THE TATE-VOLOCH CONJECTURE FOR DRINFELD MODULES

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Abstract. We study the v-adic distance from the torsion of a Drinfeld module to an affine variety.

1. Introduction

For a semi-abelian variety S and an algebraic subvariety $X \subset S$, the Manin-Mumford conjecture characterizes the subset of torsion points of S contained in X. The Tate-Voloch conjecture characterizes the distance from X of a torsion point of S not contained in X.

Let $\mathbb{C}_p$ be the completion of a fixed algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ of $\mathbb{Q}_p$. Let $\lambda(\cdot, X)$ be the p-adic proximity to X function as defined in [11] (see also our definition of v-adic distance to an affine subvariety). Tate and Voloch conjectured:

Conjecture 1.1 (Tate,Voloch)**.** Let G be a semi-abelian variety over $\mathbb{C}_p$. Let $X \subset G$ be a subvariety defined over $\mathbb{C}_p$. Then there is a constant $N \in \mathbb{N}$ such that for any torsion point $\zeta \in G(\mathbb{C}_p)$ either $\zeta \in X$ or $\lambda_p(\zeta, X) \leq N$.

The above conjecture was proved by Thomas Scanlon for all semi-abelian varieties defined over $\mathbb{Q}_p^{\mathrm{alg}}$ (see [11] and [12]).

In this paper we prove two Tate-Voloch type theorems for Drinfeld modules. Our motivation is to show that yet another question for semi-abelian varieties has a counterpart for Drinfeld modules (see [13] and [5] for the Manin-Mumford theorem for Drinfeld modules of generic characteristic and see [4] for the Mordell-Lang theorem for all Drinfeld modules).

In Section 2 we state our results. Our first result (Theorem 2.7) shows that if a torsion point of a Drinfeld module $\phi : A \to K\{\tau\}$ is close w-adically to a variety X with respect to all places w extending a fixed place v of the ground field K, then the torsion point lies on X. We prove Theorem 2.7 in Section 3. Our bound for how "close w-adically to X" means "lying on X" is effective. Our second result (Theorem 2.10) refers to proximity with respect to one fixed extension of a place v of K. We will prove Theorem 2.10 in Section 4. We also note that due to the fact that in Theorem 2.10 we work with a fixed extension of a place of K, there is a different normalization for the valuation we are working as opposed to the setting in Theorem 2.7.

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2. Statement of our main results

Before stating our results we introduce the definition of a Drinfeld module (for more details, see [3]).

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Let p be a prime number and let q be a power of p. We let C be a nonsingular projective curve defined over $\mathbb{F}_q$ and we fix a closed point ∞ on C. Then we define A as the ring of functions on C that are regular everywhere except possibly at ∞.

We let K be a field extension of $\mathbb{F}_q$ and we fix an algebraic closure of K, denoted K^{alg}. We fix a morphism $i : A \to K$. We define the operator τ as the power of the usual Frobenius with the property that for every $x \in K^{\text{alg}}$, $\tau(x) = x^q$. Then we let $K\{\tau\}$ be the ring of polynomials in τ with coefficients in K (the addition is the usual one, while the multiplication is the composition of functions).

A Drinfeld module over K is a ring morphism $\phi : A \to K\{\tau\}$ for which the coefficient of τ^0 in ϕ_a is $i(a)$ for every $a \in A$, and there exists $a \in A$ such that $\phi_a \neq i(a)\tau^0$. We call ϕ a Drinfeld module of generic characteristic if $\ker(i) = \{0\}$ and we call ϕ a Drinfeld module of finite characteristic if $\ker(i) \neq \{0\}$. In the generic characteristic case we assume i extends to an embedding of $\text{Frac}(A)$ (which is the function field of the projective nonsingular curve C) into K. In the finite characteristic case, we call $\ker(i)$ the characteristic ideal of ϕ.

For every nonzero $a \in A$, let the *a-torsion* $\phi[a]$ of ϕ be the set of all $x \in K^{\text{alg}}$ such that $\phi_a(x) = 0$. Let the torsion submodule of ϕ be $\bigcup_{a\in A\setminus\{0\}} \phi[a]$.

For every $g \geq 1$, let ϕ act diagonally on $\mathbb{G}_a^g$. An element $(x_1, \dots, x_g) \in (K^{\text{alg}})^g$ is called a torsion element of ϕ, if for every $i \in \{1, \dots, g\}$, $x_i \in \phi_{\text{tor}}$.

For each field extension L of K and for each valuation w on L we define the w-adic distance to an affine subvariety $X \subset \mathbb{G}_a^g$ defined over L.

Definition 2.1. Let I_X be the vanishing ideal in $L[X_1, \dots, X_g]$ of X. Let $R_w \subset L$ be the valuation ring of w. If $P \in \mathbb{G}_a^g(L)$, then the w-adic distance from P to X is

$$\lambda_w(P, X) := \min\{w(f(P)) \mid f \in I_X \cap R_w[X_1, \dots, X_g]\}. \tag{1}$$

We denote by M_K the set of all discrete valuations on K. Similarly, for each field extension L of K we also denote by M_L the set of all discrete valuations on L. Finally, we note that unless otherwise stated, each valuation is normalized so that its range is precisely $\mathbb{Z} \cup \{+\infty\}$ (our convention is that the valuation of 0 is $+\infty$). Our Theorem 2.7 is valid for all fields K equipped with a *coherent good set of valuations*.

Definition 2.2. We call a subset $U \subset M_K$ equipped with a function $d : U \to \mathbb{R}_{>0}$ a *good set of valuations* if the following properties are satisfied

(i) for each nonzero $x \in K$, there are finitely many $v \in U$ such that $v(x) \neq 0$.

(ii) for each nonzero $x \in K$,

$$\sum_{v\in U} d(v) \cdot v(x) = 0.$$

The positive real number $d(v)$ will be called the *degree* of the valuation v. When we say that the positive real number $d(v)$ is associated to the valuation v, we understand that the degree of v is $d(v)$.

When U is a good set of valuations, we will refer to property (ii) as the sum formula for U.

Definition 2.3. Let $v \in M_K$ of degree $d(v)$. We say that the valuation v is *coherent* if for every finite extension L of K,

$$\sum_{\substack{w \in M_L \\ w|v}} e(w|v) f(w|v) = [L : K], \tag{2}$$

where $e(w|v)$ is the ramification index and $f(w|v)$ is the relative degree between the residue field of w and the residue field of v.

Condition (2) says that v is *defectless* in L. In this case, we also let the degree of any $w \in M_L$, $w|v$ be

$$d(w) = \frac{f(w|v)d(v)}{[L : K]}. \tag{3}$$

Definition 2.4. We let U_K be a good set of valuations on K. We call U_K a *coherent* good set of valuations if for every $v \in U_K$, the valuation v is coherent.

Remark 2.5. Using the argument from page 9 of [10], we conclude that in Definition 2.4, if for each finite extension L of K we let $U_L \subset M_L$ be the set of valuations lying above valuations in U_K, then U_L is a good set of valuations.

Example 2.6. Let V be a projective, regular in codimension 1 variety defined over a finite field. Then the function field F of V is equipped with a coherent good set of valuations associated to each irreducible divisor of V. Hence every finitely generated field is equipped with at least one coherent good set of valuations (different sets of valuations correspond to different projective, regular in codimension 1 varieties with the same function field). For more details see [10] or Chapter 4 of [3].

We prove the following Tate-Voloch type theorem for Drinfeld modules.

Theorem 2.7. *Assume U_K is a coherent good set of valuations on K and let $v \in U_K$ have degree $d(v)$. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let $X \subset \mathbb{G}_a^g$ be a closed K-subvariety of the g-dimensional affine space.*

There exists a constant $C > 0$ (depending on ϕ, X and $d(v)$) such that for every finite extension L of K and for every torsion point $P \in \mathbb{G}_a^g(L)$ of ϕ, either $P \in X(L)$ or there exists $w \in M_L$ lying over v such that $\lambda_w(P, X) \leq C \cdot e(w|v)$.

Remark 2.8. There are two significant differences between our Tate-Voloch type theorem and Conjecture 1.1. We show that a torsion point of the Drinfeld module is on X if it is close to X with respect to all extensions of a fixed valuation v of K, not only with respect to one fixed extension of v. We will show in Example 2.9 that we cannot always expect proximity of P to X with respect to one fixed extension of v imply that P lies on X. The second difference between our Theorem 2.7 and Conjecture 1.1 is purely technical. Because we normalized all valuations so that their ranges equal $\mathbb{Z}$, we need to multiply by the corresponding ramification index the constant C in Theorem 2.7.

Example 2.9. Let ϕ be any Drinfeld module of generic characteristic and let v_∞ be a valuation on K extending the valuation on $\mathrm{Frac}(A)$ associated to the closed point $\infty \in C$. We let K_∞ be a completion of K with respect to v_∞. Then $\phi_{\mathrm{tor}} \subset K_\infty^{\mathrm{alg}}$ is not discrete with respect to v_∞ (see Section 4.13 of [7]). Hence there exist nonzero torsion points of ϕ arbitrarily close to $X := \{0\}$ in the v_∞-adic topology.

For the remainder of Section 2 we fix a valuation v on K (we do not require anymore that v belongs to a good set of valuations on K nor that v is coherent). We let K_v be the completion of K at v. We fix an algebraic closure K_v^{alg} of K_v and extend v to a valuation of K_v^{alg}. In this case, the value group of v is $\mathbb{Q}$. We define as in (1) the v-adic distance from a point $P \in \mathbb{G}_a^g(K_v^{\text{alg}})$ to a fixed affine variety X defined over K_v^{alg}.

Our Theorem 2.10 characterizes the distance from ϕ_{tor}^g to a fixed point of $\mathbb{G}_a^g(K_v^{\text{alg}})$. Our theorem is an analogue for Drinfeld modules of a theorem of Mattuck (see [8]).

Theorem 2.10. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let v be a place of K. If ϕ is a Drinfeld module of generic characteristic, then assume v does not lie over the valuation v_∞ of* $\text{Frac}(A)$, *which is associated to the closed point $\infty \in C$. Let $g \geq 1$.*

Then for every $Q \in \mathbb{G}_a^g(K_v^{\text{alg}})$ there exists a positive constant C depending on ϕ, v and Q such that for each $P \in \phi_{\text{tor}}^g$ either $P = Q$ or $\lambda_v(P, Q) < C$.

Note that as shown in Example 2.9, if ϕ has generic characteristic, then Theorem 2.10 does not hold if v extends the place v_∞ of $\text{Frac}(A)$. If ϕ has finite characteristic, there is no restriction on v in Theorem 2.10.

3. Proximity with respect to all extensions of a valuation v

We work under the assumption that there exists a coherent good set of valuations U_K on K. We first construct the set of local heights associated to the places in U_K and then we define the global height. All our valuations in this section are normalized so that their value group is $\mathbb{Z}$.

For each finite extension field L of K and for each place w of L lying above a place in U_K, we let $\tilde{w} : L \to \mathbb{Z}_{\leq 0}$ be defined as follows

$$\tilde{w} := \min\{w, 0\}.$$

Then the local height at w of any element $x \in L$ is $\text{h}_w(x) := -d(w)\tilde{w}(x)$. We define the global height of x as

$$\text{h}(x) := \sum_w \text{h}_w(x).$$

The above sum is a finite sum because there are finitely many w such that $w(x) < 0$ (see condition (i) of Definition 2.2). Because U_K is a coherent good set of valuations, the definition of the global height of an element x does not depend on the particular choice of the field L containing x (see for example Chapter 4 of [3]). The following two standard properties of the height will be used in our proof.

Proposition 3.1. *For each $x, y \in K^{\text{alg}}$, the following are true:*
(i) $\text{h}(xy) \leq \text{h}(x) + \text{h}(y)$.
(ii) $\text{h}(x + y) \leq \text{h}(x) + \text{h}(y)$.

Proof. The proof is immediate using the definition of height and the triangle inequality for each valuation. ☐

For a point $P := (x_1, \ldots, x_g) \in \mathbb{G}_a^g(L)$, we define the local height of P at a place w of L lying above a place in U_K, as follows:

$$\text{h}_w(P) := \max\{\text{h}_w(x_1), \ldots, \text{h}_w(x_g)\}.$$

Then the global height of P is $\mathrm{h}(P) := \sum_w \mathrm{h}_w(P)$.

Next we define the heights associated to a Drinfeld module $\phi : A \to K\{\tau\}$ (see [3] for more details). We fix a non-constant $a \in A$. For each finite extension L as above and for each place w of L as above, we define

$$V_w(x) := \lim_{n\to\infty} \frac{\tilde{w}(\phi_{a^n}(x))}{\deg(\phi_{a^n})},$$

for each $x \in L$.

Then the canonical local height of x at w with respect to ϕ is $\widehat{\mathrm{h}}_w(x) := -d(w)V_w(x)$. Finally, the canonical global height of x with respect to ϕ is $\widehat{\mathrm{h}}(x) := \sum_w \widehat{\mathrm{h}}_w(x)$. By the same reasoning as in [1] (see part 3) of Théorème 1) or in [9] (see part (2) of Proposition 1) we can show that there exists a positive constant C_0 such that for every $x \in K^{\mathrm{alg}}$,

$$|\,\mathrm{h}(x) - \widehat{\mathrm{h}}(x)| \le C_0. \tag{4}$$

Moreover, the constant C_0 is easily computable in terms of ϕ (see [9]).

For each point $P := (x_1, \dots, x_g) \in \mathbb{G}_a^g(L)$ and for each place w of L as above, we define the canonical local height of P at w as $\widehat{\mathrm{h}}_w(P) := \max\{\widehat{\mathrm{h}}_w(x_1), \dots, \widehat{\mathrm{h}}_w(x_g)\}$. The canonical global height of P is $\widehat{\mathrm{h}}(P) := \sum_w \widehat{\mathrm{h}}_w(P)$.

Using (4) and Proposition 3.1 we prove the following result.

Lemma 3.2. *Let L be a finite extension of K and let $f \in L[X_1, \dots, X_g]$. There exists a constant $C(f) > 0$ such that for every $P \in \mathbb{G}_a^g(K^{\mathrm{alg}})$, if P is a torsion point for ϕ, then $\mathrm{h}(f(P)) \le C(f)$.*

Proof. Using Proposition 3.1 (*ii*), it suffices to prove Lemma 3.2 under the assumption that f is a monomial. Hence, assume $f := cX_1^{\alpha_1} \cdot \dots \cdot X_g^{\alpha_g}$ for some $c \in L$ and $\alpha_1, \dots, \alpha_g \in \mathbb{Z}_{\ge 0}$. Let $P = (x_1, \dots, x_g)$. We know that for each i, $x_i \in \phi_{tor}$. Hence $\widehat{\mathrm{h}}(x_i) = 0$ for each i. Using (4) we conclude that $\mathrm{h}(x_i) \le C_0$ for each i. Therefore, an application of Proposition 3.1 (*i*) concludes the proof of our Lemma 3.2. ∎

We proceed to the proof of Theorem 2.7.

Proof of Theorem 2.7. Let $f_1, \dots, f_m$ be a set of polynomials in $K[X_1, \dots, X_g]$ with integral coefficients at v, which generate the vanishing ideal of X. It suffices to prove that for each such polynomial f_i and for every finite extension L of K and for every torsion point $P \in \mathbb{G}_a^g(L)$, either $f_i(P) = 0$ or there exists a place $w|v$ of L such that $w(f_i(P)) \le \frac{C(f_i)}{d(v)} e(w|v)$, where $C(f_i)$ is the constant corresponding to f_i as in Lemma 3.2. Then we obtain Theorem 2.7 with $C := \max_i \frac{C(f_i)}{d(v)}$.

Assume for some $i \in \{1, \dots, m\}$ and for some torsion point $P \in \mathbb{G}_a^g(L)$, $w(f_i(P)) > \frac{C(f_i)}{d(v)} e(w|v)$ for every place $w|v$ of L. Then

$$\sum_{w|v} d(w) \cdot w(f_i(P)) > \frac{C(f_i)}{d(v)} \sum_{w|v} d(w)e(w|v) = \frac{C(f_i)}{d(v)} \sum_{w|v} \frac{d(v)f(w|v)e(w|v)}{[L:K]} = C(f_i) > 0 \tag{5}$$

because $\sum_{w|v} f(w|v)e(w|v) = [L : K]$, as v is a coherent valuation. If $f_i(P) \ne 0$, then (5) yields that the set S of places of L lying above places in U_K for which $f_i(P)$ is non-integral, is non-empty. Moreover, using (5) and the sum formula for the nonzero element $f_i(P) \in L$, we conclude

$$\sum_{w \in S} d(w) \cdot w(f_i(P)) < -C(f_i). \tag{6}$$

Therefore, by the definition of the local heights we get

$$\sum_{w \in S} \mathrm{h}_w(f_i(P)) > C(f_i). \tag{7}$$

Using the definition of the global height and (7) we conclude $\mathrm{h}(f_i(P)) > C(f_i)$. This last inequality contradicts Lemma 3.2 because P is a torsion point. This shows that $f_i(P) = 0$ assuming $f_i(P)$ is close w-adically to 0 for each $w|v$. This concludes the proof of our Theorem 2.7. □

Remark 3.3. Theorem 2.7 cannot be strenghtened to ask that proximity of P to X with respect to one extension w of v would guarantee that $P \in X$ (see Example 2.9). However, even if we want to strenghten Theorem 2.7 by assuming proximity with respect to only one extension of v (under the extra assumption that v does not lie over v_∞), our proof would not extend. We use in a crucial way in (5) that P is close to X with respect to all extensions of v. If we would know this information about only one place w, this would not guarantee that $f_i(P)$ has "sufficiently many zeros" (as described in (5)). In turn, this would not yield that $f_i(P)$ has "sufficiently many poles" (as in (6)) and hence, we would not obtain a contradiction regarding the height of $f_i(P)$ (as in (7)). We believe that the question of proximity with respect to one extension of a valuation v (which does not extend v_∞) is a difficult question and we also believe answering this question would involve new methods.

Remark 3.4. Because the constants $C(f_i)$ from the proof of Theorem 2.7 are easily computable in terms of the polynomials f_i and in terms of the constant from (4), then the constant C from the conclusion of Theorem 2.7 is effective.

4. Proximity with respect to a fixed extension of the valuation v

In this Section 4 we work under the hypothesis that the valuation v of K does not extend the valuation v_∞ of $\mathrm{Frac}(A)$ in case $\phi : A \to K\{\tau\}$ is a Drinfeld module of generic characteristic. We also work with a fixed completion K_v of K at v and with its algebraic closure K_v^{alg}. In this section, the value group of our valuation v is $\mathbb{Q}$, while its restriction to K has value group $\mathbb{Z}$.

We first reduce Theorem 2.10 to the following Lemma 4.1.

Lemma 4.1. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module and let v be a discrete valuation on K. If ϕ has generic characteristic, assume moreover that v does not lie over the place v_∞ of* $\mathrm{Frac}(A)$. *Then there exists a positive constant C_v depending only on ϕ and v such that in the ball*

$$\{x \in K_v^{\mathrm{alg}} \mid v(x) \geq C_v\}$$

there are no nonzero torsion points of ϕ.

Lemma 4.1 shows that for each place v which does not lie over v_∞ (if ϕ has generic characteristic), ϕ_{tor} is discrete in the v-adic topology. If ϕ has finite characteristic, then ϕ_{tor} is discrete with respect to each valuation v (without any restriction). Moreover, as it will be shown in the proof of Lemma 4.1, the constant C_v is easily computable in terms of ϕ and v.

Proof of Theorem 2.10. We prove Theorem 2.10 using the result of Lemma 4.1. Let $Q := (y_1, \dots, y_g)$ and let $L := K_v(Q)$. Let $\beta_i := \max\{0, -v(y_i)\}$ for each $i \in \{1, \dots, g\}$. For each i, let $\gamma_i \in L$ be an element of valuation equal to β_i. Then for each $i \in \{1, \dots, g\}$, the linear polynomial $\gamma_i(X_i - y_i) \in L[X_1, \dots, X_g]$ has integral coefficients at v and vanishes at Q.

We know (see Lemma 5.2.5 of [3] or Lemma 4.12 of [6]) that there exists an absolute constant $M_v \le 0$ depending only on ϕ and v such that for every torsion point $x \in \phi_{\mathrm{tor}}$, $v(x) \ge M_v$ (because otherwise, x has positive local height at v, contradicting the fact that each local height of a torsion point is 0). Then for each point $P := (x_1, \dots, x_g) \in \phi^g_{\mathrm{tor}}$, if for some $i \in \{1, \dots, g\}$,

$$v(y_i) = -\beta_i < M_v \le v(x_i),$$

then $v(x_i - y_i) = v(y_i)$. In this case, $\lambda_v(P, Q) \le v(\gamma_i(x_i - y_i)) = 0$. Therefore, in case for some $i \in \{1, \dots, g\}$, $v(y_i) < M_v$, we obtained an absolute upper bound for the v-adic distance of a torsion point to Q.

Assume from now on in this proof that for every $i \in \{1, \dots, g\}$, $v(y_i) \ge M_v$. Hence $\beta_i \le -M_v$. We compute the v-adic distance between a torsion point $P := (x_1, \dots, x_g) \in \phi^g_{\mathrm{tor}}$ and Q. We obtain:

$$\lambda_v(P, Q) \le \min_{i=1}^{g} v(\gamma_i(x_i - y_i)) = \min_{i=1}^{g} (\beta_i + v(y_i - x_i)) \le -M_v + \min_{i=1}^{g} v(x_i - y_i). \tag{8}$$

Therefore, in order to prove Theorem 2.10 it suffices to show that

$$\min_{i=1}^{g} v(x_i - y_i)$$

is uniformly bounded from above when $(x_1, \dots, x_g) \in \phi^g_{\mathrm{tor}} \setminus \{(y_1, \dots, y_g)\}$. But Lemma 4.1 shows that for each i, there is at most one torsion point of ϕ in the ball

$$\{x \in K_v^{\mathrm{alg}} \mid v(x - y_i) \ge C_v\}, \tag{9}$$

because otherwise there would be at least one nonzero torsion point of ϕ in $\{x \in K_v^{\mathrm{alg}} \mid v(x) \ge C_v\}$ after translating the ball in (9) by a torsion point of ϕ which lies inside the ball from (9). Therefore, $\lambda_v(P, Q)$ is indeed uniformly bounded from above for $P \in \phi^g_{\mathrm{tor}} \setminus \{Q\}$ because there is at most one torsion point $P \in \phi^g_{\mathrm{tor}}$ such that $\lambda_v(P, Q) > -M_v + C_v$. ☐

Remark 4.2. As discussed in Remark 3.3, the problem of proximity to an arbitrary variety X of a torsion point with respect to a single extension of a valuation v seems to be a difficult question. We note that the methods involved in our proof of Theorem 2.10 do not easily generalize to the case of higher dimensional varieties X because then the vanishing ideal of X would not be necessarily generated by linear polynomials. This would prevent us to have a good control (as we had in (8)) on computing the v-adic distance to X of a torsion point.

We proceed to the proof of Lemma 4.1.

Proof of Lemma 4.1. We first choose $t \in A$ satisfying certain properties according to the two cases we have: ϕ has generic characteristic or not.

Case (i). ϕ has generic characteristic.

Let $\mathfrak{p}$ be the nonzero prime ideal of A which is contained in the maximal ideal of the valuation ring of v (we are using the fact that v does not lie over v_∞ to derive that all the elements of A are integral at v). We fix $t \in \mathfrak{p} \setminus \{0\}$.

Case (ii). ϕ has finite characteristic.

Let $\mathfrak{p}$ be the characteristic ideal of ϕ. By the hypothesis for our *Case (ii)*, $\mathfrak{p}$ is nonzero. We fix $t \in \mathfrak{p} \setminus \{0\}$.

Let $\phi_t = \sum_{i=r_0}^{r} a_i \tau^i$, where $a_{r_0} \neq 0$. In finite characteristic, $r_0 \geq 1$, while in generic characteristic, $r_0 = 0$ and $v(a_0) \geq 1$ (by our choice of t). We let C_v be the smallest positive integer larger than all of the numbers from the following set:

$$S := \{-\frac{v(a_{r_0})}{q^{r_0}-1}\} \cup \{\frac{v(a_{r_0}) - v(a_i)}{q^i - q^{r_0}} \mid r_0 < i \leq r\}.$$

We note that if ϕ has generic characteristic, then $r_0 = 0$ and so, $q^{r_0} = 1$. Then the denominator of the first fraction contained in S is 0. So, because the numerator $-v(a_0) \leq -1$, that fraction equals $-\infty$ and so, any integer is larger than it, i.e. if ϕ has generic characteristic, we may disregard the first fraction in the definition of S. As we will see in our proof, that first fraction will only be used in the finite characteristic case.

Claim 4.3. If $x \in K_v^{\text{alg}} \setminus \{0\}$ satisfies $v(x) \geq C_v$, then $v(\phi_t(x)) = v\left(a_{r_0} x^{q^{r_0}}\right) > v(x) \geq C_v$. In particular, $\phi_t(x) \neq 0$.

Proof of Claim 4.3. Because $v(x) \geq C_v$, then for each $i \in \{r_0 + 1, \ldots, r\}$, $v(x) > \frac{v(a_{r_0}) - v(a_i)}{q^i - q^{r_0}}$. Hence

$$v(a_i) + q^i v(x) > v(a_{r_0}) + q^{r_0} v(x) \text{ and so,}$$

$$v\left(a_i x^{q^i}\right) > v\left(a_{r_0} x^{q^{r_0}}\right) \text{ for each } i > r_0. \tag{10}$$

Inequality (10) shows that $v(\phi_t(x)) = v\left(a_{r_0} x^{q^{r_0}}\right)$. In particular, this shows $\phi_t(x)$ does not equal 0, because its valuation is not $+\infty$ (both x and a_{r_0} are nonzero numbers). Hence

$$v(\phi_t(x)) = v(a_{r_0}) + q^{r_0} v(x). \tag{11}$$

If ϕ has generic characteristic, then (11) shows that $v(\phi_t(x)) = v(a_0) + v(x) \geq 1 + v(x) > C_v$. If ϕ has finite characteristic, then using that

$$v(x) \geq C_v > -\frac{v(a_{r_0})}{q^{r_0} - 1},$$

we conclude $v(\phi_t(x)) = v(a_{r_0}) + q^{r_0} v(x) > v(x) \geq C_v$. □

Claim 4.3 shows that for every nonzero $x \in K_v^{\text{alg}}$ satisfying $v(x) \geq C_v$, the sequence $\{v(\phi_{t^n}(x))\}_{n \geq 0}$ is strictly increasing. Hence, $x \notin \phi_{\text{tor}}$, because if x were torsion, then the sequence $\{\phi_{t^n}(x)\}_{n \geq 0}$ would contain only finitely many distinct elements. This concludes the proof of Lemma 4.1. □

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