Outline

Week 12: Vector differential equations

Course Notes: 6.3

Goals: be able to solve a linear system of differential equations; find characteristics of electrical networks involving inductors and capacitors using methods learned this term.

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C

Example: a population's growth rate is 0.3 times the number of individuals in that population per year.

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C

Example: a population's growth rate is 0.3 times the number of individuals in that population per year.

$$y'(t) = 0.3y(t)$$

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C

Example: a population's growth rate is 0.3 times the number of individuals in that population per year.

$$y'(t) = 0.3y(t)$$

$$y(t) = Ce^{0.3t} \qquad \text{for some constant } C$$

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \qquad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C
Example: a population's growth rate is 0.3 times the number of individuals in that population per year.

$$y'(t) = 0.3y(t)$$

$$y(t) = Ce^{0.3t} \qquad \text{for some constant } C$$

At year $t = 0$, there are 100 individuals.

Differential Equations

We're going to doing this in a linear-systems context soon.

$$y'(t) = \lambda y(t), \quad \lambda \text{ constant}$$

Solutions: $y(t) = Ce^{\lambda t}$, constant C
Example: a population's growth rate is 0.3 times the number of individuals in that population per year.

$$y'(t) = 0.3y(t)$$

$$y(t) = Ce^{0.3t} \quad \text{for some constant } C$$

At year $t = 0$, there are 100 individuals.

$$y(t) = 100e^{0.3t}$$

Differential Equations

Example: a radioactive substance decays at a rate of 2% of its mass every year.

Differential Equations

Example: a radioactive substance decays at a rate of 2% of its mass every year.

$$y'(t) = -0.02y(t)$$

Differential Equations

Example: a radioactive substance decays at a rate of 2% of its mass every year.

$$y'(t) = -0.02y(t)$$

$$y(t) = Ce^{-0.02t} \quad \text{where } C \text{ is the amount at } t = 0$$

Systems of Linear Differential Equations

$$\begin{aligned} y_1'(t) &= a y_1(t) + b y_2(t) \\ y_2'(t) &= c y_1(t) + d y_2(t) \end{aligned}$$

Systems of Linear Differential Equations

$$
\begin{aligned}
y_1'(t) &= a y_1(t) + b y_2(t)\\
y_2'(t) &= c y_1(t) + d y_2(t)
\end{aligned}
$$

Example: y_1 population of lynx, y_2 population of hares

Systems of Linear Differential Equations

$$\begin{aligned} y_1'(t) &= a y_1(t) + b y_2(t) \\ y_2'(t) &= c y_1(t) + d y_2(t) \end{aligned}$$

Example: y_1 population of lynx, y_2 population of hares

$$\mathbf{y}' := \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{y} := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\mathbf{y}' = A\mathbf{y}$$

Systems of Linear Differential Equations

$$\begin{aligned} y_1'(t) &= a y_1(t) + b y_2(t) \\ y_2'(t) &= c y_1(t) + d y_2(t) \end{aligned}$$

Example: y_1 population of lynx, y_2 population of hares

$$\mathbf{y}' := \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{y} := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\mathbf{y}' = A\mathbf{y}$$

Note: there isn't something weird going on with "differentiating a vector." We're just differentiating each (totally standard) equation inside the vector.

Guessing Solutions: Eigenvectors

Differential Equation:

$$\mathbf{y}' = A\mathbf{y}$$

Let's take a guess from our previous examples: what if

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \mathbf{y} = e^{\lambda t}\mathbf{x} = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix}$$

for some constant λ and some constant vector $\mathbf{x}$?

Guessing Solutions: Eigenvectors

Differential Equation:

$$\mathbf{y}' = A\mathbf{y}$$

Let's take a guess from our previous examples: what if

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \mathbf{y} = e^{\lambda t}\mathbf{x} = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix}$$

for some constant λ and some constant vector $\mathbf{x}$?

Then: $\mathbf{y}' = \lambda e^{\lambda t}\mathbf{x}$

So, if $\mathbf{y}' = A\mathbf{y}$: $\lambda e^{\lambda t}\mathbf{x} = A(e^{\lambda t}\mathbf{x})$

Hence: $\lambda\mathbf{x} = A\mathbf{x}$

so λ and $\mathbf{x}$ are an eigenvalue/eigenvector pair of A.

Guessing Solutions: Eigenvectors

Differential Equation:

$$\mathbf{y}' = A\mathbf{y}$$

Let's take a guess from our previous examples: what if

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \mathbf{y} = e^{\lambda t}\mathbf{x} = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix}$$

for some constant λ and some constant vector $\mathbf{x}$?

We've successfully guessed a solution!

$$\mathbf{y} = e^{\lambda t}\mathbf{x}$$

where λ, $\mathbf{x}$ are an eigenvalue/eigenvector pair of A

Systems of Linear Differential Equations: Adding Solutions

Adding Solutions

Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are both solutions to the system of differential equations $A\mathbf{y} = \mathbf{y}'$.

Systems of Linear Differential Equations: Adding Solutions

Adding Solutions

Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are both solutions to the system of differential equations $A\mathbf{y} = \mathbf{y}'$. Then $(\mathbf{y}_1 + \mathbf{y}_2)$ is *also* a solution.

Systems of Linear Differential Equations: Adding Solutions

Adding Solutions

Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are both solutions to the system of differential equations $A\mathbf{y} = \mathbf{y}'$. Then $(\mathbf{y}_1 + \mathbf{y}_2)$ is *also* a solution.

Further, $(c_1\mathbf{y}_1 + c_2\mathbf{y}_2)$ is *also* a solution, for any constants c_1 and c_2.
(Home exercise: prove this is true!)

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Example: $\mathbf{y}' = \mathbf{I}\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^2$

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Example: $\mathbf{y}' = \mathbf{I}\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^2$

$\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Example: $\mathbf{y}' = \mathbf{I}\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^2$

$\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\mathbf{y}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

General Question: Is there a solution to $\mathbf{y}' = A\mathbf{y}$ that also has $\mathbf{y}(0) = \mathbf{y}_0$, for some constant vector $\mathbf{y}_0$?

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

General Question: Is there a solution to $\mathbf{y}' = A\mathbf{y}$ that also has $\mathbf{y}(0) = \mathbf{y}_0$, for some constant vector $\mathbf{y}_0$? Suppose it has the form above:

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k \;{}^{?}\!=^{?}\; \mathbf{y}_0$$

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

General Question: Is there a solution to $\mathbf{y}' = A\mathbf{y}$ that also has $\mathbf{y}(0) = \mathbf{y}_0$, for some constant vector $\mathbf{y}_0$? Suppose it has the form above:

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k \overset{?}{=} \mathbf{y}_0$$

If the eigenvectors of A form a *basis* then there is always exactly one solution to $\mathbf{y}' = A\mathbf{y}$ with any desired initial condition $\mathbf{y}(0) = \mathbf{y}_0$, and it has the form given above.

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \dots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Example: $\mathbf{y}' = \mathbf{I}\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^2$

$\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \dots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Example: $\mathbf{y}' = \mathbf{I}\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^2$

$\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Suppose we have initial conditions $\mathbf{y}(0) = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$.

Solutions to Systems of Linear Differential Equations

Theorem

Suppose A is an n-by-n matrix with eigenvalues and vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

is a solution to the equation $\mathbf{y}' = A\mathbf{y}$.

Example: $\mathbf{y}' = \mathbf{I}\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^2$

$\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Suppose we have initial conditions $\mathbf{y}(0) = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$.

$$\mathbf{y} = 7e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7e^t \\ -3e^t \end{bmatrix}.$$

Example

Find the solution to the system of linear differential equations

$$\begin{array}{rcrcrcr} y_1'(t) & = & y_1(t) & + & 4y_2(t) & + & 5y_3(t) \\ y_2'(t) & = & & & 2y_2(t) & + & 6y_3(t) \\ y_3'(t) & = & & & & & 3y_3(t) \end{array}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 0 \\ 11 \\ 2 \end{bmatrix}$$

Example

Find the solution to the system of linear differential equations

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) + 5y_3(t) \\ y_2'(t) &= 2y_2(t) + 6y_3(t) \\ y_3'(t) &= 3y_3(t) \end{aligned}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 0 \\ 11 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

solving $\mathbf{y}' = A\mathbf{y}$

Example

Find the solution to the system of linear differential equations

$$\begin{array}{rclcrcr} y_1'(t) & = & y_1(t) & + & 4y_2(t) & + & 5y_3(t) \\ y_2'(t) & = & & & 2y_2(t) & + & 6y_3(t) \\ y_3'(t) & = & & & & & 3y_3(t) \end{array}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 0 \\ 11 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

solving $\mathbf{y}' = A\mathbf{y}$

$\lambda_1 = 1,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $\lambda_3 = 3,\ \mathbf{x}_3 = \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$

The form of the solution will be:

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

That is:

$$\mathbf{y}(t) = c_1 e^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}4\\1\\0\end{bmatrix} + c_3 e^{3t}\begin{bmatrix}29\\12\\2\end{bmatrix}$$

To find the constants c_1, c_2, c_3 we solve:

$$\begin{bmatrix}0\\11\\2\end{bmatrix} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}4\\1\\0\end{bmatrix} + c_3\begin{bmatrix}29\\12\\2\end{bmatrix}$$

So $c_1 = -25$, $c_2 = -1$, and $c_3 = 1$. Our solution is:

$$\mathbf{y}(t) = -25e^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} - 1e^{2t}\begin{bmatrix}4\\1\\0\end{bmatrix} + e^{3t}\begin{bmatrix}29\\12\\2\end{bmatrix} = \boxed{\begin{bmatrix}-25e^{t} - 4e^{2t} + 29e^{3t}\\ -e^{2t} + 12e^{3t}\\ 2e^{3t}\end{bmatrix}}$$

Example

Find the solution to the system of linear differential equations

$$\begin{aligned} y_1'(t) &= y_1(t) \\ y_2'(t) &= 3y_1(t) - y_2(t) \end{aligned}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Example

Find the solution to the system of linear differential equations

$$\begin{array}{rcrcr} y_1'(t) & = & y_1(t) & & \\ y_2'(t) & = & 3y_1(t) & - & y_2(t) \end{array}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

Example

Find the solution to the system of linear differential equations

$$\begin{aligned} y_1'(t) &= y_1(t) \\ y_2'(t) &= 3y_1(t) - y_2(t) \end{aligned}$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$ $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\lambda_2 = -1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The form of the solution will be:

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_k e^{\lambda_k t}\mathbf{x}_k$$

That is:

$$\mathbf{y}(t) = c_1 e^{t}\begin{bmatrix}2\\3\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\1\end{bmatrix}$$

To find the constants c_1, c_2 we solve:

$$\begin{bmatrix}4\\1\end{bmatrix} = c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}$$

So $c_1 = 2$, $c_2 = -5$. Our solution is:

$$\mathbf{y}(t) = 2e^{t}\begin{bmatrix}2\\3\end{bmatrix} - 5e^{-t}\begin{bmatrix}0\\1\end{bmatrix} = \boxed{\begin{bmatrix}4e^{t}\\6e^{t} - 5e^{-t}\end{bmatrix}}$$

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
2. Find eigenvalues and corresponding eigenvectors

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
2. Find eigenvalues and corresponding eigenvectors
   $\lambda_1 = 0,\ \begin{bmatrix} -2 & 1 \end{bmatrix}^T;\qquad \lambda_2 = 3,\ \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

Quick Recap

$$\begin{array}{rcl} y_1'(t) & = & y_1(t) + 2y_2(t) \\ y_2'(t) & = & y_1(t) + 2y_2(t) \end{array} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
2. Find eigenvalues and corresponding eigenvectors
   $\lambda_1 = 0,\ \begin{bmatrix} -2 & 1 \end{bmatrix}^T$; $\lambda_2 = 3,\ \begin{bmatrix} 1 & 1 \end{bmatrix}^T$
3. The general solution is $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots c_n e^{\lambda_n t}\mathbf{x_n}$

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
2. Find eigenvalues and corresponding eigenvectors
   $\lambda_1 = 0,\ \begin{bmatrix} -2 & 1 \end{bmatrix}^T;\qquad \lambda_2 = 3,\ \begin{bmatrix} 1 & 1 \end{bmatrix}^T$
3. The general solution is $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots c_n e^{\lambda_n t}\mathbf{x_n}$
   $$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
2. Find eigenvalues and corresponding eigenvectors
   $\lambda_1 = 0,\ \begin{bmatrix} -2 & 1 \end{bmatrix}^T;\quad \lambda_2 = 3,\ \begin{bmatrix} 1 & 1 \end{bmatrix}^T$
3. The general solution is $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots c_n e^{\lambda_n t}\mathbf{x_n}$
   $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
4. Find the values of c_i that fit the initial conditions. That gives you the particular solution.

Quick Recap

$$\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= y_1(t) + 2y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1. Create the matrix of coefficients $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
2. Find eigenvalues and corresponding eigenvectors
   $\lambda_1 = 0, \begin{bmatrix} -2 & 1 \end{bmatrix}^T; \quad \lambda_2 = 3, \begin{bmatrix} 1 & 1 \end{bmatrix}^T$
3. The general solution is $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x_1} + c_2 e^{\lambda_2 t}\mathbf{x_2} + \cdots c_n e^{\lambda_n t}\mathbf{x_n}$
   $$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
4. Find the values of c_i that fit the initial conditions. That gives you the particular solution.
   $$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 2e^{3t} \\ 1 + 2e^{3t} \end{bmatrix}$$

More Practice

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

More Practice

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix};$$

More Practice

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}; \qquad \lambda_1 = 7,\ \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \lambda_2 = -1,\ \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

More Practice

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}; \quad \lambda_1 = 7,\ \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \lambda_2 = -1,\ \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

General solution:

$$\mathbf{y}(t) = c_1 e^{7t}\mathbf{x}_2 + c_2 e^{-t}\mathbf{x}_2$$

More Practice

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}; \quad \lambda_1 = 7,\ \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \lambda_2 = -1,\ \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

General solution:

$$\mathbf{y}(t) = c_1 e^{7t}\mathbf{x}_2 + c_2 e^{-t}\mathbf{x}_2$$

Particular solution:

$$\mathbf{y}(t) = 4e^{-1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8e^{-t} \\ 4e^{-t} \end{bmatrix}$$

More Practice

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}; \quad \lambda_1 = 7,\ \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \lambda_2 = -1,\ \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

General solution:

$$\mathbf{y}(t) = c_1 e^{7t}\mathbf{x}_2 + c_2 e^{-t}\mathbf{x}_2$$

Particular solution:

$$\mathbf{y}(t) = 4e^{-1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8e^{-t} \\ 4e^{-t} \end{bmatrix}$$

Note: $\lim_{t\to\infty} \mathbf{y}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; if we'd had different initial conditions, these limits might have been infinite.

End Behaviour

Constants c are determined by *initial conditions*, i.e. $y(0)$; λ and $\mathbf{x}$ are an eigenvalue-eigenvector pair.

$\lambda = 1$	$ce^{t}\mathbf{x}$
$\lambda = -1$	$ce^{-t}\mathbf{x}$
$\lambda = 0$	$c\mathbf{x}$

End Behaviour

Constants c are determined by *initial conditions*, i.e. $y(0)$; λ and $\mathbf{x}$ are an eigenvalue-eigenvector pair.

$\lambda = 1$	$ce^{t}\mathbf{x}$	$\xrightarrow{t\to\infty} \pm\infty$	if $c \neq 0$
$\lambda = -1$	$ce^{-t}\mathbf{x}$		
$\lambda = 0$	$c\mathbf{x}$		

End Behaviour

Constants c are determined by *initial conditions*, i.e. $y(0)$; λ and $\mathbf{x}$ are an eigenvalue-eigenvector pair.

$\lambda = 1$	$ce^{t}\mathbf{x}$	$\xrightarrow{t\to\infty} \pm\infty$	if $c \neq 0$
$\lambda = -1$	$ce^{-t}\mathbf{x}$	$\xrightarrow{t\to\infty} 0$	
$\lambda = 0$	$c\mathbf{x}$		

End Behaviour

Constants c are determined by *initial conditions*, i.e. $y(0)$; λ and $\mathbf{x}$ are an eigenvalue-eigenvector pair.

$$
\begin{array}{llll}
\lambda = 1 & ce^{t}\mathbf{x} & \xrightarrow{t\to\infty} \pm\infty & \text{if } c \neq 0 \\
\lambda = -1 & ce^{-t}\mathbf{x} & \xrightarrow{t\to\infty} 0 & \\
\lambda = 0 & c\mathbf{x} & \xrightarrow{t\to\infty} c\mathbf{x} & \text{if } c \neq 0
\end{array}
$$

End Behaviour

Constants c are determined by *initial conditions*, i.e. $y(0)$; λ and $\mathbf{x}$ are an eigenvalue-eigenvector pair.

$\lambda = 1$	$ce^{t}\mathbf{x}$	$\xrightarrow{t\to\infty} \pm\infty$	if $c \neq 0$
$\lambda = -1$	$ce^{-t}\mathbf{x}$	$\xrightarrow{t\to\infty} 0$	
$\lambda = 0$	$c\mathbf{x}$	$\xrightarrow{t\to\infty} c\mathbf{x}$	if $c \neq 0$

Positive real eigenvalues lead to solutions that can diverge to $\pm\infty$ (depending on initial conditions);
Negative real eigenvalues lead to solutions that can converge to 0 (depending on initial conditions);
An eigenvalue of zero leads to solutions that can converge to a nonzero constant (depending on initial conditions);

Visualizing End Behaviour

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \implies \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1e^{7t} - 2c_2e^{-t} \\ 3c_1e^{7t} + c_2e^{-t} \end{bmatrix}$$

Visualizing End Behaviour

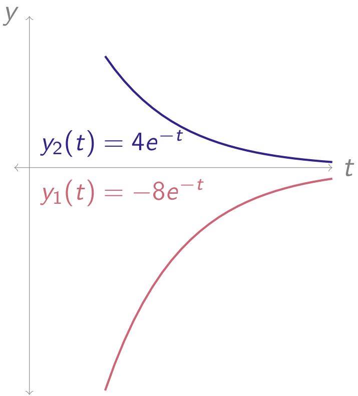
$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \implies \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1e^{7t} - 2c_2e^{-t} \\ 3c_1e^{7t} + c_2e^{-t} \end{bmatrix}$$

If $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$, then $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -8e^{-t} \\ 4e^{-t} \end{bmatrix}$

Visualizing End Behaviour

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \implies \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1e^{7t} - 2c_2e^{-t} \\ 3c_1e^{7t} + c_2e^{-t} \end{bmatrix}$$

If $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$, then $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -8e^{-t} \\ 4e^{-t} \end{bmatrix}$

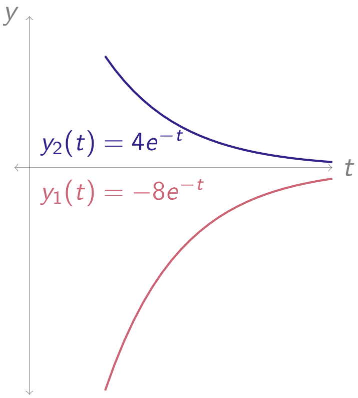


Visualizing End Behaviour

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \implies \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1e^{7t} - 2c_2e^{-t} \\ 3c_1e^{7t} + c_2e^{-t} \end{bmatrix}$$

If $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$, then $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -8e^{-t} \\ 4e^{-t} \end{bmatrix}$

If $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$, then $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 6e^{7t} - 2e^{-t} \\ 9e^{7t} + e^{-t} \end{bmatrix}$

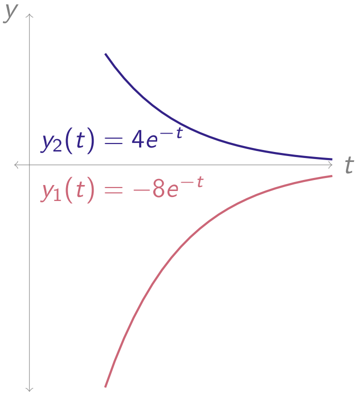


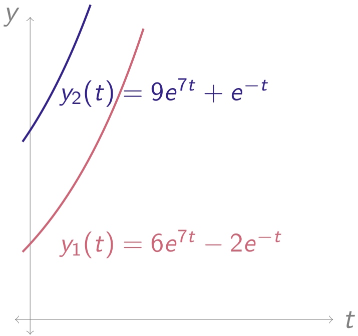
Visualizing End Behaviour

$$\begin{aligned} y_1'(t) &= y_1(t) + 4y_2(t) \\ y_2'(t) &= 3y_1(t) + 5y_2(t) \end{aligned} \implies \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1e^{7t} - 2c_2e^{-t} \\ 3c_1e^{7t} + c_2e^{-t} \end{bmatrix}$$

If $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$, then $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -8e^{-t} \\ 4e^{-t} \end{bmatrix}$

If $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$, then $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 6e^{7t} - 2e^{-t} \\ 9e^{7t} + e^{-t} \end{bmatrix}$





Complex Eigenvalues

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Complex Eigenvalues

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Complex Eigenvalues

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Eigenvectors: $\mathbf{x_1} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

Complex Eigenvalues

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Eigenvectors: $\mathbf{x_1} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

General solution: $\mathbf{y}(t) = c_1 e^{4it}\mathbf{x}_1 + c_2 e^{-4it}\mathbf{x}_2$
for some constants c_1 and c_2.

Complex Eigenvalues

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Eigenvectors: $\mathbf{x_1} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

General solution: $\mathbf{y}(t) = c_1 e^{4it}\mathbf{x}_1 + c_2 e^{-4it}\mathbf{x}_2$
for some constants c_1 and c_2.

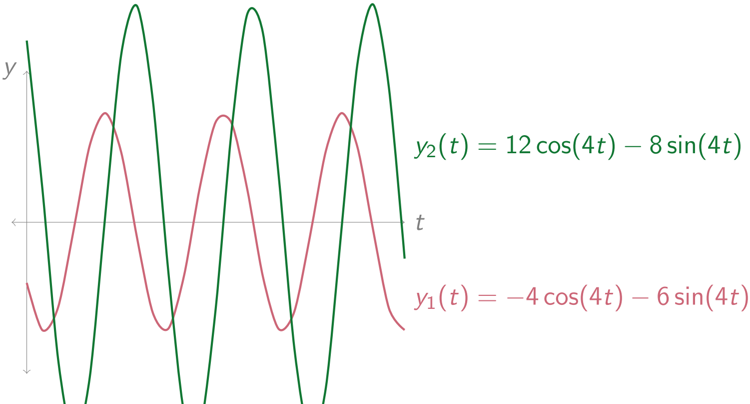
Particular solution: $\mathbf{y}(t) = (3+2i)e^{4it}\mathbf{x}_1 + (3-2i)e^{-4it}\mathbf{x}_2$

Complex Eigenvalues: Particular Solution

$$
\begin{aligned}
\mathbf{y}(t) &= (3+2i)e^{4it}\mathbf{x}_1 + (3-2i)e^{-4it}\mathbf{x}_2 \\
&= (3+2i)[\cos(4t) + i\sin(4t)]\mathbf{x}_1 + (3-2i)[\cos(-4t) + i\sin(-4t)]\mathbf{x}_2 \\
&= (3+2i)[\cos(4t) + i\sin(4t)]\mathbf{x}_1 + (3-2i)[\cos(4t) - i\sin(4t)]\mathbf{x}_2 \\
&= (3+2i)[\cos(4t) + i\sin(4t)]\begin{bmatrix} i \\ 2 \end{bmatrix} + (3-2i)[\cos(4t) - i\sin(4t)]\begin{bmatrix} i \\ 2 \end{bmatrix} \\
&= \cdots \\
&= \begin{bmatrix} -4\cos(4t) - 6\sin(4t) \\ 12\cos(4t) - 8\sin(4t) \end{bmatrix}
\end{aligned}
$$

Complex Eigenvalues: Particular Solution

$$\mathbf{y}(t) = (3+2i)e^{4it}\mathbf{x}_1 + (3-2i)e^{-4it}\mathbf{x}_2$$
$$= \begin{bmatrix} -4\cos(4t) - 6\sin(4t) \\ 12\cos(4t) - 8\sin(4t) \end{bmatrix}$$



Complex Eigenvalues: Closer Look

You should be able to follow this explanation, but you don't have to memorize it

Suppose $\lambda_1 = \overline{\lambda_2}$ and $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

Complex Eigenvalues: Closer Look

You should be able to follow this explanation, but you don't have to memorize it

Suppose $\lambda_1 = \overline{\lambda_2}$ and $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

Then $e^{\lambda_1 t}\mathbf{x_1} = \overline{e^{\lambda_2 t}\mathbf{x}_2}$.

Complex Eigenvalues: Closer Look

You should be able to follow this explanation, but you don't have to memorize it

Suppose $\lambda_1 = \overline{\lambda_2}$ and $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

Then $e^{\lambda_1 t}\mathbf{x_1} = \overline{e^{\lambda_2 t}\mathbf{x}_2}$.
Let $f = Re(e^{\lambda_1 t}\mathbf{x_1})$ and $g = Im(e^{\lambda_1 t}\mathbf{x_1})$.
Example: $Re\begin{bmatrix} a+bi \\ c+di \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $Im\begin{bmatrix} a+bi \\ c+di \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$.

Complex Eigenvalues: Closer Look

You should be able to follow this explanation, but you don't have to memorize it

Suppose $\lambda_1 = \overline{\lambda_2}$ and $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

Then $e^{\lambda_1 t}\mathbf{x_1} = \overline{e^{\lambda_2 t}\mathbf{x}_2}$.
Let $f = Re(e^{\lambda_1 t}\mathbf{x_1})$ and $g = Im(e^{\lambda_1 t}\mathbf{x_1})$.
Example: $Re\begin{bmatrix} a+bi \\ c+di \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $Im\begin{bmatrix} a+bi \\ c+di \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$.

$$\begin{aligned} c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 &= c_1(f+gi) + c_2(f-gi) \\ &= (c_1+c_2)f + i(c_1-c_2)g \\ &= af + bg \\ &= a \cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b \cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1) \end{aligned}$$

where a and b are arbitrary constants, possibly complex

Complex Eigenvalues: Closer Look

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Complex Eigenvalues: Closer Look

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Complex Eigenvalues: Closer Look

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Eigenvectors: $\mathbf{x_1} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

Complex Eigenvalues: Closer Look

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \mathbf{y}(t) \qquad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 4i$, $\lambda_2 = -4i$

Eigenvectors: $\mathbf{x_1} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$

General solution:

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{4it}\mathbf{x}_1 + c_2 e^{-4it}\mathbf{x}_2 \\ &= c_1 e^{4it}\mathbf{x}_1 + c_2 \overline{e^{4it}\mathbf{x}_1} \\ &= a \cdot \mathrm{Re}(e^{4it}\mathbf{x}_1) + b \cdot \mathrm{Im}(e^{4it}\mathbf{x}_1) \\ &= a \begin{bmatrix} -2\sin(4t) \\ 4\cos(4t) \end{bmatrix} + b \begin{bmatrix} -2\cos(4t) \\ -4\sin(4t) \end{bmatrix} \end{aligned}$$

where a and b are arbitrary constants

Shorcut

Suppose we're solving $\mathbf{y}' = A\mathbf{y}$, and A has a complex pair of eigenvalues and eigenvectors $\lambda_1 = \overline{\lambda_2}$, $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

To find the solutions corresponding to these eigenvalues and eigenvectors, $c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ is equivalent to $a \cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b \cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1)$.
That is:

Shorcut

Suppose we're solving $\mathbf{y}' = A\mathbf{y}$, and A has a complex pair of eigenvalues and eigenvectors $\lambda_1 = \overline{\lambda_2}$, $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

To find the solutions corresponding to these eigenvalues and eigenvectors, $c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ is equivalent to $a \cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b \cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1)$.
That is:

1. Choose a single solution, like $e^{\lambda_1 t}x_1$

Shorcut

Suppose we're solving $\mathbf{y}' = A\mathbf{y}$, and A has a complex pair of eigenvalues and eigenvectors $\lambda_1 = \overline{\lambda_2}$, $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

To find the solutions corresponding to these eigenvalues and eigenvectors, $c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ is equivalent to $a \cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b \cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1)$.
That is:

1. Choose a single solution, like $e^{\lambda_1 t} x_1$
2. Separate it into its real and imaginary part

Shorcut

Suppose we're solving $\mathbf{y}' = A\mathbf{y}$, and A has a complex pair of eigenvalues and eigenvectors $\lambda_1 = \overline{\lambda_2}$, $\mathbf{x}_1 = \overline{\mathbf{x}_2}$.

To find the solutions corresponding to these eigenvalues and eigenvectors, $c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ is equivalent to $a \cdot \mathrm{Re}(e^{\lambda_1 t}\mathbf{x}_1) + b \cdot \mathrm{Im}(e^{\lambda_1 t}\mathbf{x}_1)$.
That is:

1. Choose a single solution, like $e^{\lambda_1 t} x_1$
2. Separate it into its real and imaginary part
3. The general solution is any linear combination of the real and imaginary part

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$,

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ One solution:

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ One solution:

$$e^{(1+i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^t e^{it} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= e^t(\cos(t) + i\sin(t)) \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} -e^t \cos t \\ e^t \sin t \end{bmatrix}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ One solution:

$$e^{(1+i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} -e^t \cos t \\ e^t \sin t \end{bmatrix}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; \qquad \text{solve } \mathbf{y}' = A\mathbf{y}$$

Eigenvalues: $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ One solution:

$$e^{(1+i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} -e^t \cos t \\ e^t \sin t \end{bmatrix}$$

General Solution:

$$= c_1 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + c_2 \begin{bmatrix} -e^t \cos t \\ e^t \sin t \end{bmatrix}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & \frac{1}{4} \\ -5 & -2 \end{bmatrix}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & \frac{1}{4} \\ -5 & -2 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = -1 + \frac{1}{2}i, \qquad \lambda_2 = -1 - \frac{1}{2}i$

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & \frac{1}{4} \\ -5 & -2 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = -1 + \frac{1}{2}i,\qquad \lambda_2 = -1 - \frac{1}{2}i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} 2+i \\ -10 \end{bmatrix},\qquad \mathbf{x}_2 = \begin{bmatrix} 2-i \\ -10 \end{bmatrix}$

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & \frac{1}{4} \\ -5 & -2 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = -1 + \frac{1}{2}i$, $\qquad \lambda_2 = -1 - \frac{1}{2}i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} 2+i \\ -10 \end{bmatrix}$, $\qquad \mathbf{x}_2 = \begin{bmatrix} 2-i \\ -10 \end{bmatrix}$

Choosing one:

$$\begin{aligned} e^{\lambda_1 t}\mathbf{x}_1 &= e^{(-1+\frac{1}{2}i)t}\mathbf{x}_1 = e^{-t}e^{it/2}\mathbf{x}_1 = e^{-t}(\cos(t/2) + i\sin(t/2))\mathbf{x}_1 \\ &= e^{-t}(\cos(t/2) + i\sin(t/2)) \begin{bmatrix} 2+i \\ -10 \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} 2\cos(t/2) - \sin(t/2) \\ -10\cos(t/2) \end{bmatrix} + ie^{-t} \begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -10\sin(t/2) \end{bmatrix} \end{aligned}$$

Complex Eigenvalues

$$A = \begin{bmatrix} 0 & \frac{1}{4} \\ -5 & -2 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = -1 + \frac{1}{2}i, \qquad \lambda_2 = -1 - \frac{1}{2}i$

Eigenvectors: $\mathbf{x}_1 = \begin{bmatrix} 2+i \\ -10 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 2-i \\ -10 \end{bmatrix}$

Choosing one:

$$e^{\lambda_1 t}\mathbf{x}_1 = e^{-t}\begin{bmatrix} 2\cos(t/2) - \sin(t/2) \\ -10\cos(t/2) \end{bmatrix} + ie^{-t}\begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -10\sin(t/2) \end{bmatrix}$$

General solution:

$$\frac{c_1}{e^t}\begin{bmatrix} 2\cos(t/2) - \sin(t/2) \\ -10\cos(t/2) \end{bmatrix} + \frac{c_2}{e^t}\begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -10\sin(t/2) \end{bmatrix}$$

End Behaviour

$\lambda = 1$	$ce^{t}\mathbf{x}$	$\xrightarrow{t\to\infty} \pm\infty$	if $c \neq 0$
$\lambda = -1$	$ce^{-t}\mathbf{x}$	$\xrightarrow{t\to\infty} 0$	
$\lambda = 0$	$c\mathbf{x}$	$\xrightarrow{t\to\infty} c\mathbf{x}$	if $c \neq 0$

$\lambda = i$	$c(\cos t + i\sin t)\mathbf{x}$	oscillating
$\lambda = 1 + i$	$ce^{t}(\cos t + i\sin t)\mathbf{x}$	oscillating, growing
$\lambda = -1 + i$	$ce^{-t}(\cos t + i\sin t)\mathbf{x}$	oscillating, decaying

Bigger Matrices

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}$$

Bigger Matrices

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}$$

$$\lambda_1 = 0 \qquad \lambda_2 = 1 + i \qquad \lambda_3 = 1 - i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

Bigger Matrices

$$\lambda_1 = 0 \qquad \lambda_2 = 1 + i \qquad \lambda_3 = 1 - i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

General Solution:

$$c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$$

Bigger Matrices

$$\lambda_1 = 0 \qquad \lambda_2 = 1+i \qquad \lambda_3 = 1-i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

General Solution:

$$c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$$

Bigger Matrices

$$\lambda_1 = 0 \qquad \lambda_2 = 1 + i \qquad \lambda_3 = 1 - i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

General Solution:

$$c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 + c_3 e^{\lambda_3 t} \mathbf{x}_3$$

$$e^{\lambda_1 t} \mathbf{x}_1 = e^{0t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Bigger Matrices

$$\lambda_1 = 0 \qquad \lambda_2 = 1 + i \qquad \lambda_3 = 1 - i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

General Solution:

$$c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$$

Bigger Matrices

$$\lambda_1 = 0 \qquad \lambda_2 = 1 + i \qquad \lambda_3 = 1 - i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

General Solution:

$$c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$$

We get to use our shortcut, because $\mathbf{x_1}$ and $\mathbf{x_2}$ are complex conjugates of one another.

$$e^{\lambda_2 t}\mathbf{x}_2 = e^{(1+i)t}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = e^t e^{it}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$= e^t(\cos t + i \sin t)\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \cos t \\ -e^t \sin t \\ 0 \end{bmatrix} + i\begin{bmatrix} e^t \sin t \\ e^t \cos t \\ 0 \end{bmatrix}$$

Bigger Matrices

$$\lambda_1 = 0 \qquad \lambda_2 = 1 + i \qquad \lambda_3 = 1 - i$$

$$\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

General Solution:

$$c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$$

$$\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^t \sin t \\ e^t \cos t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 e^t \cos t + c_3 e^t \sin t \\ -c_2 e^t \sin t + c_3 e^t \cos t \\ c_1 \end{bmatrix}$$

$$\begin{aligned}
y_1'(t) &= 3y_1(t) + 0y_2(t) + 0y_3(t) \\
y_2'(t) &= 0y_1(t) + 2y_2(t) - 4y_3(t) \\
y_3'(t) &= 0y_1(t) + 1y_2(t) + 2y_3(t)
\end{aligned}$$

$$\begin{aligned} y_1'(t) &= 3y_1(t) + 0y_2(t) + 0y_3(t) \\ y_2'(t) &= 0y_1(t) + 2y_2(t) - 4y_3(t) \\ y_3'(t) &= 0y_1(t) + 1y_2(t) + 2y_3(t) \end{aligned}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \lambda_1 = 3,\ \lambda_2 = 2+2i,\ \lambda_3 = 2-2i$$

$$\begin{aligned} y_1'(t) &= 3y_1(t) + 0y_2(t) + 0y_3(t) \\ y_2'(t) &= 0y_1(t) + 2y_2(t) - 4y_3(t) \\ y_3'(t) &= 0y_1(t) + 1y_2(t) + 2y_3(t) \end{aligned}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \lambda_1 = 3,\ \lambda_2 = 2+2i,\ \lambda_3 = 2-2i$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ -i \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 2 \\ i \end{bmatrix}$$

$$\begin{aligned} y_1'(t) &= 3y_1(t) + 0y_2(t) + 0y_3(t) \\ y_2'(t) &= 0y_1(t) + 2y_2(t) - 4y_3(t) \\ y_3'(t) &= 0y_1(t) + 1y_2(t) + 2y_3(t) \end{aligned}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \lambda_1 = 3,\ \lambda_2 = 2 + 2i,\ \lambda_3 = 2 - 2i$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ -i \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 2 \\ i \end{bmatrix}$$

General Solution:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2e^{2t}\cos(2t) \\ e^{2t}\sin(2t) \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2e^{2t}\sin(2t) \\ -e^{2t}\cos(2t) \end{bmatrix}$$