Outline

Week 12: Vector differential equations

Course Notes: 6.3

Goals: be able to solve a linear system of differential equations; find characteristics of electrical networks involving inductors and capacitors using methods learned this term.

Notes

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C

Notes

Differential Equations

Example: a radioactive substance decays at a rate of 2% of its mass every year.

Notes

Systems of Linear Differential Equations

$$\begin{aligned} y_1'(t) &= a y_1(t) + b y_2(t) \\ y_2'(t) &= c y_1(t) + d y_2(t) \end{aligned}$$

$$\mathbf{y}' := \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{y} := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\mathbf{y}' = A\mathbf{y}$$

Notes

Guessing Solutions: Eigenvectors

Differential Equation:

$$\mathbf{y}' = A\mathbf{y}$$

Let's take a guess from our previous examples: what if

$$\mathbf{y} = e^{\lambda t}\mathbf{x}$$

for some constant λ and some constant vector $\mathbf{x}$?

Notes

Systems of Linear Differential Equations: Adding Solutions

Adding Solutions

Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are both solutions to the system of differential equations $A\mathbf{y} = \mathbf{y}'$.

Notes

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

General Question: Is there a solution to $\mathbf{y}' = A\mathbf{y}$ that also has $\mathbf{y}(0) = \mathbf{y}_0$, for some constant vector $\mathbf{y}_0$?

Notes

Example

Find the solution to the system of linear differential equations

$$\begin{array}{rcrcrcr} y_1'(t) & = & y_1(t) & + & 4y_2(t) & + & 5y_3(t) \\ y_2'(t) & = & & & 2y_2(t) & + & 6y_3(t) \\ y_3'(t) & = & & & & & 3y_3(t) \end{array}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 0 \\ 11 \\ 2 \end{bmatrix}$$

Notes

The form of the solution will be:

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

That is:

$$\mathbf{y}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$$

To find the constants c_1, c_2, c_3 we solve:

$$\begin{bmatrix} 0 \\ 11 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$$

So $c_1 = -25$, $c_2 = -1$, and $c_3 = 1$. Our solution is:

$$\mathbf{y}(t) = -25e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1e^{2t} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -25e^{t} - 4e^{2t} + 29e^{3t} \\ -e^{2t} + 12e^{3t} \\ 2e^{3t} \end{bmatrix}}$$

Notes

Example

Find the solution to the system of linear differential equations

$$\begin{aligned} y_1'(t) &= y_1(t) \\ y_2'(t) &= 3y_1(t) - y_2(t) \end{aligned}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Notes

The form of the solution will be:

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

That is:

$$\mathbf{y}(t) = c_1 e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To find the constants c_1, c_2 we solve:

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $c_1 = 2$, $c_2 = -5$. Our solution is:

$$\mathbf{y}(t) = 2e^t \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 5e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4e^t \\ 6e^t - 5e^{-t} \end{bmatrix}}$$

Notes

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients
2. Find eigenvalues and corresponding eigenvectors
3. The general solution is $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots c_n e^{\lambda_n t}\mathbf{x_n}$
4. Find the values of c_i that fit the initial conditions. That gives you the particular solution.

Notes

End Behaviour

$\lambda = 1$	$ce^{t}\mathbf{x}$	$\xrightarrow{t\to\infty} \pm\infty$
$\lambda = -1$	$ce^{-t}\mathbf{x}$	
$\lambda = 0$	$c\mathbf{x}$	$\xrightarrow{t\to\infty} c\mathbf{x}$

Notes

Complex Eigenvalues

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Eigenvectors: $\mathbf{x_1} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

General solution: $\mathbf{y}(t) = c_1 e^{4it}\mathbf{x}_1 + c_2 e^{-4it}\mathbf{x}_2$
for some constants c_1 and c_2.

Particular solution: $\mathbf{y}(t) = (3+2i)e^{4it}\mathbf{x}_1 + (3-2i)e^{-4it}\mathbf{x}_2$

Notes

Complex Eigenvalues: Particular Solution

$$\begin{aligned}
\mathbf{y}(t) &= (3+2i)e^{4it}\mathbf{x}_1 + (3-2i)e^{-4it}\mathbf{x}_2 \\
&= (3+2i)[\cos(4t) + i\sin(4t)]\mathbf{x}_1 + (3-2i)[\cos(-4t) + i\sin(-4t)]\mathbf{x}_2 \\
&= (3+2i)[\cos(4t) + i\sin(4t)]\mathbf{x}_1 + (3-2i)[\cos(4t) - i\sin(4t)]\mathbf{x}_2 \\
&= (3+2i)[\cos(4t) + i\sin(4t)]\begin{bmatrix} i \\ 2 \end{bmatrix} + (3-2i)[\cos(4t) - i\sin(4t)]\begin{bmatrix} i \\ 2 \end{bmatrix} \\
&= \cdots \\
&= \begin{bmatrix} -4\cos(4t) - 6\sin(4t) \\ 12\cos(4t) - 8\sin(4t) \end{bmatrix}
\end{aligned}$$

Notes

Complex Eigenvalues: Closer Look

Suppose $\lambda_1 = \overline{\lambda_2}$ and $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

Then $e^{\lambda_1 t}\mathbf{x}_1 = \overline{e^{\lambda_2 t}\mathbf{x}_2}$

$$\begin{aligned} c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 &= c_1(f+gi) + c_2(f-gi) \\ &= (c_1+c_2)f + i(c_1-c_2)g \\ &= af + bg \\ &= a\cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b\cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1) \end{aligned}$$

where a and b are arbitrary constants

Notes

Complex Eigenvalues: Closer Look

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix}\mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Notes

Shorcut

Suppose we're solving $\mathbf{y}' = A\mathbf{y}$, and A has a complex pair of eigenvalues and eigenvectors $\lambda_1 = \overline{\lambda_2}$, $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

To find the solutions corresponding to these eigenvalues and eigenvectors, $c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ is equivalent to $a\cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b\cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1)$.
That is:

1. Choose a single solution, like $e^{\lambda_1 t}x_1$
2. Separate it into its real and imaginary part
3. The general solution is any linear combination of the real and imaginary part

Notes

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Notes

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & \frac{1}{4} \\ -5 & -2 \end{bmatrix}$$

Notes

End Behaviour

$\lambda = 1$	$ce^t\mathbf{x}$	$\xrightarrow{t\to\infty} \pm\infty$	if $c \neq 0$
$\lambda = -1$	$ce^{-t}\mathbf{x}$	$\xrightarrow{t\to\infty} 0$	
$\lambda = 0$	$c\mathbf{x}$	$\xrightarrow{t\to\infty} c\mathbf{x}$	if $c \neq 0$

$\lambda = i$	$c(\cos t + i \sin t)\mathbf{x}$	oscillating
$\lambda = 1 + i$	$ce^t(\cos t + i \sin t)\mathbf{x}$	oscillating, growing
$\lambda = -1 + i$	$ce^{-t}(\cos t + i \sin t)\mathbf{x}$	oscillating, decaying

Notes

Bigger Matrices

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}$$

Notes

Bigger Matrices

$\lambda_1 = 0$ $\lambda_2 = 1 + i$ $\lambda_3 = 1 - i$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

Notes

$$\begin{aligned} y_1'(t) &= 3y_1(t) + 0y_2(t) + 0y_3(t) \\ y_2'(t) &= 0y_1(t) + 2y_2(t) - 4y_3(t) \\ y_3'(t) &= 0y_1(t) + 1y_2(t) + 2y_3(t) \end{aligned}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \lambda_1 = 3,\ \lambda_2 = 2 + 2i,\ \lambda_3 = 2 - 2i$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ -i \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 2 \\ i \end{bmatrix}$$

Notes