Outline

Week 10: Eigenvalues and eigenvectors

Course Notes: 6.1

Goals: Understand how to find eigenvector/eigenvalue pairs, and use them to simplify calculations involving matrix powers.

Notes

This Will Look Strange: A preview of Why We Bother

Eigenvectors: we'll use a **very specific** phenomenon to make hard calculations easier. Here is a preview of what we can do.

For random walks, we often wanted to compute a high power of a transition matrix. We used Matlab for this.

$$\text{Let } P = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}$$

Fact 1: $P\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} = 1\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$ Fact 2: $P\begin{bmatrix} x \\ -x \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} x \\ -x \end{bmatrix}$

$P^n\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

$P^n\begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix} = P^n\left(\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix}\right) = P^n\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} + P^n\begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix} =$
$\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} + \left(-\frac{1}{4}\right)^n\begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} + (-1)^n\frac{1}{5\cdot 4^n} \\ \frac{2}{5} - (-1)^n\frac{1}{5\cdot 4^n} \end{bmatrix}$ No Matlab!

Notes

When Matrix Multiplication looks like Scalar Multiplication

$$\begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}\begin{bmatrix} x \\ -x \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} x \\ -x \end{bmatrix} \qquad \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}\begin{bmatrix} x \\ 2x/3 \end{bmatrix} = 1\begin{bmatrix} x \\ 2x/3 \end{bmatrix}$$

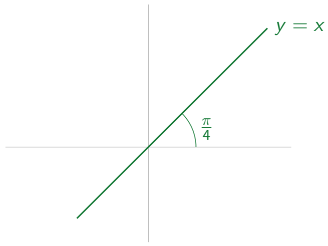
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

Recall:
Two vectors that are scalar multiples of one another are parallel.

If we interpret matrix multiplication as a **linear transformation**, we're looking for **a vector whose image is parallel to itself**.

Notes

Reflections, Revisited



$$Ref_{\frac{\pi}{4}} = \begin{bmatrix} \cos(2\frac{\pi}{4}) & \sin(2\frac{\pi}{4}) \\ \sin(2\frac{\pi}{4}) & -\cos(2\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Notes

Eigenvectors and Eigenvalues

Given a matrix A, a scalar λ, and a NONZERO vector $\mathbf{x}$ with

$$A\mathbf{x} = \lambda\mathbf{x}$$

we say $\mathbf{x}$ is an *eigenvector* of A with *eigenvalue* λ.

Notice we omit zero vectors! These are not particularly useful, and they behave differently from nonzero vectors with this property.

Notes

Rotation Matrix Eigenvalues

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Search for eigenvectors:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

e.g. $\theta = 135^\circ$

Notes

Finding Eigenvectors, Given Eigenvalues

Computation! We'll learn this in two stages. Given: 7 is an eigenvalue of the matrix $\begin{bmatrix}1&2&4\\2&4&1\\4&1&2\end{bmatrix}$.
What is an eigenvector associated to that eigenvalue?

An eigenvector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ with eigenvalue 7 would satisfy this equation:

$$\begin{bmatrix}1&2&4\\2&4&1\\4&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=7\begin{bmatrix}x\\y\\z\end{bmatrix}$$

In equation form:

$$\begin{array}{rcrcrcr} x&+&2y&+&4x&=&7x\\ 2x&+&4y&+&z&=&7y\\ 4x&+&y&+&2z&=&7z\end{array}$$

Notes

Finding Eigenvectors, Given Eigenvalues

In equation form:

$$\begin{array}{rcrcrcr} x&+&2y&+&4x&=&7x\\ 2x&+&4y&+&z&=&7y\\ 4x&+&y&+&2z&=&7z\end{array}$$

In better equation form:

$$\begin{array}{rcrcrcr} -6x&+&2y&+&4x&=&0\\ 2x&-&3y&+&z&=&0\\ 4x&+&y&-&5z&=&0\end{array}$$

Gaussian elimination on augmented matrix:

$$\left[\begin{array}{ccc|c}-6&2&4&0\\2&-3&1&0\\4&1&-5&0\end{array}\right]\rightarrow\rightarrow\rightarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

Notes

Finding Eigenvectors, Given Eigenvalues

Gaussian elimination on augmented matrix:

$$\left[\begin{array}{ccc|c}-6&2&4&0\\2&-3&1&0\\4&1&-5&0\end{array}\right]\rightarrow\rightarrow\rightarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

Solutions:

Notes

Finding Eigenvectors, Given Eigenvalues

The solutions to:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 7 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

are precisely the vectors:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Notes

Finding Eigenvectors from Eigenvalues

The matrix $\begin{bmatrix} 3 & 6 \\ 6 & -2 \end{bmatrix}$ has eigenvalues -6 and 7.
Find an eigenvector associated to each eigenvalue.

$\lambda = -6$, $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ $\qquad$ $\lambda = 7$, $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
(or any scalar multiple of those vectors)

Basis of $\mathbb{R}^2$: $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

Notes

Finding Eigenvectors from Eigenvalues

The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has eigenvalues -2 and 5.
Find an eigenvector associated to each eigenvalue.

Notes

From the 2015 final exam

The matrix below represents rotation in 3D about a line through the origin.

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

Find a vector in the direction of the line of rotation.

Notes

Generalized Eigenvector Finding

Given matrix A with eigenvalue λ, find an associated eigenvector $\mathbf{x}$

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

Eigenvectors $\mathbf{x}$ associated with eigenvalue λ are precisely the nonzero solutions to this homogeneous system.

So, we set up a homogeneous system of equations, where the coefficient matrix is formed by subtracting λ from every entry in the main diagonal of A. of

Notes

How do we Find Eigenvalues, Though?

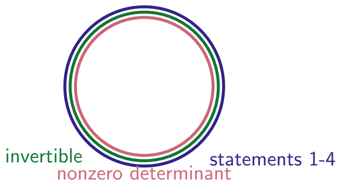
First, a reminder....

Notes

Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible
6) $\det(A) \neq 0$



Notes

How do we Find Eigenvalues, Though?

A matrix; λ eigenvalue, $\mathbf{x}$ eigenvector (so $\mathbf{x} \neq \mathbf{0}$)

$$
\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\
(A - \lambda I)\mathbf{x} &= \mathbf{0} \\
(A - \lambda I)\mathbf{x} &= \mathbf{0} \text{ has a nonzero solution} \\
\det(A - \lambda I) &= 0
\end{aligned}
$$

Note: we're not taking the determinant of A!

Notes

Find Eigenvalues and Associated Eigenvectors

$$\lambda \text{ eigenvalue } \Leftrightarrow \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Notes

Find Eigenvalues and Associated Eigenvectors

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Notes

Using Eigenvalues to Compute Matrix Powers

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\lambda = 1,\ \mathbf{k}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \lambda = 2,\ \mathbf{k}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda = 3,\ \mathbf{k}_3 = \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 47 \\ 16 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix} = 2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3$$

Notes

Using Eigenvalues to Compute Matrix Powers

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\lambda = 1,\ \mathbf{k}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \lambda = 2,\ \mathbf{k}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda = 3,\ \mathbf{k}_3 = \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$$

$$\mathbf{x} = 2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3$$

Notes

To Compute $A^n\mathbf{x}$:

- Find eigenvalues of A

  Find all solutions λ to the equation $\det(A - \lambda I) = 0$.
  These are your eigenvalues.
- Find associated eigenvectors of A

  These will be the nonzero solutions $\mathbf{x}$ to $(A - \lambda I)\mathbf{x} = 0$
- Write $\mathbf{x}$ as a linear combination of eigenvectors of A

  For eigenvectors $k_1, k_2, \ldots, k_n$, solve the linear system of equations $\mathbf{x} = a_1k_1 + a_2k_2 + \cdots + a_nk_n$ for constants $a_1, a_2, \ldots, a_n$.
- Matrix multiplication turns into scalar multiplication

  $A\mathbf{x} = A\left(a_1k_1 + a_2k_2 + \cdots + a_nk_n\right) =$
  $\left(a_1\lambda_1^nk_1 + a_2\lambda_2^nk_2 + \cdots + a_n\lambda_n^nk_n\right)$

Notes

Compute

$$\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}^{101} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1. Eigenvalues:
2. Eigenvectors:
3. Write $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ as a linear combination of eigenvectors:
4. Evaluate.

Your final answer should consist of **real, integer** entries.

Notes

Notes