# Outline

Week 11: Eigenvalues and eigenvectors:
complex numbers and random walks

Course Notes: 6.2

Goals: More practice finding eigenvalues and eigenvectors; expanding these to the complex numbers; using them in the context of random walks.

Note: because these computations get pretty long, we will skip many of the repetitive parts in lecture. We'll focus on newer material, and leave it to you to review older content.

# Random Walks - Review

- $\mathbf{x}_n = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix}$ and $p_1 + p_2 + \cdots + p_k = 1$

  Probability vector, at time $n$

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- $\mathbf{x}_n = P^n \mathbf{x}_0$

# Random Walks and Eigenvalues

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|---|---|---|
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$$\lim_{n\to\infty} P^n\mathbf{x}_0 = \frac{1}{5}\mathbf{k}_2 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

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What if $\mathbf{x}_0$ were different?

# Eigenvalues of Probability Transition Matrices

**Theorem**

If $P$ is a transition matrix (non-negative entries with all columns summing to one) that in addition has all positive entries then $P$ has an eigenvalue 1 with a single eigenvector $k_1$ that can be chosen to be a probability vector. All other eigenvalues satisfy $|\lambda| < 1$ with eigenvectors with components that sum to zero. Thus,

$$\lim_{n \to \infty} x_n = k_1$$

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[Proof, of sorts]

In short: that last example was typical. As long as a probability matrix has no zeroes:

- Probability matrices have 1 as an eigenvalue
- There will be some equilibrium that the system will reach in the long run, regardless of initial state, corresponding to an eigenvector of 1.

# Equilibrium Probability

Which of these random walk models seems likely to have an equilibrium probability? Is it clear what would it be?

In any given day, your odds of dying are 1 in 1,000.

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Equilibrium probability: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; everybody dies.

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No equilibrium probability–depends on initial stages.

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Equilibrium probability: $\begin{bmatrix} .5 \\ .5 \end{bmatrix}$; long-term average.

# Equilibrium Probability

Find the equilibrium probability of the system.

In a relationship or not, by year.

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| single | .4 | .25 |
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| $\lambda_1 = 1$ | $\mathbf{k_1} = \begin{bmatrix} 1 \\ \frac{12}{5} \end{bmatrix}$ |
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For any $\mathbf{x}_0 = a_1\mathbf{k_1} + a_2\mathbf{k_2}$:

$$P^n\mathbf{x}_0 = a_2\mathbf{k_1} + a_2\left(\frac{3}{20}\right)^n\mathbf{k_2} \xrightarrow{n\to\infty} a_2\mathbf{k_1}$$

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$$\lim_{n\to\infty} P^n\mathbf{x}_0 = \begin{bmatrix} 5/17 \\ 12/17 \end{bmatrix} \text{ regardless of } \mathbf{x}_0$$

# Computation Practice

$$P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix}, \ a \in [0,1]$$

1. Find all eigenvalues of $P$, and an associated eigenvector to each.
2. Write $\mathbf{x}_0$ as a linear combination of eigenvectors of $P$.
3. Calculate $\mathbf{x}_n$, where $n$ is some positive integer.
4. Find the equilibrium probability of $P$.
   (If this is the only thing we want to find, we can skip all other steps, and simply find the eigenvector associated to eigenvalue 1, then scale it to be a probability vector.)

# 1. Find Eigenvalues and Eigenvectors

By our theorem, we know that 1 will be an eigenvalue. However, for the sake of practice, let's find them the old-fashioned way.

Eigenvalues of $P$ are precisely those scalars $\lambda$ such that $\det(P - \lambda I) = 0$. So we set the determinant equal to zero:

$$\det \begin{bmatrix} 1/3-\lambda & 1/2 \\ 2/3 & 1/2-\lambda \end{bmatrix} = \left(\frac{1}{3} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \lambda^2 - \frac{5}{6}\lambda - \frac{1}{6}$$

And find $\lambda_1 = 1$ (as expected) and $\lambda_2 = -\frac{1}{6}$.

To find the associated eigenvectors, we set $P\mathbf{x} = \lambda\mathbf{x}$. (Next Slide)

# 1. Find Eigenvalues and Eigenvectors

$\boxed{\lambda_1 = 1}$

$$\begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} -2/3 & 1/2 \\ 2/3 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions to this system are of the form $s \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ for some scalar $s$. Any vector of this form will do.

$\boxed{\lambda_2 = -\frac{1}{6}}$

$$\begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} x \\ y \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions to this system are of the form $s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for some scalar $s$. Any vector of this form will do.

# 2. Basis Vectors

To find $\mathbf{x}_0$ as a combination of eigenvectors, we have to CHOOSE our eigenvectors. I like integers, so I'll use $\mathbf{k_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{k_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Your vectors may be scalar multiples of these.

The equation we have to solve is:

$$\begin{bmatrix} a \\ 1-a \end{bmatrix} = x \begin{bmatrix} 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 1-a \end{bmatrix}$$

Since $a$ is a constant, we can solve this using an augmented matrix and row reduction.

$$\left[\begin{array}{cc|c} 3 & 1 & a \\ 4 & -1 & 1-a \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 3 & 1 & a \\ 7 & 0 & 1 \end{array}\right]$$

So $x = \frac{1}{7}$ and $y = a - \frac{3}{7}$. That is, $\mathbf{x}_0 = \frac{1}{7}\mathbf{k_1} + (a - \frac{3}{7})\mathbf{k_2}$.

# 3. Find $\mathbf{x}_n$

Recall $\mathbf{x}_n = P^n \mathbf{x}_0$. With our previous work, the answer is an easy calculation.

$$
\begin{aligned}
\mathbf{x}_n = P^n \mathbf{x}_0 &= P^n \left( \frac{1}{7}\mathbf{k_1} + \left( \frac{3}{7} - a \right) \mathbf{k_2} \right) \\
&= \frac{1}{7} P^n \mathbf{k_1} + \left( \frac{3}{7} - a \right) P^n \mathbf{k_2} \\
&= \frac{1}{7} \mathbf{k_1} + \left( \frac{3}{7} - a \right) \left( -\frac{1}{6} \right)^n \mathbf{k_2} \\
&= \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} + \begin{bmatrix} \left(\frac{3}{7} - a\right)(-1/6)^n \\ -\left(\frac{3}{7} - a\right)(-1/6)^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{7} + \left(\frac{3}{7} - a\right)(-1/6)^n \\ \frac{4}{7} - \left(\frac{3}{7} - a\right)(-1/6)^n \end{bmatrix}
\end{aligned}
$$

# 4. Find the Equilibrium Probability

Recall the equilibrium probability is $\lim_{n\to\infty} \mathbf{x}_n = \lim_{n\to\infty} P^n \mathbf{x}_0$. With our previous work, the answer is an easy calculation. Using an intermediate result from the last slide:

$$\begin{aligned}\lim_{n\to\infty} \mathbf{x}_n &= \lim_{n\to\infty} \left[\frac{1}{7}\mathbf{k_1} + \left(\frac{3}{7} - a\right)\left(-\frac{1}{6}\right)^n \mathbf{k_2}\right] \\ &= \frac{1}{7}\mathbf{k_1} \\ &= \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}\end{aligned}$$

ALTERNATELY, our theorem tells us that the equilibrium probability will always be an eigenvector associated with the eigenvalue $\lambda = 1$. Since our eigenvectors were of the form $s[3, 4]$, we can find the equilibrium probability by figuring out which value of $s$ gives us a vector whose entries sum to one; $s = 7$ is that scalar.

# Complex Eigenvalues: A Neat Trick

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\lambda_1 = i, \mathbf{k}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad \lambda_2 = -i, \mathbf{k}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

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If the entries of $A$ are all real, its eigenvalues and eigenvectors are complex conjugates of one another.

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$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} \quad \Leftrightarrow \quad \overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}} \quad \Leftrightarrow \quad A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

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(If $\lambda$ and $\mathbf{x}$ are real-valued, this statement is true but not interesting, because $\overline{\lambda} = \lambda$ and $\overline{\mathbf{x}} = \mathbf{x}$.)

# Complex Eigenvalues

Suppose $A$ is a 3-by-3 matrix with all real entries, whose eigenvectors form a basis of $\mathbb{R}^3$.

- If $A$ has eigenvalue-eigenvector pair $\lambda = (2+i)$, $\mathbf{k} = \left[\begin{smallmatrix} 3i+4 \\ 2i-9 \end{smallmatrix}\right]$, give another eigenvalue-eigenvector pair.

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- If $\lambda_1 = 1$ and $\lambda_2 = 5 + 4i$, what is $\lambda_3$?
- If $A$ has eigenvalue-eigenvector pair $\lambda = 5 + \sqrt{-2}$, $\mathbf{k} = \left[\begin{smallmatrix} 3 \\ 18+\sqrt{-3} \end{smallmatrix}\right]$, give another eigenvalue-eigenvector pair.

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Find all eigenvalues and eigenvectors of :

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$$\text{Roots:} \quad \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i$$

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$$\lambda_1 = -2 - 3i,$$

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$$\text{Roots:} \quad \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i$$

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# Eigenfact or Eigenfiction?

Let $A$ be a matrix with eigenvalue $\lambda$ and associated eigenvector $\mathbf{k}$.

True or False:

1. $2\mathbf{k}$ is an eigenvector of $A$, associated with $\lambda$.
2. It is possible that $\mathbf{k}$ is an eigenvalue of $A$ associated with a *different* eigenvalue (that is, other than $\lambda$).
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4. $\mathbf{k}$ might be the zero vector.
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6. If $A$ has only real entries, then $\lambda$ is real.
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# Explanations

1. Check the definition of eigenvalues and eigenvectors:

$$A\mathbf{k} = \lambda\mathbf{k} \implies 2A\mathbf{k} = 2\lambda\mathbf{k} \implies A(2\mathbf{k}) = \lambda(2\mathbf{k})$$

Indeed, any nonzero multiple of $\mathbf{k}$ is an eigenvector of $A$, associated with $\lambda$.

2. $\mathbf{k}$ is not the zero vector, so it has some nonzero entry, say $\mathbf{k} = [k_1, k_2, \ldots, k_n]^T$ and $k_i \neq 0$. If $\lambda$ and $\gamma$ are both eigenvalues associated with $\mathbf{k}$, then:

$$\lambda\mathbf{k} = A\mathbf{k} = \gamma\mathbf{k} \implies \lambda\mathbf{k} = \gamma\mathbf{k}$$

so, in particular, $\lambda k_i = \gamma k_i$; since $k_i \neq 0$, this implies $\lambda = \gamma$. That is, the two eigenvalues were actually the same.

# Explanations

3. Consider $A = I_2$ (the $2 \times 2$ identity matrix), $\lambda = 1$, $\mathbf{k}_1 = [1, 0]^T$ and $\mathbf{k}_2 = [0, 1]^T$.

4. By definition, eigenvectors are nonzero.

5. For example, $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $\lambda = 0$, and $\mathbf{k} = [10]^T$.

6. We've seen several examples of real-valued matrices with complex eigenvalues

# Explanations

7. If $A$ and $\mathbf{k}$ are real-valued, then $Ak$ has only real entries as well. Therefore, $\lambda\mathbf{k}$ has only real entries. If $\lambda$ isn't real, then the entries of $\lambda\mathbf{k}$ contain at least one nonzero entry, and this entry isn't real because it's a nonreal complex number multiplied by a nonzero real number. This is a contradiction.