

Outline

Week 12: Application of vector differential equations to electrical networks

Course Notes: 6.4

Goals: determine behaviour of LCR (inductor-capacitor-resistor) networks using vector differential equations

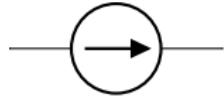
Circuit Components



Resistor



Voltage Source



Current Source

Circuit Components



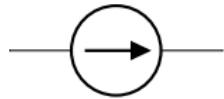
Resistor



Voltage Source



Capacitor

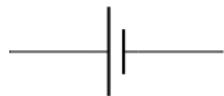


Current Source

Circuit Components



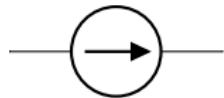
Resistor



Voltage Source



Capacitor



Current Source

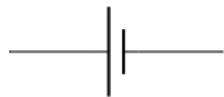


Inductor

Circuit Components



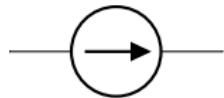
Resistor



Voltage Source



Capacitor
acts like a changing voltage source



Current Source

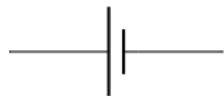


Inductor

Circuit Components



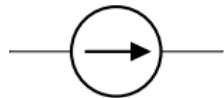
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Voltage Source



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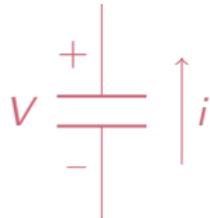


Current Source



Inductor
acts like a changing current source

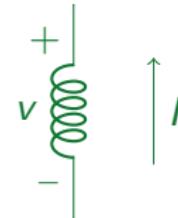
Differential Equations



V: Voltage drop

i: current

C: capacitance (constant)



v: voltage drop

I: current

L: inductance (constant)

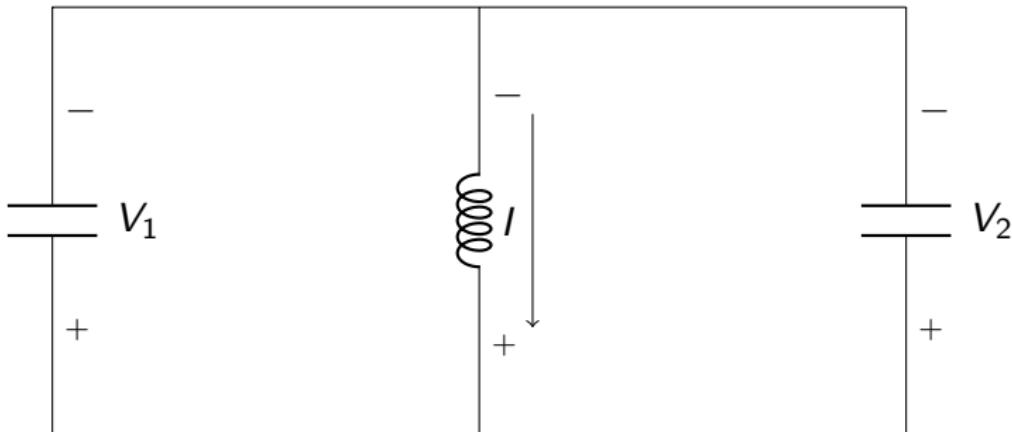
$$\frac{dV}{dt} = \frac{-i}{C}$$

$$\frac{dI}{dt} = \frac{-v}{L}$$

Beyond Kirhoff: Initial Practice

C_1, C_2, L known (and constant)

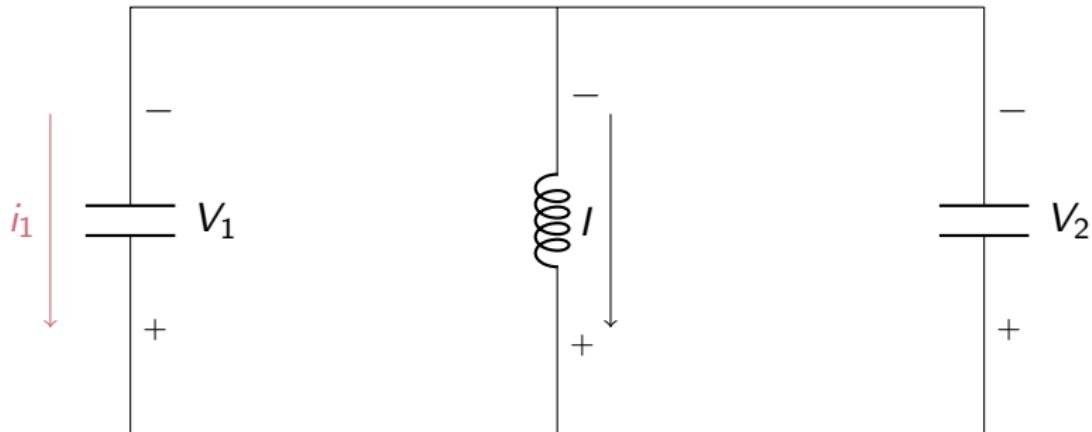
V_1, V_2 , and I unknown (functions of time)



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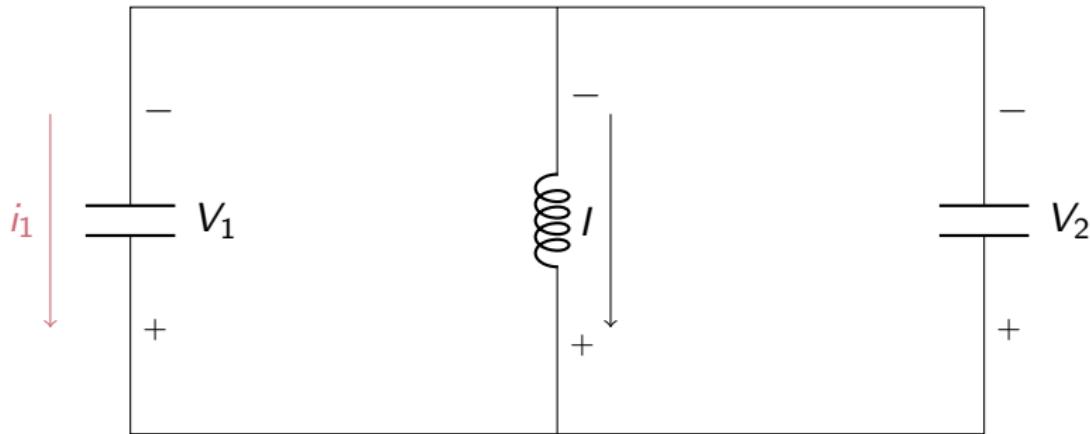
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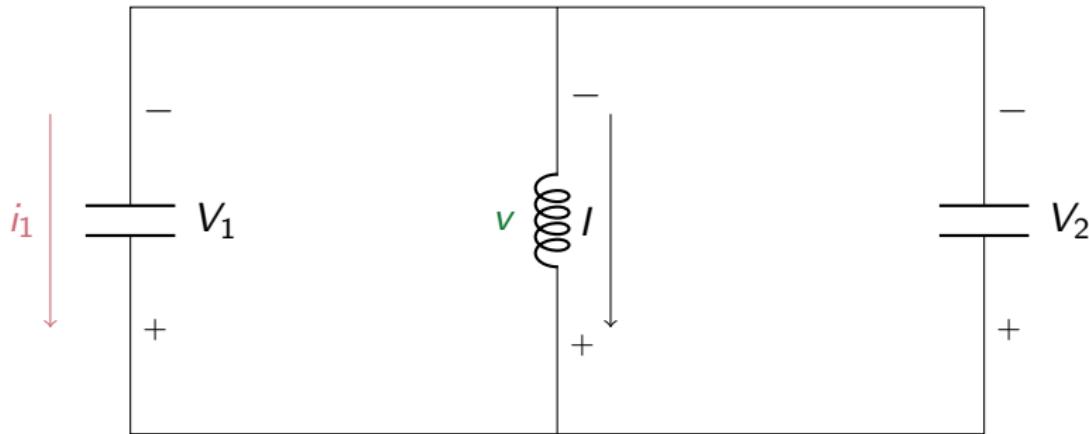


$$\frac{dV_1}{dt} = -\frac{i_1}{C_1}$$

Beyond Kirhoff: Initial Practice

C_1, C_2, L known (and constant)

V_1, V_2 , and I unknown (functions of time)

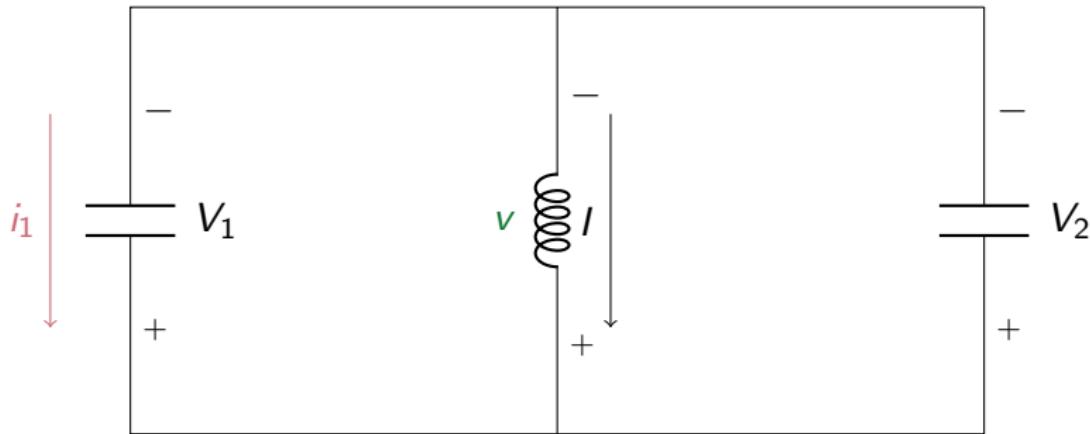


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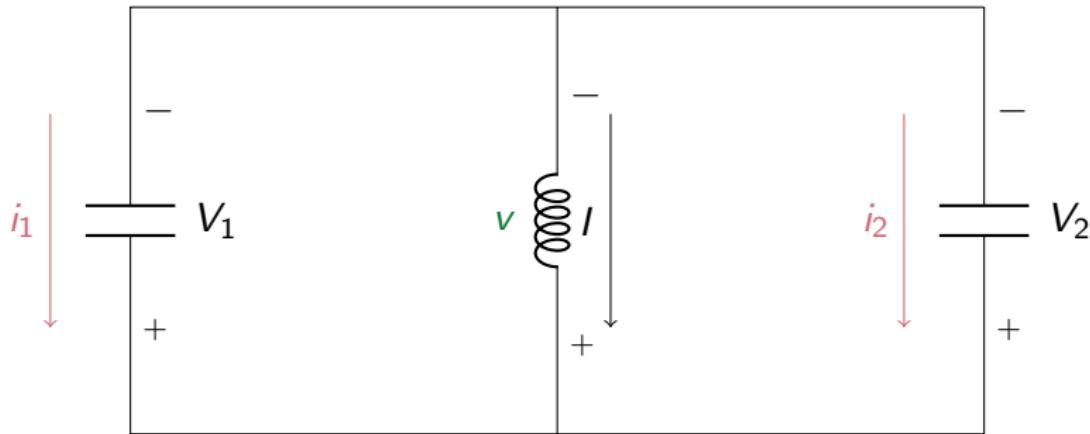
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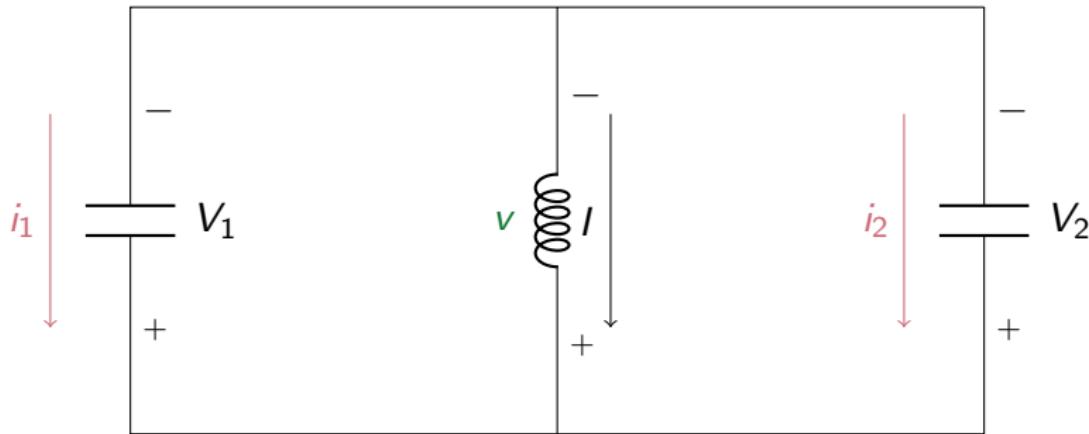
$$\frac{dV_1}{dt} = -\frac{i_1}{C_1}$$

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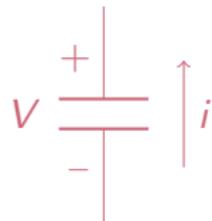


$$\frac{dV_1}{dt} = -\frac{i_1}{C_1}$$

$$\frac{dl}{dt} = -\frac{v}{L}$$

$$\frac{dV_2}{dt} = -\frac{i_2}{C_2}$$

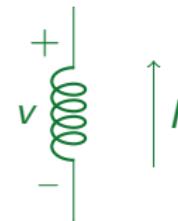
Differential Equations



V: Voltage drop

i: current

C: capacitance (constant)



v: voltage drop

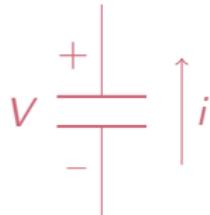
I: current

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Differential Equations

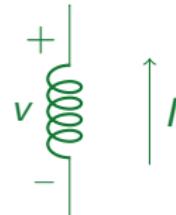


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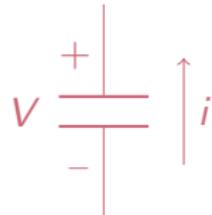
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$$\frac{dI}{dt} = \frac{-v}{L}$$

- Current goes – to + through the component

Differential Equations

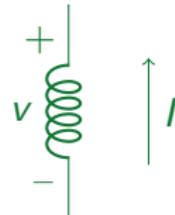


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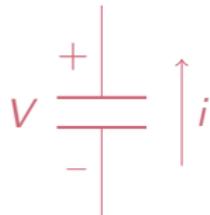
I: current

L: inductance (constant)

$$\frac{dI}{dt} = \frac{-v}{L}$$

- Current goes – to + through the component
- Negative something over something

Differential Equations

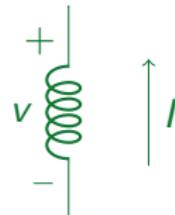


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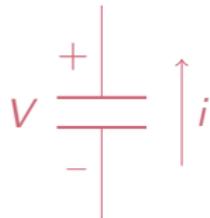
I: current

L: inductance (constant)

$$\frac{dl}{dt} = \frac{-v}{L}$$

- Current goes – to + through the component
- Negative something over something
- Capacitors have capacitance, inductors have inductance

Differential Equations

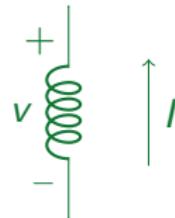


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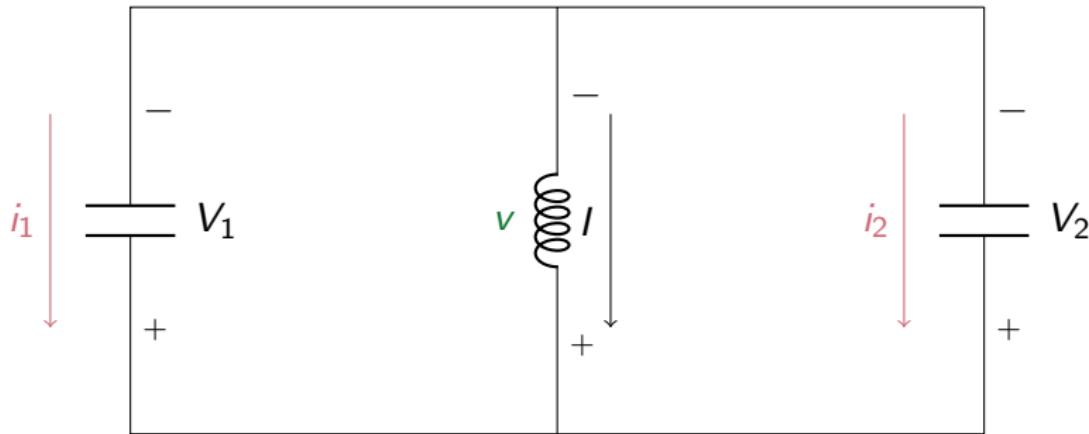
$$\frac{dl}{dt} = \frac{-v}{L}$$

- Current goes – to + through the component
- Negative something over something
- Capacitors have capacitance, inductors have inductance
- Voltage goes with current

Beyond Kirhoff: Initial Practice

C_1, C_2, L known (and constant)

V_1, V_2 , and I unknown (functions of time)



$$\frac{dV_1}{dt} = -\frac{i_1}{C_1}$$

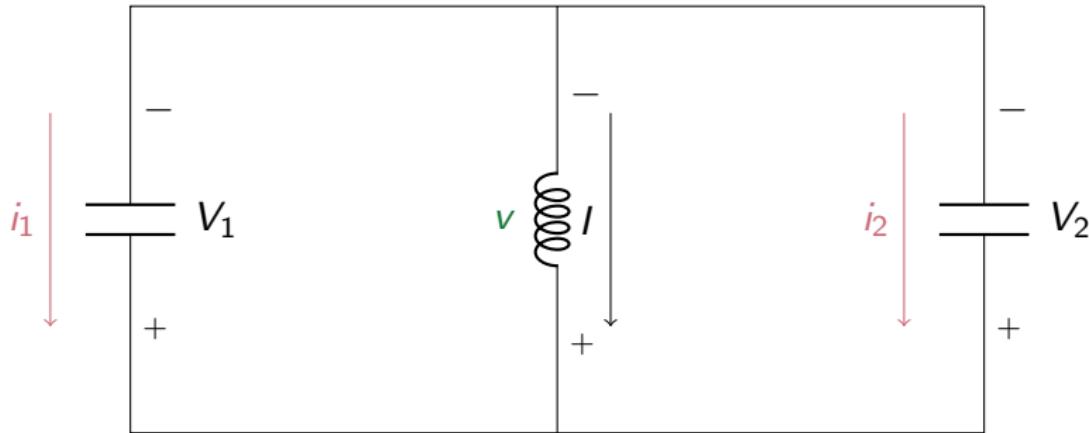
$$\frac{dI}{dt} = -\frac{v}{L}$$

$$\frac{dV_2}{dt} = -\frac{i_2}{C_2}$$

Beyond Kirhoff: Initial Practice

C_1, C_2, L known (and constant)

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$$\frac{dV_1}{dt} = -\frac{i_1}{C_1}$$

$$\frac{di}{dt} = -\frac{v}{L}$$

$$\frac{dV_2}{dt} = -\frac{i_2}{C_2}$$

Now that we know the new part, let's do a full example.

Changing:

V , v : voltage drops

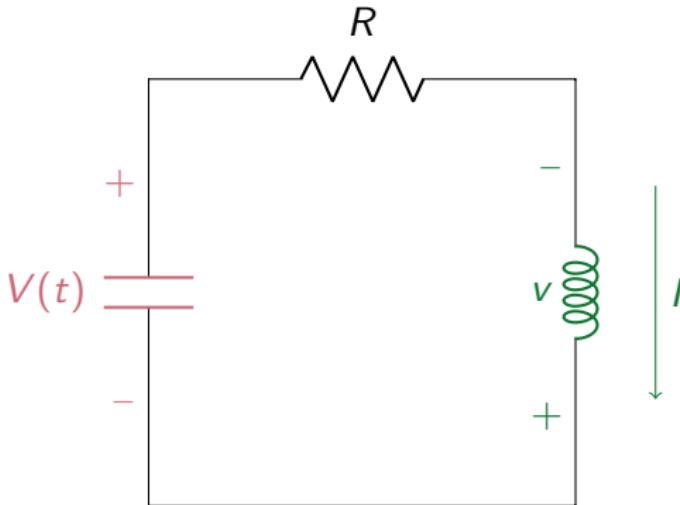
I : current

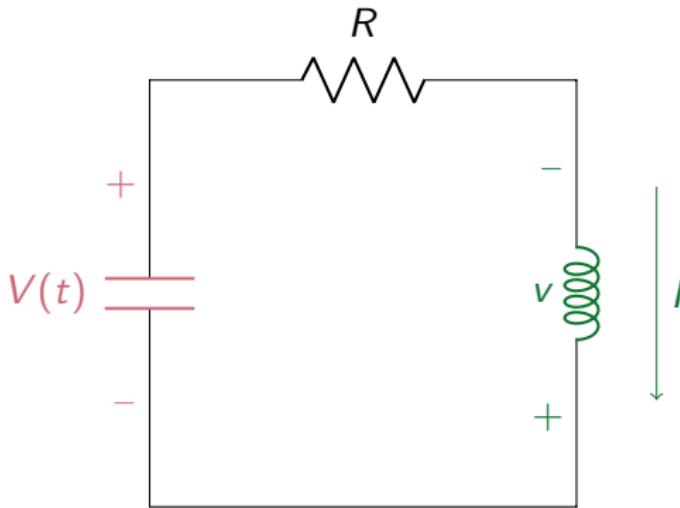
Constant:

R : resistance

C : capacitance

L : inductance



Changing: V , v : voltage drops I : current**Constant:** R : resistance C : capacitance L : inductance

Goal: find equations for $V(t)$ (voltage across capacitor) and $I(t)$ (current through inductor).

Changing:

V , v : voltage drops

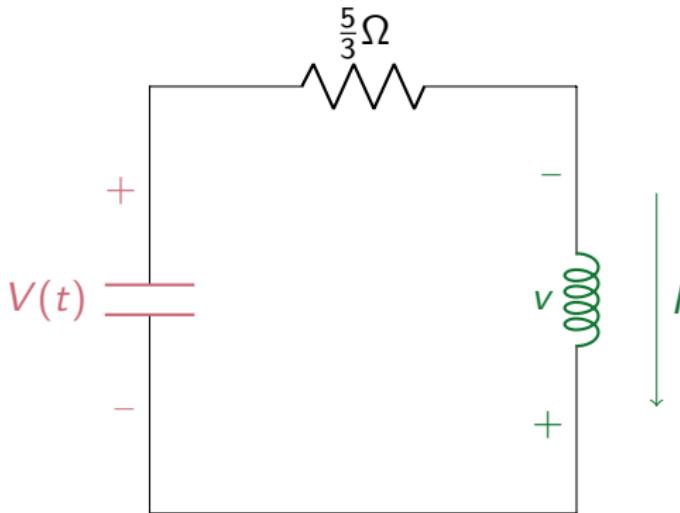
I : current

Constant:

$R = \frac{5}{3}$: resistance

$C = \frac{1}{2}$: capacitance

$L = \frac{1}{3}$: inductance



Goal: find equations for $V(t)$ (voltage across capacitor) and $I(t)$ (current through inductor).

Changing:

V , v : voltage drops

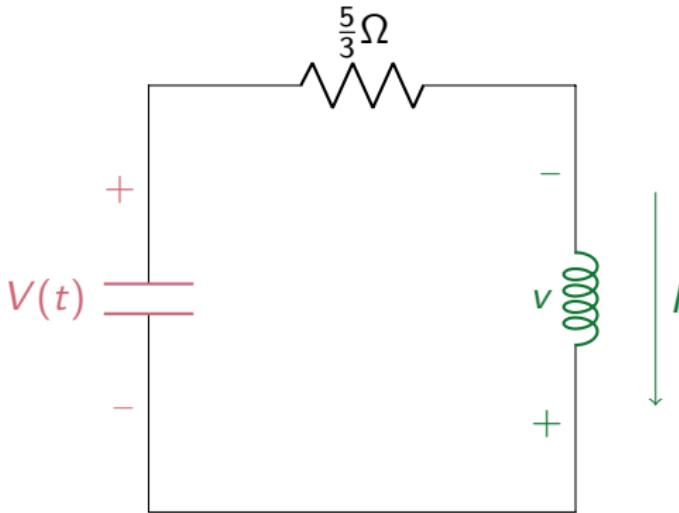
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Goal: find equations for $V(t)$ (voltage across capacitor) and $I(t)$ (current through inductor).

Kirkhoff: $-v - V + \frac{5}{3}I = 0 \Rightarrow v = \frac{5}{3}I - V$

Changing:

V , v : voltage drops

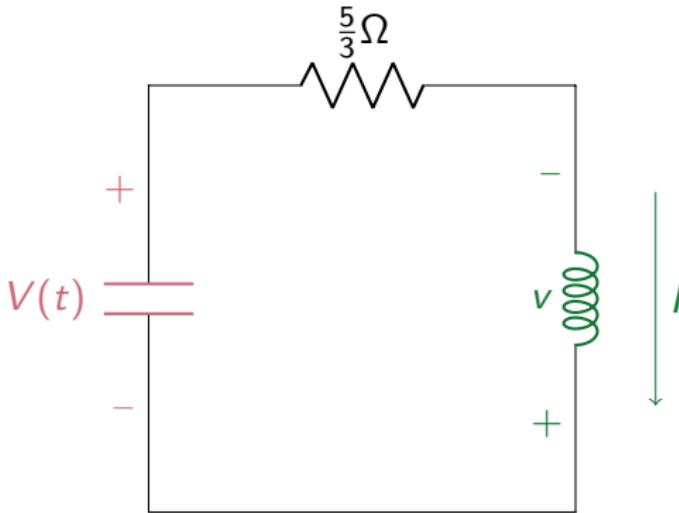
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$$\text{Kirkhoff: } -v - V + \frac{5}{3}I = 0 \quad \Rightarrow \quad v = \frac{5}{3}I - V$$

Differential Equations:

$$\frac{dV}{dt} = \frac{-I}{C} = -2I \quad \frac{dI}{dt} = \frac{-v}{L} = 3(V - \frac{5}{3}I) = 3V - 5I$$

Differential Equations: $\frac{dV}{dt} = -2I$, $\frac{dI}{dt} = 3V - 5I$

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Find: $\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$

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$$\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}' = \begin{bmatrix} -2I \\ 3V - 5I \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$$

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$$\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}' = \begin{bmatrix} -2I \\ 3V - 5I \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$$

Differential Equations: $\frac{dV}{dt} = -2I$, $\frac{dI}{dt} = 3V - 5I$

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$$\lambda_1 = -2, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3 \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Differential Equations: $\frac{dV}{dt} = -2I$, $\frac{dI}{dt} = 3V - 5I$

Find: $\begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$

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$$\lambda_1 = -2, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_2 = -3 \mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} V(t) \\ I(t) \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{-3t} \\ c_1 e^{-2t} + 3c_2 e^{-3t} \end{bmatrix}$$

Changing:

V , v : voltage drops

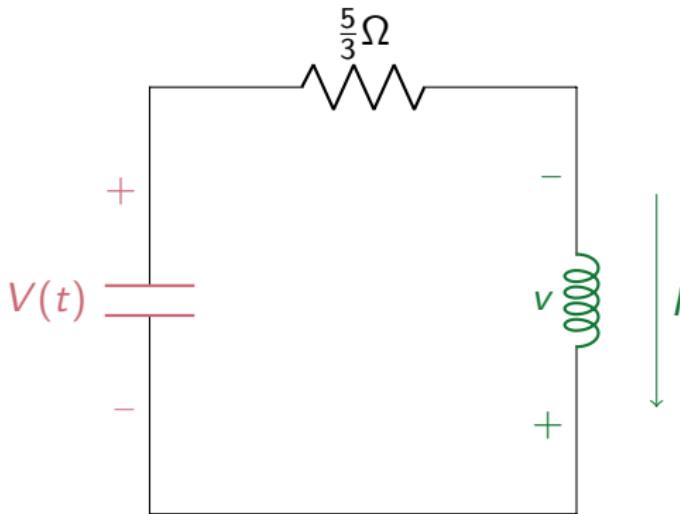
I : current

Constant:

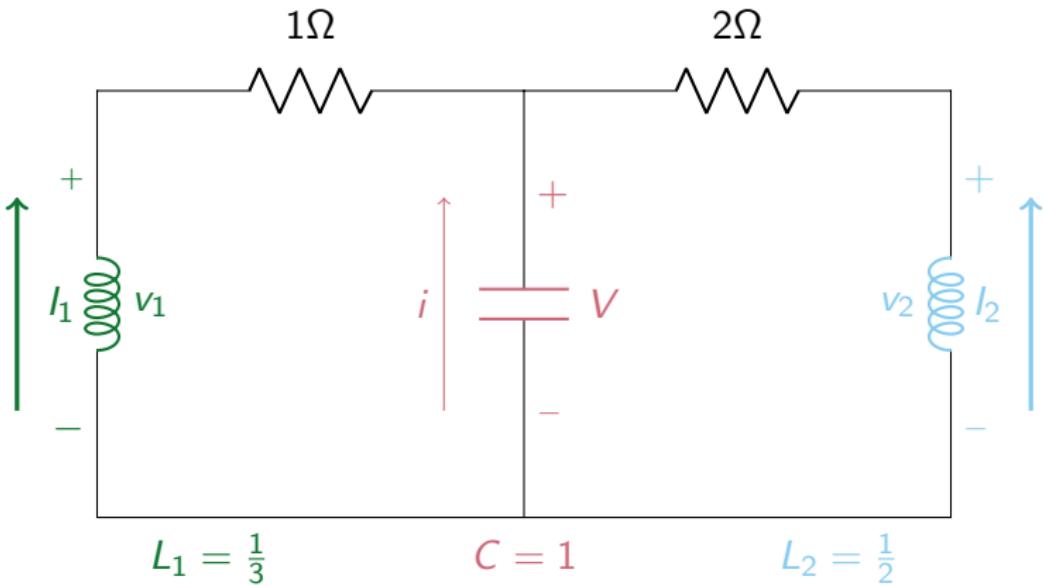
$R = \frac{5}{3}$: resistance

$C = \frac{1}{2}$: capacitance

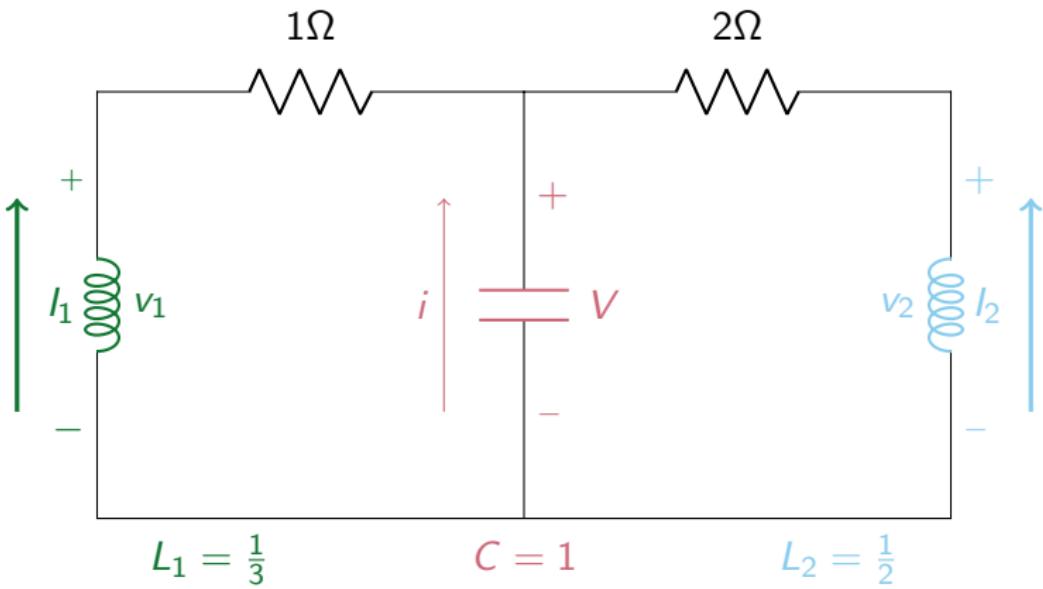
$L = \frac{1}{3}$: inductance



Goal: find equations for $V(t)$ (voltage across capacitor) and $I(t)$ (current through inductor).



Want to find: $i_1(t)$, $i_2(t)$, and $V(t)$.



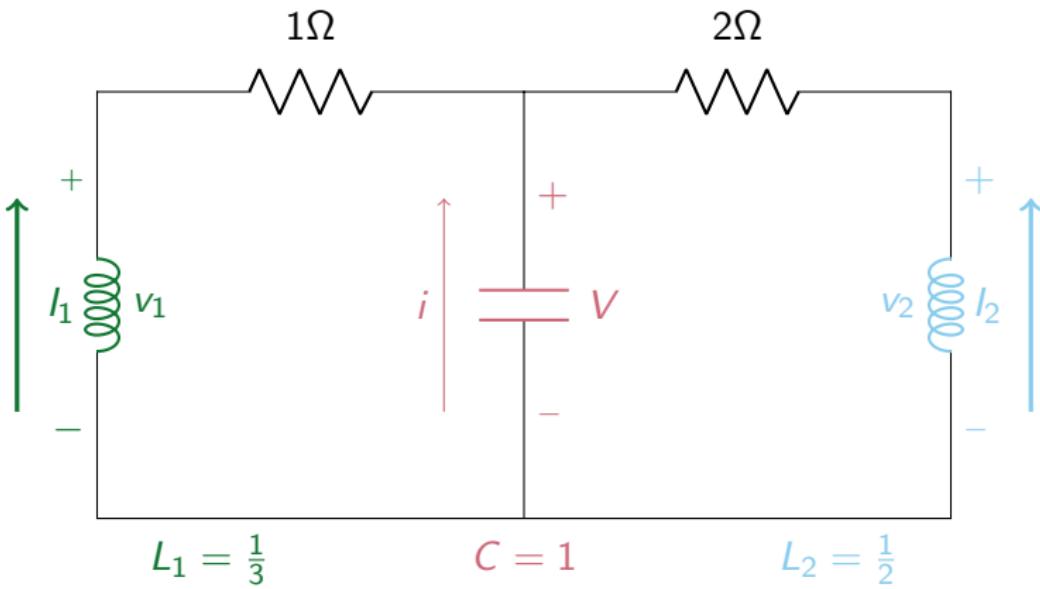
Want to find: $i_1(t)$, $i_2(t)$, and $V(t)$.

Differential:

$$\frac{di_1}{dt} = \frac{-v_1}{L_1} = -3v_1$$

$$\frac{di_2}{dt} = \frac{-v_2}{L_2} = -2v_2$$

$$\frac{dV}{dt} = \frac{-i}{C} = -i$$



Want to find: $i_1(t)$, $i_2(t)$, and $V(t)$.

Differential:

$$\frac{di_1}{dt} = \frac{-v_1}{L_1} = -3v_1$$

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$$\frac{dV}{dt} = \frac{-i}{C} = -i$$

Kirkhoff:

$$-v_1 + 1i_1 + V = 0$$

$$-v_2 + 2i_2 + V = 0$$

$$i = -i_1 - i_2$$

Differential:

$$\frac{di_1}{dt} = \frac{-v_1}{L_1} = -3v_1$$

$$\frac{dl_2}{dt} = \frac{-v_2}{L_2} = -2v_2$$

$$\frac{dV}{dt} = \frac{-i}{C} = -i$$

Kirkhoff:

$$-v_1 + 1l_1 + V = 0$$

$$-v_2 + 2l_2 + V = 0$$

$$i = -l_1 - l_2$$

Combined:

$$\frac{dl_1}{dt} = -3v_1 = -3(l_1 + V) = -3l_1 - 3V$$

$$\frac{dl_2}{dt} = -2v_2 = -2(2l_1 + V) = -4l_2 - 2V$$

$$\frac{dV}{dt} = -i = l_1 + l_2$$

$$\begin{bmatrix} l_1 \\ l_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ V \end{bmatrix}$$

Now we need the eigenvalues and eigenvectors of the matrix.

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

$$\lambda_1 \approx -1.6 - 1.5i,$$

$$x_1 \approx \begin{bmatrix} 1 \\ 0.46 - 0.13i \\ -.46 + .49i \end{bmatrix}$$

$$\lambda_2 \approx -1.6 + 1.5i,$$

$$x_2 \approx \begin{bmatrix} 1 \\ 0.46 + 0.13i \\ -.46 - .49i \end{bmatrix}$$

$$\lambda_3 \approx -3.7,$$

$$x_3 \approx \begin{bmatrix} 1 \\ -1.9 \\ 0.25 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

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$$\lambda_3 \approx -3.7,$$

$$\mathbf{x}_3 \approx \begin{bmatrix} 1 \\ -1.9 \\ 0.25 \end{bmatrix}$$

$$c_1 e^{-1.6t} e^{-1.5it} \mathbf{x}_1 + c_2 e^{-1.6t} e^{1.5it} \mathbf{x}_2 + c_3 e^{-3.7t} \mathbf{x}_3$$

$$\begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ V \end{bmatrix}$$

$$\lambda_1 \approx -1.6 - 1.5i,$$

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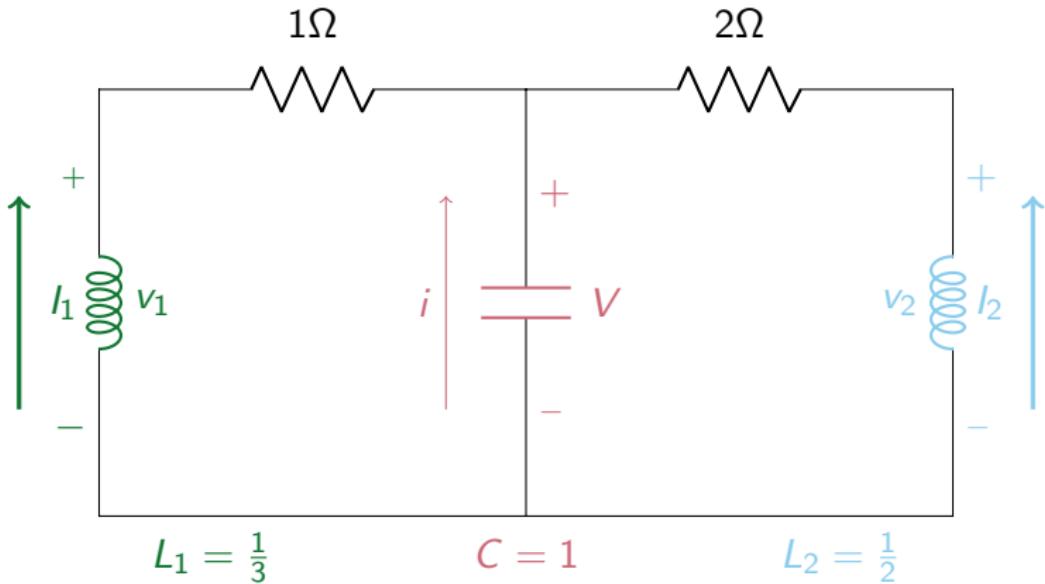
$$\mathbf{x}_2 \approx \begin{bmatrix} 1 \\ 0.46 + 0.13i \\ -.46 - .49i \end{bmatrix}$$

$$\lambda_3 \approx -3.7,$$

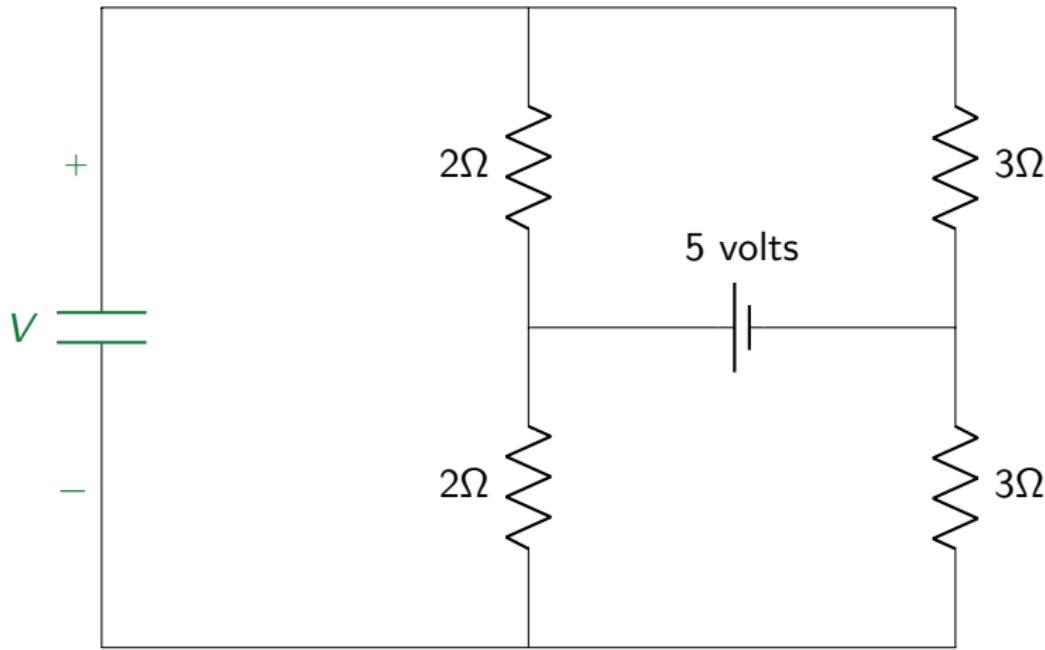
$$\mathbf{x}_3 \approx \begin{bmatrix} 1 \\ -1.9 \\ 0.25 \end{bmatrix}$$

$$c_1 e^{-1.6t} e^{-1.5it} \mathbf{x}_1 + c_2 e^{-1.6t} e^{1.5it} \mathbf{x}_2 + c_3 e^{-3.7t} \mathbf{x}_3$$

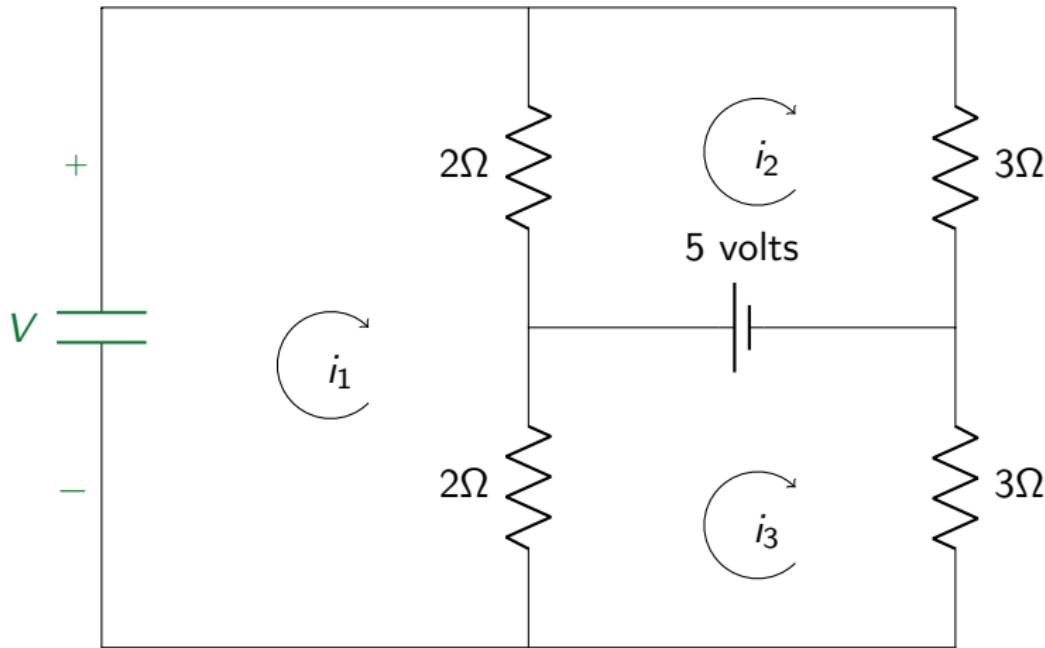
Regardless of initial conditions, the solutions will decay.
They may or may not oscillate while decaying.



Want to find: $i_1(t)$, $i_2(t)$, and $V(t)$.



Capacitance: $\frac{1}{12}$



Capacitance: $\frac{1}{12}$

Differential equation:

$$\frac{dV}{dt} = -\frac{i_1}{C} = -12i_1$$

Differential equation:

$$\frac{dV}{dt} = -\frac{i_1}{C} = -12i_1$$

Kirchhoff:

$$-V + 2(i_1 - i_2) + 2(i_1 - i_3) = 0$$

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Differential equation:

$$\frac{dV}{dt} = -\frac{i_1}{C} = -12i_1$$

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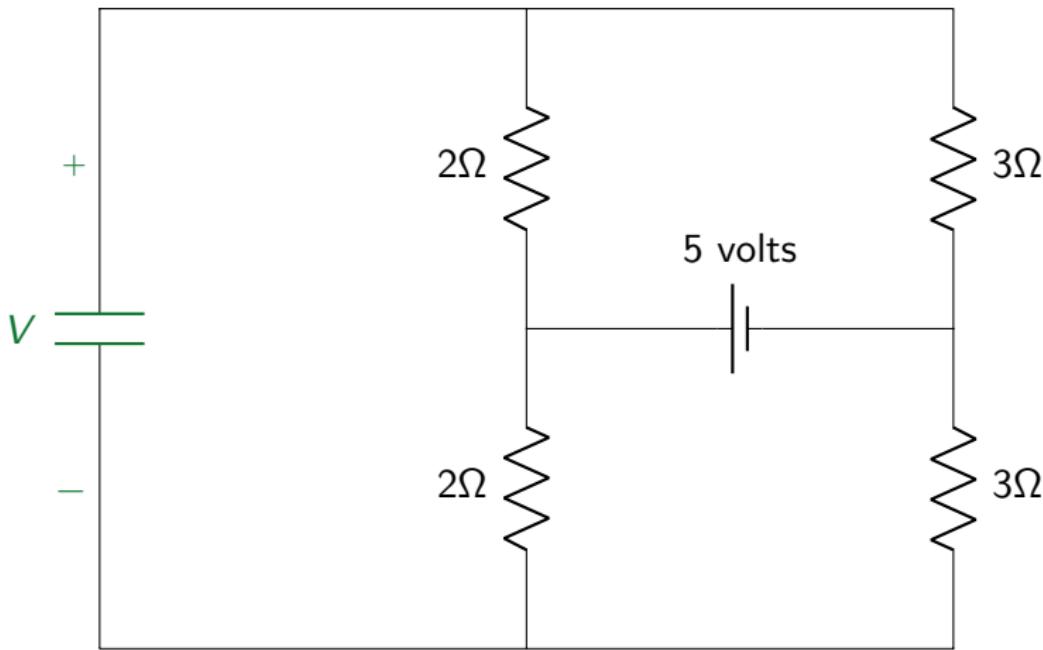
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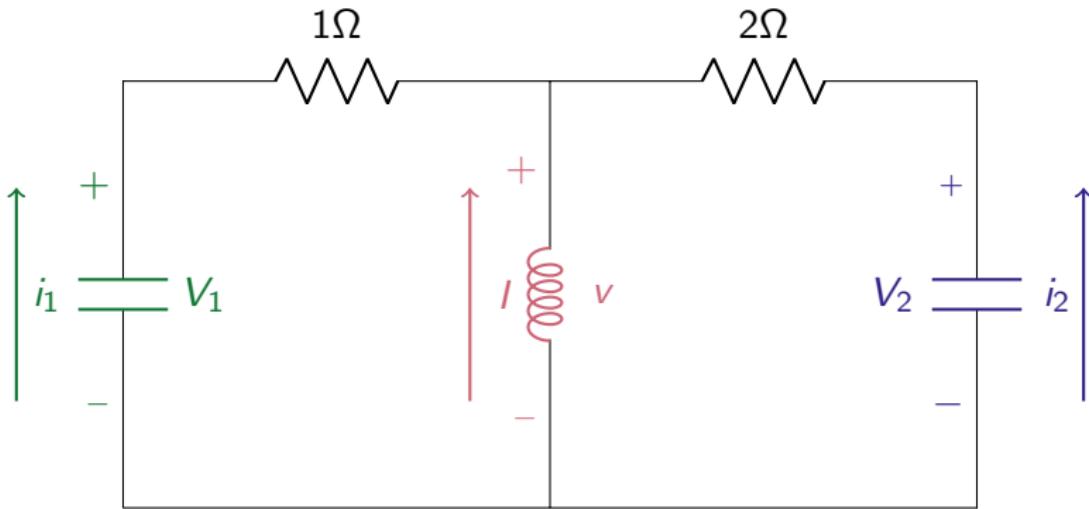
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$$V(t) = Ce^{-5t}$$



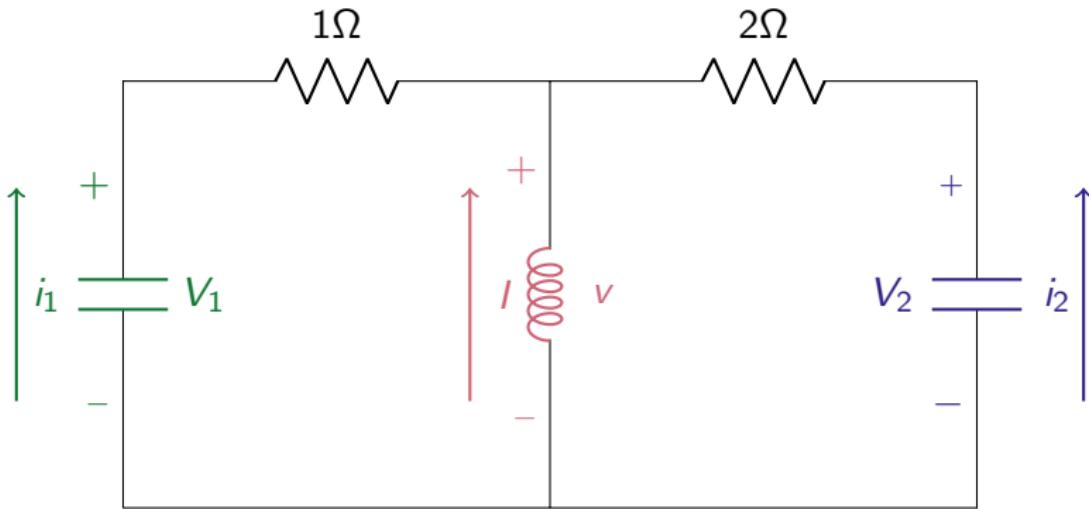
Capacitance: $\frac{1}{12}$
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Capacitances: $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{3}$

Inductance: $L = \frac{1}{3}$

Find $V_1(t)$, $V_2(t)$, and $I(t)$.



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Differential Equations:

$$\frac{dV_1}{dt} = -\frac{i_1}{C_1}$$

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$$\frac{dI}{dt} = -\frac{v}{L}$$

We use Kirkhoff's Laws to solve for i_1 , i_2 , and v in terms of V_1 , V_2 , and I .

Left Loop: $-V_1 + i_1 + v = 0$

Right Loop: $-V_2 + 2i_2 + v = 0$

Inductor (like a current source): $i_1 + i_2 = -I$

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$$\left[\begin{array}{ccc|c} i_1 & i_2 & v \\ \hline 1 & 0 & 1 & V_1 \\ 0 & 2 & 1 & V_2 \\ 1 & 1 & 0 & -I \end{array} \right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3}(V_1 - V_2 - 2I) \\ 0 & 1 & 0 & \frac{1}{3}(-V_1 + V_2 - I) \\ 0 & 0 & 1 & \frac{1}{3}(2V_1 + V_2 + 2I) \end{array} \right]$$

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$$\frac{dV_1}{dt} = -\frac{i_1}{C_1} = -6 \left(\frac{1}{3} \right) (V_1 - V_2 - 2I) = -2V_1 + 2V_2 + 4I$$

$$\frac{dV_2}{dt} = -\frac{i_2}{C_2} = -3 \left(\frac{1}{3} \right) (-V_1 + V_2 - I) = V_1 - V_2 + I$$

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In vector differential equation form:

$$\begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}' = \begin{bmatrix} -2 & 2 & 4 \\ 1 & -1 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}$$

Eigenvalues and Eigenvectors:

$$\lambda_1 = -2, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = \overline{\lambda_3} = \frac{-3 + 3\sqrt{3}i}{2}, \quad \mathbf{x}_2 = \overline{\mathbf{x}_3} = \begin{bmatrix} 4 \\ 1 - \sqrt{3}i \\ 2\sqrt{3}i \end{bmatrix}$$

Solutions all decay; some oscillate.

General Solution:

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ I(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix} + c_2 e^{-\frac{3}{2}t} \begin{bmatrix} 4 \cos\left(\frac{3\sqrt{3}}{2}t\right) \\ \cos\left(\frac{3\sqrt{3}}{2}t\right) + 3 \sin\left(\frac{3\sqrt{3}}{2}t\right) \\ -2\sqrt{3} \sin\left(\frac{3\sqrt{3}}{2}t\right) \end{bmatrix} \\ + c_3 e^{-\frac{3}{2}t} \begin{bmatrix} 4 \sin\left(\frac{3\sqrt{3}}{2}t\right) \\ \sin\left(\frac{3\sqrt{3}}{2}t\right) - 3 \cos\left(\frac{3\sqrt{3}}{2}t\right) \\ 2\sqrt{3} \cos\left(\frac{3\sqrt{3}}{2}t\right) \end{bmatrix}$$