Outline

Week 10: Eigenvalues and eigenvectors

Course Notes: 6.1

Goals: Understand how to find eigenvector/eigenvalue pairs, and use them to simplify calculations involving matrix powers.

This Will Look Strange: A preview of Why We Bother

Eigenvectors: we'll use a **very specific** phenomenon to make hard calculations easier. Here is a preview of what we can do.

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$$\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} + \left(-\frac{1}{4} \right)^n \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} + (-1)^n \frac{1}{5 \cdot 4^n} \\ \frac{2}{5} - (-1)^n \frac{1}{5 \cdot 4^n} \end{bmatrix}$$

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When Matrix Multiplication looks like Scalar Multiplication

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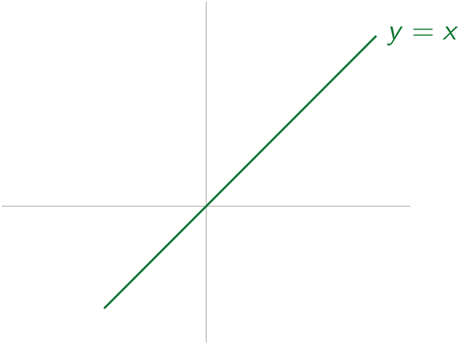
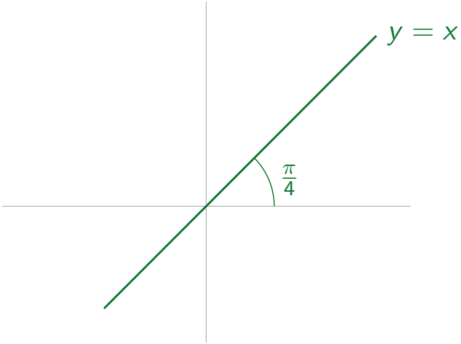
Recall:
Two vectors that are scalar multiples of one another are parallel.

If we interpret matrix multiplication as a **linear transformation**, we're looking for **a vector whose image is parallel to itself**.

Reflections, Revisited



Reflections, Revisited



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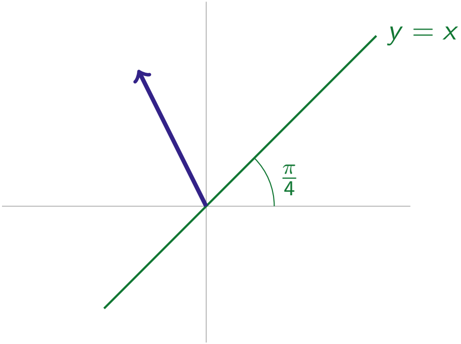

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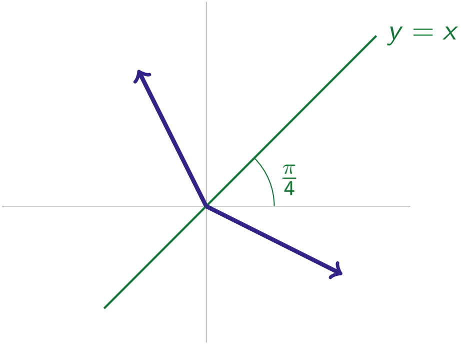

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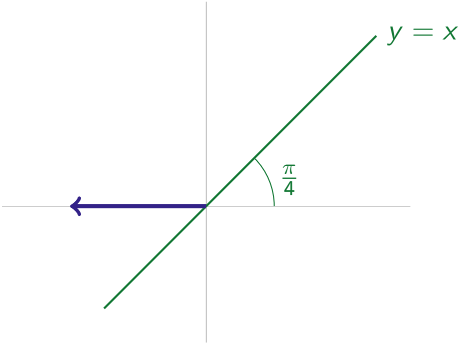

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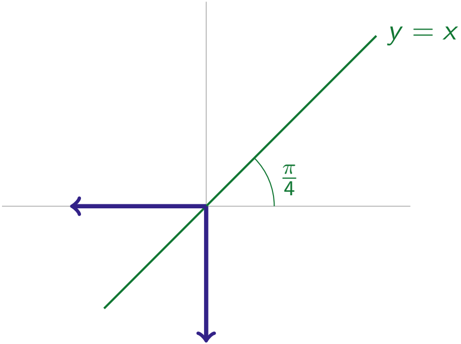

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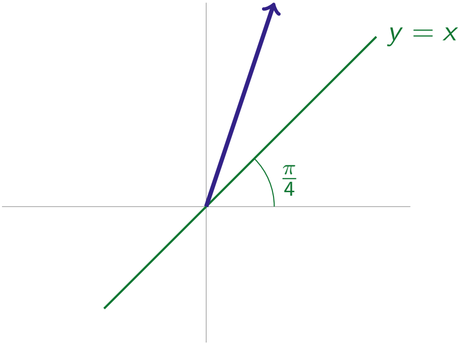

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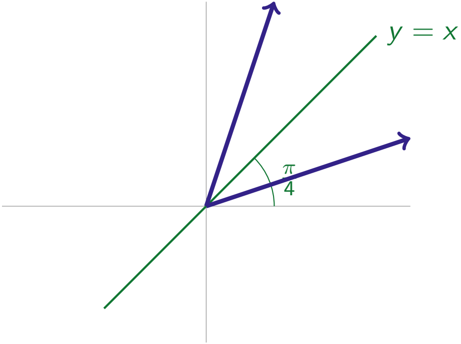

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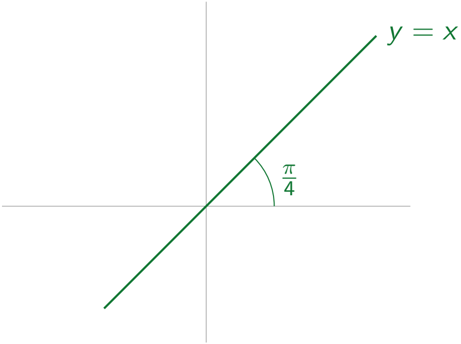

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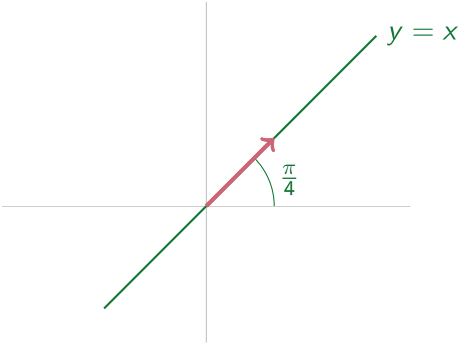

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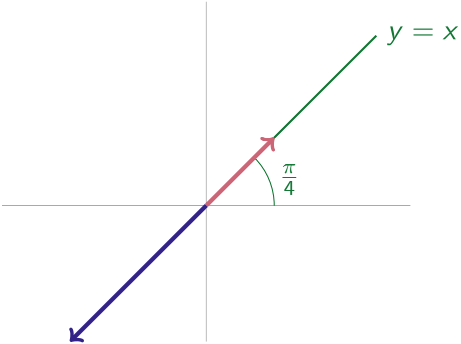

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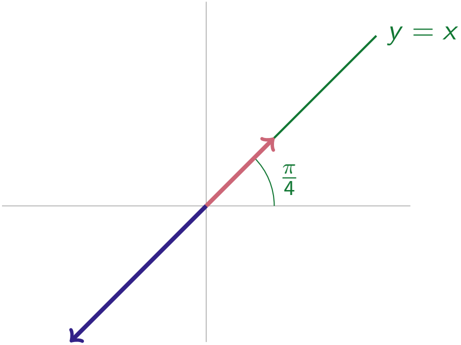

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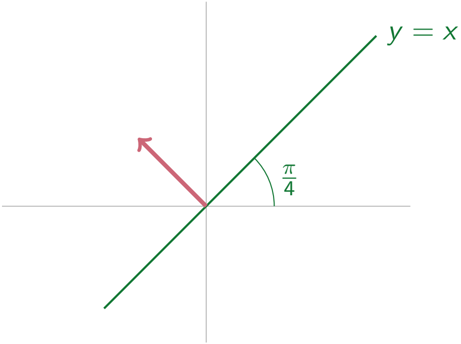

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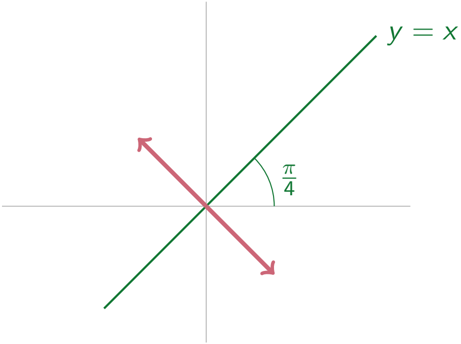

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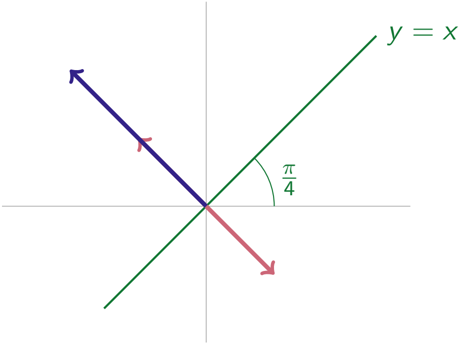

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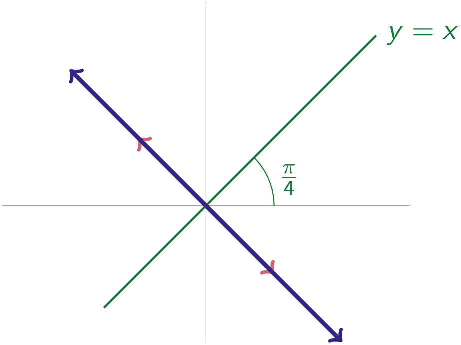

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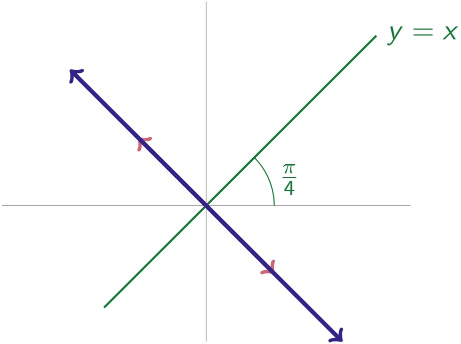

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Eigenvectors and Eigenvalues

Given a matrix A, a scalar λ, and a NONZERO vector $\mathbf{x}$ with

$$A\mathbf{x} = \lambda\mathbf{x}$$

we say $\mathbf{x}$ is an *eigenvector* of A with *eigenvalue* λ.

Notice we omit zero vectors! These are not particularly useful, and they behave differently from nonzero vectors with this property.

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$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ x \end{bmatrix}$$

The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has **eigenvector** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with **eigenvalue** 1.

By convention, we choose one representative eigenvector, with the understanding that all its nonzero scalar multiples are eigenvectors as well.

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The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has **eigenvector** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with **eigenvalue** -1.

Rotation Matrix Eigenvalues

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Search for eigenvectors:

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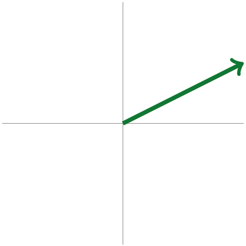
e.g. $\theta = 135^\circ$



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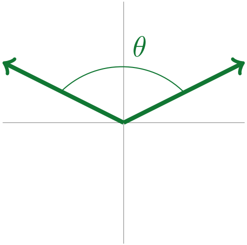
Search for eigenvectors:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

e.g. $\theta = 135^\circ$



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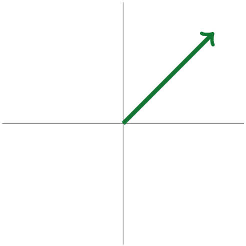

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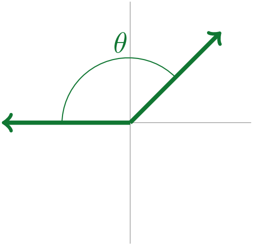

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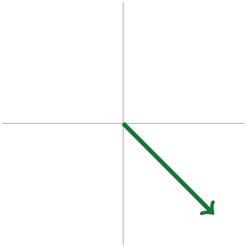

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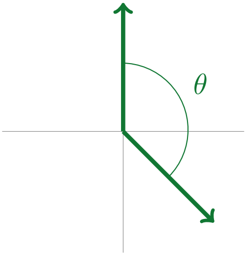

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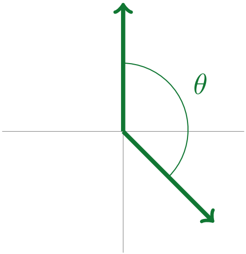

For most values of θ, the matrix Rot_θ has no (real) eigenvalues.

Finding Eigenvectors, Given Eigenvalues

Computation! We'll learn this in two stages.

Finding Eigenvectors, Given Eigenvalues

Given: 7 is an eigenvalue of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

What is an eigenvector associated to that eigenvalue?

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In equation form:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 4x & = & 7x \\ 2x & + & 4y & + & z & = & 7y \\ 4x & + & y & + & 2z & = & 7z \end{array}$$

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In better equation form:

$$\begin{array}{rcrcrcr} -6x & + & 2y & + & 4x & = & 0 \\ 2x & - & 3y & + & z & = & 0 \\ 4x & + & y & - & 5z & = & 0 \end{array}$$

Finding Eigenvectors, Given Eigenvalues

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Gaussian elimination on augmented matrix:

$$\left[\begin{array}{ccc|c} -6 & 2 & 4 & 0 \\ 2 & -3 & 1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right] \rightarrow \rightarrow \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Finding Eigenvectors, Given Eigenvalues

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Solutions:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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So these are the solutions to:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 7 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finding Eigenvectors, Given Eigenvalues

The solutions to:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 7 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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are precisely the vectors:

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So, we can choose $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as an example of an eigenvector of $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$ with eigenvalue 7.

Finding Eigenvectors from Eigenvalues

The matrix $\begin{bmatrix} 3 & 6 \\ 6 & -2 \end{bmatrix}$ has eigenvalues -6 and 7.

Find an eigenvector associated to each eigenvalue.

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Basis of $\mathbb{R}^2$: $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

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From the 2015 final exam

The matrix below represents rotation in 3D about a line through the origin.

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

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In 2015 (as today!) we do not talk about computing 3D rotations in class. This is a computation you can do just by visualizing the matrix as a transformation and understanding the definition of an eigenvector.

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Given matrix A with eigenvalue λ, find an associated eigenvector $\mathbf{x}$

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Eigenvectors $\mathbf{x}$ associated with eigenvalue λ are precisely the nonzero solutions to this homogeneous system.

So, we set up a homogeneous system of equations, where the coefficient matrix is formed by subtracting λ from every entry in the main diagonal of A.

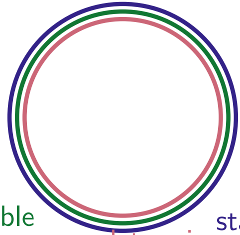
How do we Find Eigenvalues, Though?

First, a reminder....

Solutions to Systems of Equations

Let A be an n-by-n matrix. The following statements are equivalent:

1) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.
2) $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions.
3) The rank of A is n.
4) The reduced form of A has no zeroes along the main diagonal.
5) A is invertible
6) $\det(A) \neq 0$



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Note: we're not taking the determinant of A!

Find Eigenvalues and Associated Eigenvectors

$$\lambda \text{ eigenvalue } \Leftrightarrow \det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

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$\lambda = 1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ NOT a basis of $\mathbb{R}^2$!

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$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix} \right\} \text{ Basis of } \mathbb{R}^3$$

Using Eigenvalues to Compute Matrix Powers

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$$\mathbf{x} = \begin{bmatrix} 47 \\ 16 \\ 2 \end{bmatrix}$$

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$$A^{10}\mathbf{x} = A^{10}(2\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)$$

Using Eigenvalues to Compute Matrix Powers

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

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$4(2^{10})\mathbf{k}_2 + 3^{10}\mathbf{k}_3$

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To Compute $A^n\mathbf{x}$:

- Find eigenvalues of A

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  Find all solutions λ to the equation $\det(A - \lambda I) = 0$.
  These are your eigenvalues.

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  These will be the nonzero solutions $\mathbf{x}$ to $(A - \lambda I)\mathbf{x} = 0$

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  For eigenvectors $k_1, k_2, \ldots, k_n$, solve the linear system of equations $\mathbf{x} = a_1k_1 + a_2k_2 + \cdots + a_nk_n$ for constants $a_1, a_2, \ldots, a_n$.

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- Matrix multiplication turns into scalar multiplication

  $A\mathbf{x} = A\left(a_1k_1 + a_2k_2 + \cdots + a_nk_n\right) =$
  $\left(a_1\lambda_1^n k_1 + a_2\lambda_2^n k_2 + \cdots + a_n\lambda_n^n k_n\right)$

Compute

$$\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}^{101} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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1. Eigenvalues:
2. Eigenvectors:
3. Write $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ as a linear combination of eigenvectors:
4. Evaluate.

Your final answer should consist of **real, integer** entries.

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Compute

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2. Eigenvectors: i is associated to $\left[\begin{smallmatrix} -3-i \\ 5 \end{smallmatrix}\right]$;
   $-i$ is associated to $\left[\begin{smallmatrix} -3+i \\ 5 \end{smallmatrix}\right]$
3. Write $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ as a linear combination of eigenvectors:
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   $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1+8i}{10}\begin{bmatrix} -3-i \\ 5 \end{bmatrix} + \frac{1-8i}{10}\begin{bmatrix} -3+i \\ 5 \end{bmatrix}$
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3. Write $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ as a linear combination of eigenvectors: $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \frac{1+8i}{10}\left[\begin{smallmatrix} -3-i \\ 5 \end{smallmatrix}\right] + \frac{1-8i}{10}\left[\begin{smallmatrix} -3+i \\ 5 \end{smallmatrix}\right]$
4. Evaluate.

$$\begin{aligned}
\left[\begin{smallmatrix} 3 & 2 \\ -5 & -3 \end{smallmatrix}\right]^{101}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] &= \left[\begin{smallmatrix} 3 & 2 \\ -5 & -3 \end{smallmatrix}\right]^{101}\left(\frac{1+8i}{10}\left[\begin{smallmatrix} -3-i \\ 5 \end{smallmatrix}\right] + \frac{1-8i}{10}\left[\begin{smallmatrix} -3+i \\ 5 \end{smallmatrix}\right]\right) \\
&= (i^{101})\frac{1+8i}{10}\left[\begin{smallmatrix} -3-i \\ 5 \end{smallmatrix}\right] + ((-i)^{101})\frac{1-8i}{10}\left[\begin{smallmatrix} -3+i \\ 5 \end{smallmatrix}\right] \\
&= i\frac{1+8i}{10}\left[\begin{smallmatrix} -3-i \\ 5 \end{smallmatrix}\right] - i\frac{1-8i}{10}\left[\begin{smallmatrix} -3+i \\ 5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 \\ -8 \end{smallmatrix}\right]
\end{aligned}$$

Your final answer should consist of **real, integer** entries.

There exists a 2-by-2 matrix A with the following eigenvalue-eigenvector pairs:

$$A\begin{bmatrix}1\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} \qquad A\begin{bmatrix}1\\-1\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix}$$

What is A? What is A^n?

First, we note that for $A = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$, $A\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}a\\c\end{smallmatrix}\right]$, and $A\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}b\\d\end{smallmatrix}\right]$. So, we can pick off the columns of A by finding its product with the standard basis vectors.

$$A\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = A\left(\frac{2}{3}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right] + \frac{1}{3}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) = 0 + (3)(1/3)\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$$
$$A\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = A\left(\frac{-1}{3}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right] + \frac{1}{3}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) = 0 + 3(1/3)\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$$

So, $A = \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}$ Similarly, $A^n = 3^{n-1}\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}$