ .

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Differential:

$$\begin{aligned} \frac{di_1}{dt} &= \frac{-v_1}{L_1} = -3v_1 \\ \frac{di_2}{dt} &= \frac{-v_2}{L_2} = -2v_2 \\ \frac{dV}{dt} &= \frac{i}{C} = -i \end{aligned}$$

Kirkhoff:

$$\begin{aligned} -v_1 + 1i_1 + V &= 0 \\ -v_2 + 2i_2 + V &= 0 \\ i &= -i_1 - i_2 \end{aligned}$$

Combined:

$$\begin{aligned} \frac{di_1}{dt} &= -3v_1 = -3(i_1 + V) = -3i_1 - 3V \\ \frac{di_2}{dt} &= -2v_2 = -2(2i_2 + V) = -4i_2 - 2V \\ \frac{dV}{dt} &= -i = i_1 + i_2 \end{aligned}$$

$$\begin{bmatrix} i_1 \\ i_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V \end{bmatrix}$$

Now we need the eigenvalues and eigenvectors of the matrix.

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$$\begin{bmatrix} i_1 \\ i_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V \end{bmatrix}$$

$$\begin{aligned} \lambda_1 &\approx -1.6 - 1.5i, & \mathbf{x}_1 &\approx \begin{bmatrix} 1 \\ 0.46 - 0.13i \\ -.46 + .49i \end{bmatrix} \\ \lambda_2 &\approx -1.6 + 1.5i, & \mathbf{x}_2 &\approx \begin{bmatrix} 1 \\ 0.46 + 0.13i \\ -.46 - .49i \end{bmatrix} \\ \lambda_3 &\approx -3.7, & \mathbf{x}_3 &\approx \begin{bmatrix} 1 \\ -1.9 \\ 0.25 \end{bmatrix} \end{aligned}$$

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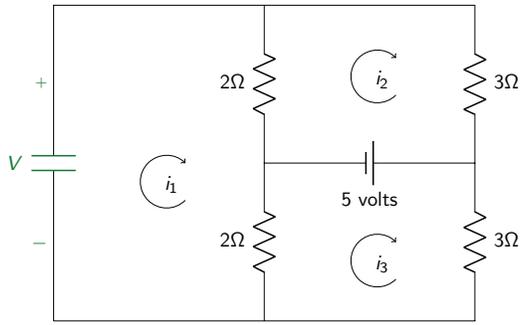
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Capacitance:  $\frac{1}{12}$

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Differential equation:

Kirkhoff:

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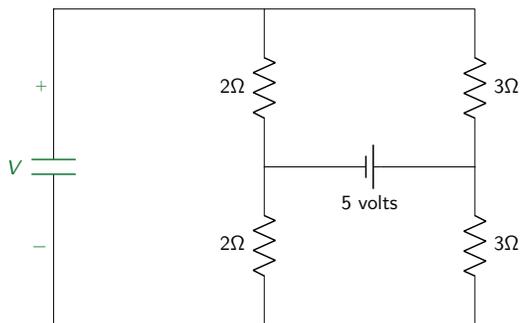
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Capacitance:  $\frac{1}{12}$   
 $V(t) = Ce^{-5t}$

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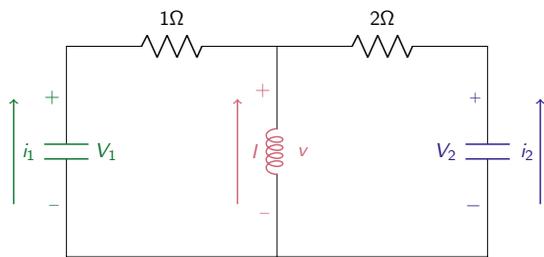
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Capacitances:  $C_1 = \frac{1}{6}$ ,  $C_2 = \frac{1}{3}$   
 Inductance:  $L = \frac{1}{3}$   
 Find  $V_1(t)$ ,  $V_2(t)$ , and  $I(t)$ .

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We use Kirkhoff's Laws to solve for  $i_1$ ,  $i_2$ , and  $v$  in terms of  $V_1$ ,  $V_2$ , and  $I$ .  
 Left Loop:  $-V_1 + i_1 + v = 0$   
 Right Loop:  $-V_2 + 2i_2 + v = 0$   
 Inductor (like a current source):  $i_1 + i_2 = -I$

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$$\frac{dV_1}{dt} = -\frac{i_1}{C_1} = -6 \left(\frac{1}{3}\right) (V_1 - V_2 - 2I) = -2V_1 + 2V_2 + 4I$$

$$\frac{dV_2}{dt} = -\frac{i_2}{C_2} = -3 \left(\frac{1}{3}\right) (-V_1 + V_2 - I) = V_1 - V_2 + I$$

$$\frac{dI}{dt} = -\frac{v}{L} = -3 \left(\frac{1}{3}\right) (2V_1 + V_2 + 2I) = -2V_1 - V_2 - 2I$$

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General Solution:

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ I(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix} + c_2 e^{-\frac{3}{2}t} \begin{bmatrix} 4 \cos\left(\frac{3\sqrt{3}}{2}t\right) \\ \cos\left(\frac{3\sqrt{3}}{2}t\right) + 3 \sin\left(\frac{3\sqrt{3}}{2}t\right) \\ -2\sqrt{3} \sin\left(\frac{3\sqrt{3}}{2}t\right) \end{bmatrix} \\ + c_3 e^{-\frac{3}{2}t} \begin{bmatrix} 4 \sin\left(\frac{3\sqrt{3}}{2}t\right) \\ \sin\left(\frac{3\sqrt{3}}{2}t\right) - 3 \cos\left(\frac{3\sqrt{3}}{2}t\right) \\ 2\sqrt{3} \cos\left(\frac{3\sqrt{3}}{2}t\right) \end{bmatrix}$$

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