Assignment #2
To be handed in Friday, October 23

1. Question 1

    (a) Let X be a Gaussian variable with mean $m = 10^6\$$ and standard deviation $\sigma = 3 \times 10^5\$$. Compute $V@R(X)$ and $AV@R(X)$ at the level $\lambda = 1\%$

    (b) Let Y be another variable, independent from X and with the same law. Compute $V@R(X+Y)$ and $AV@R(X+Y)$ at the level $\lambda = 1\%$

2. Question 2

    I have to pay back $10^6\$$ exactly one year from now, and I have a zero-coupon bond, maturing in exactly ten years, with face value $1.4 \times 10^6\$$. On the day the loan comes due, I will sell the bond, and use the proceeds to pay off the loan. What is the present (ie discounted using today's interest rate) $V@R$ of my position at the level 1%? The yield curve is assumed to be flat and to remain so. The interest rate today is 3%. The interest rate one year from now will be $r\%$, where r is a lognormal random variable with mean $m = 3\%$ and standard deviation $\sigma = 2\%$.

3. Question 3.

    Let X be a random variable, $F_X(x)$ its distribution function and $q_X(u)$ its quantile, $0 \leq u \leq 1$. It is assumed that X is uniformly bounded and F is continuous and strictly increasing.

    (a) Show that $U(\omega) = F_X(X(\omega))$ is a uniformly distributed random variable, i.e. $P[U \leq \lambda] = \lambda$

    (b) Show that $q_X(U(\omega)) = X(\omega)$

    (c) Show that, for any bounded measurable function $f(x)$, we have:

    $$\int_{-\infty}^{\infty} f(X)\, dP = \int_0^1 f(q_X(u))\, du$$

4. Question 4

    (a) Prove that, given two sets of n numbers:

    $$\begin{aligned} a_1, a_2, ..., a_{n-1}, a_n \text{ with } a_i &< a_j \text{ when } i < j \\ b_1, b_2, ...b_{n-1}, b_n \text{ with } b_i &\neq b_j \text{ when i} \neq j \end{aligned}$$

    and a permutation σ of $\{1, 2, ..., n-1, n\}$, the sum:

    $$S_\sigma := \sum_{i=1}^{n} a_i b_{\sigma(i)}$$

is largest when the $b_{\sigma(i)}$ are ordered:

$$b_{\sigma(i)} < b_{\sigma(j)} \text{ whenever } i < j$$

(b) We consider the interval $I = [0,1]$ which we divide into n equal subintervals $I_k := \left[\frac{k}{n}, \frac{k+1}{n}\right]$. A n-step function on I is a function which is constant on each of the I_k.. Given two n-step functions X and Z, we shall say that $X\tilde{}Z$ if X and Z have the same law.

Let X and Y be two strictly increasing n-step functions. Show that, for all $Z\tilde{}X$, we have

$$\int_0^1 ZY \leq \int_0^1 XY$$

(c) Show the same result when X and Y are L^2 functions on $[0,1]$. It is called the Hardy-Littlewood inequality (Hint: use the fact that the set of all n-step functions, $n \geq 1$, is dense in L^2)