Dedicated to Joe Silverman on his 60th birthday.

UNLIKELY INTERSECTION FOR TWO-PARAMETER FAMILIES OF POLYNOMIALS

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Abstract. Let c_1, c_2, c_3 be distinct complex numbers, and let $d \geq 3$ be an integer. We show that the set of all pairs $(a, b) \in \mathbb{C} \times \mathbb{C}$ such that each c_i is preperiodic for the action of the polynomial $x^d + ax + b$ is not Zariski dense in the affine plane.

1. Introduction

The results of this paper are in the context of the *unlikely intersections* problem in arithmetic dynamics, and more generally in arithmetic geometry. The philosophy of the unlikely intersections principle in arithmetic geometry says that an *event* that is *unlikely* to occur in a geometric setting must be explained by a (rigid) arithmetic property. Roughly speaking, in this context, an event is said to be "unlikely" when the number of conditions it satisfies is very large relative to the number of parameters of the underlying space. For more details, see the Pink-Zilber Conjecture [Pin], the various results (such as [BMZ99]) in this direction, and also the beautiful book of Zannier [Zan12].

At the suggestion of Zannier (whose question was motivated by [MZ08, MZ10, MZ13]), Baker-DeMarco proved a first result [BD11] for the unlikely intersection principle this time in arithmetic dynamics. Baker and DeMarco [BD11] proved that given complex numbers a and b, and an integer $d \geq 2$, if there exist infinitely many $t \in \mathbb{C}$ such that both a and b are preperiodic under the action of $z \mapsto z^d + t$, then $a^d = b^d$. Several results followed (see [BD13, GHT13, GHT15, DWYa, DWYb, GKN, GKNY, GNY]), each time

2010 *Mathematics Subject Classification.* Primary: 37P05 Secondary: 37P30, 11G50, 14G40.

Key words and phrases. Arithmetic dynamics, unlikely intersection, canonical heights, equidistribution.

The first author was partially supported by NSERC. The second author was partially supported by NSC Grant 102-2115-M-003-002-MY2 and he also acknowledges the support from NCTS. The third author was partially supported by NSF Grants DMS-0854839 and DMS-1200749.

the setting being the following: given two starting points for two families of (one-parameter) algebraic dynamical systems, there exist infinitely many parameters (or more generally, a Zariski dense set of parameters, as considered in [GHT15]) for which both points are preperiodic at the same time if and only if there is a (precise, global) relation between the two families of dynamical systems and the two starting points.

We note that all results known so far regarding dynamical unlikely intersection problems are in the context of simultaneous preperiodicity of two points in a one-parameter families of dynamical systems, except for [GHT15, Theorem 1.4] which is the first instance regarding dynamical systems under the action of a family of endomorphisms of $\mathbb{P}^2$ parameterized by points of a higher dimensional variety. One might ask more generally for dynamical unlikely intersection problems involving the simultaneous preperiodicity of $n+1$ points in an n-parameter family of dynamical systems, where n is any positive integer. In this paper, we consider the general family of polynomial maps on $\mathbb{P}^1$ of degree $d \geq 3$ in *normal form* (i.e., polynomials of the form $z^d + a_{d-2}z^{d-2} + \cdots + a_0$ with parameters $a_{d-2}, \dots, a_0$). The dimension of the space of such maps is $d-1$. We pose the following question about simultaneous preperiodicity of d constant points for polynomials in this family.

Question 1.1. *Let $d \geq 3$ be an integer, let $c_1, \dots, c_d$ be distinct complex numbers, and let $\mathbf{f_a}(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$ be a family of degree d polynomials in normal form parametrized by $\mathbf{a} = (a_{d-2}, \dots, a_0) \in \mathbb{A}^{d-1}(\mathbb{C})$. Is it true that the set of parameters $\mathbf{a}$ such that each c_i is preperiodic under the action of the polynomial $\mathbf{f_a}$ is not Zariski dense in $\mathbb{A}^{d-1}$?*

Remark 1.2. In the case where $d = 2$, it follows from the main result of [BD11] that the set of complex numbers t such that c_1, c_2 are preperiodic under the action of the polynomial $\mathbf{f}_t(z) = z^2 + t$ is Zariski dense in $\mathbb{A}^1$ if and only if $c_1^2 = c_2^2$. Hence, in this case the set of parameters t such that both $c, -c$ (which are distinct if $c \neq 0$) are preperiodic under the action of $\mathbf{f}_t$ is Zariski dense in the complex affine line.

Remark 1.3. We note that Question 1.1 is indeed meaningful when we deal with d starting points c_i since when dealing with at most $d-1$ starting points, then there is a Zariski dense set of parameters $(a_{d-2}, \dots, a_0) \in \mathbb{A}^{d-1}(\mathbb{C})$ such that each starting point is preperiodic under the action of $\mathbf{f_a}(z)$. This assertion follows from the main result of [GNT] (see also Remark 5.2 which provides more explanation in the special case $d = 3$). On the other hand, if we were to consider more than d starting points, then obviously we expect to have *very few* parameters $\mathbf{a}$ such that all starting points are preperiodic under the action of $\mathbf{f_a}$. It would be interesting to determine the *smallest* number ℓ such that for *any* ℓ starting points there are only finitely many parameters $(a_{d-2}, \dots, a_0)$ with the property that each of the ℓ starting points is preperiodic under the action of $\mathbf{f_a}$.

In this paper we are able to answer positively Question 1.1 when $d = 3$; actually, we can prove a stronger result, as follows.

Theorem 1.4. *Let $c_1, c_2, c_3 \in \mathbb{C}$ be distinct complex numbers, and let $d \geq 3$ be an integer. Then the set of all pairs $(a_1, a_0) \in \mathbb{C} \times \mathbb{C}$ such that each c_i is preperiodic for the action of $z \mapsto z^d + a_1 z + a_0$ is not Zariski dense in $\mathbb{A}^2$.*

Theorem 1.4 implies that there are at most finitely many plane curves containing all pairs of parameters $\mathbf{a} = (a_1, a_0) \in \mathbb{C}^2$ such that all c_i (for $i = 1, 2, 3$) are preperiodic under the action of the polynomial $\mathbf{f_a}(z) = z^d + a_1 z + a_0$. The result is best possible as shown by the following example: if $c \in \mathbb{C}$ is a nonzero number, and $\zeta \in \mathbb{C}$ is a $(d-1)$-st root of unity, then there exist infinitely many $a_1 \in \mathbb{C}$ such that 0, c and $\zeta \cdot c$ are preperiodic for the polynomial $z^d + a_1 z$. The idea is that c is preperiodic for $z^d + a_1 z$ if and only if $\zeta \cdot c$ is preperiodic for $z^d + a_1 z$, and there exist infinitely many $a_1 \in \mathbb{C}$ such that c is preperiodic under the action of $z^d + a_1 z$ (by [GHT13, Proposition 9.1] applied to the family of polynomials $\mathbf{g}_t(z) := z^d + tz$ and the starting point $\mathbf{g}_t(c) = ct + c^d$). There are other more complicated examples showing that the locus of $\mathbf{a} = (a_1, a_0) \in \mathbb{A}^2$ where all the c_i's are preperiodic for $\mathbf{f_a}$ can be 1-dimensional. For example, if $d = 3$ and $c_1 + c_2 + c_3 = 0$, then letting

$$-a_1 := c_1^2 + c_1 c_2 + c_2^2 = c_1^2 + c_1 c_3 + c_3^2 = c_2^2 + c_2 c_3 + c_3^2,$$

we see that $\mathbf{f_a}(c_1) = \mathbf{f_a}(c_2) = \mathbf{f_a}(c_3)$ and thus there are infinitely many $a_0 \in \mathbb{C}$ such that each c_i is preperiodic under the action of $\mathbf{f_a}$ (for $\mathbf{a} = (a_1, a_0)$ with a_1 as above). It is conceivable that *all special curves* for parameters $\mathbf{a} = (a_1, a_0) \in \mathbb{A}^2(\mathbb{C})$ such that each c_i for $1 \leq i \leq 3$ is preperiodic for $\mathbf{f_a}(z)$ is given by *orbit relations* as in the examples described above; however this is a separate and more delicate question.

Also, one cannot expect that Theorem 1.4 can be extended to any 2-parameter family of polynomials and three starting points. Indeed, for any nonzero $c_1 \in \mathbb{C}$ and any $c_2 \in \mathbb{C}$, there exists a Zariski dense set of points $(a_1, a_0) \in \mathbb{A}^2(\mathbb{C})$ such that the points c_1, $-c_1$ and c_2 are preperiodic under the action of the polynomial $z^4 + a_1 z^2 + a_0$. We view the 2-parameter family of polynomials $\mathbf{f}_{a_1, a_0}(z) := z^d + a_1 z + a_0$ as the natural extension of the family of cubic polynomials in normals form, thus explaining why the conclusion of Theorem 1.4 holds for this 2-parameter family of polynomials, while it fails for other 2-parameter families of polynomials. Also, the family of polynomials from Theorem 1.4 is the generalization of the 1-parameter family of polynomials $\mathbf{g}_t(z) := z^d + t$ considered by Baker and DeMarco in [BD11].

Even though we believe Question 1.1 should be true in general, we were not able to fully extend our method to the general case. As we will explain in the next section, there are significant arithmetic complications arising in the last step of our strategy of proof when we deal with families of polynomials depending on more than 2 parameters. On the other hand, the last step in our proof is inductive in that it reduces to applying one-dimensional results of Baker and DeMarco [BD13] to a line in our two-dimensional parameter space. Thus, we are hopeful that there is a more general inductive argument that will allow one to obtain a full result in arbitrary dimension.

We describe briefly the contents of our paper. In Section 2 we discuss the strategy of our proof and also state in Theorem 2.1 a by-product of our proof regarding the variation of the canonical height in an m-parameter family of endomorphisms of $\mathbb{P}^m$ for any $m \geq 2$. In Section 3 we introduce our notation and state the necessary background results used in our proof. In Section 4 we prove Theorem 2.1, and based on our result in Section 5 we prove a general unlikely intersection statement for the dynamics of polynomials in normal form of arbitrary degree (see Theorem 5.1). We conclude in Section 6 by proving Theorem 1.4 using Theorem 5.1.

Acknowledgments. We thank both referees for their many useful comments.

2. Our method of proof

In the section, we give a sketch of the method used in the proof of our main result. We first prove that if there exist a Zariski dense set of points $\mathbf{a} = (a_1, a_0) \in \mathbb{A}^2(\mathbb{C})$ such that c_1, c_2, c_3 are simultaneously preperiodic under the action of

$$\mathbf{f_a}(z) := z^d + a_1 z + a_0,$$

then for *each* point $\mathbf{a} \in \mathbb{A}^2(\mathbb{C})$, if any two of the points c_i are preperiodic under the action of $\mathbf{f_a}$, then also the third point c_i is preperiodic. We prove this statement using the powerful equidistribution theorem of Yuan [Yua08] for generic sequences of points of small height on projective varieties X endowed with a metrized line bundle (we also use the function field version of this equidistribution theorem proven by Gubler [Gub08]). Such equidistribution statements were previously obtained when X is $\mathbb{P}^1$ by Baker-Rumely [BR06] and Favre-Rivera-Letelier [FRL06, FRL04], and when X is an arbitrary curve by Chambert-Loir [CL06] and Thuillier [Thu]. Our method is similar to the one employed in [GHT15] and it extends to polynomials in normal form of arbitrarily degree $d \geq 3$; i.e., by the same technique we prove (see Theorem 5.1) that given d distinct numbers $c_1, \ldots, c_d \in \mathbb{C}$, if there exist a Zariski dense set of points $\mathbf{a} = (a_{d-2}, \ldots, a_0) \in \mathbb{A}^{d-1}(\mathbb{C})$ such that each c_i is preperiodic under the action of $\mathbf{f_a}(z)$, then for *each* $\mathbf{a} \in \mathbb{A}^{d-1}(\mathbb{C})$ such that $d-1$ of the points c_i are preperiodic under the action of $\mathbf{f_a}$, then *all* the d points c_i are preperiodic.

Now, for the 2-parameter family of polynomials $\mathbf{f_a}(z) := z^d + a_1 z + a_0$, assuming there exists a Zariski dense set of points $\mathbf{a} \in \mathbb{A}^2(\mathbb{C})$ such that each c_i (for $i = 1, 2, 3$) is preperiodic under the action of $\mathbf{f_a}$, we consider the line L contained in the parameter space $\mathbb{A}^2$ along which c_1 is fixed by $\mathbf{f_a}(z)$ for each $\mathbf{a} \in L(\mathbb{C})$. Then we have a 1-parameter family of polynomials $\mathbf{g}_t$ (which is $\mathbf{f_a}$ with $\mathbf{a}$ moving along the line L), and moreover, for each parameter t, the point c_2 is preperiodic for $\mathbf{g}_t$ if and only if c_3 is preperiodic for $\mathbf{g}_t$. Applying [BD13, Theorem 1.3] (combined with [Ngu15, Proposition 2.3]), we obtain that $\mathbf{g}_t^m(c_2) = \mathbf{g}_t^m(c_3)$ for some positive integer m. This yields that the starting points c_i are not all distinct, giving a contradiction.

The above argument becomes much more complicated for families of polynomials in normal form parametrized by arbitrary many variables. One could still employ the same strategy and work along the line $L \subset \mathbb{A}^{d-1}$ along which each of c_i, for $i = 1, \dots, d-2$ are fixed by $\mathbf{f_a}$ (with $\mathbf{a} \in L$). Then [BD13, Theorem 1.3] still yields a relation of the form $\mathbf{g}_t^m(c_{d-1}) = \zeta \cdot \mathbf{g}_t^m(c_d)$ (for some root of unity ζ, some positive integer m, where $\mathbf{g}_t$ is $\mathbf{f_a}$ where $\mathbf{a} := (a_{d-2}, \dots, a_0)$ is moving along the line L). However this relation does not seem to be sufficient for deriving the desired conclusion. We suspect that in order to derive a contradiction one would have to analyze more general curves in the parameter space along which $(d-2)$ of the points c_i are persistently preperiodic. However this creates additional problems since one would have to prove a generalization of [BD13, Theorem 1.3] which seems difficult because that result relies (among other ingredients) on a deep theorem of Medvedev-Scanlon [MS14] regarding the shape of periodic plane curves under the action of one-variable polynomials acting on each affine coordinate.

As a by-product of our method we obtain a result on the variation of the canonical height in an m-parameter family of endomorphisms of $\mathbb{P}^m$ defined over a product formula field K (for more details on product formula fields, see Section 3). The family of endomorphisms of $\mathbb{P}^m$ we consider here is a product of the family of polynomials $\mathbf{f_t}(z) = z^d + t_1 z^{m-1} + t_2 z^{m-2} + \cdots + t_m$ where $d > m \geq 2$ and $t_1, \dots, t_m$ are parameters. Let $\phi := \mathbf{f_t} \times \cdots \times \mathbf{f_t} : \mathbb{A}^m \to \mathbb{A}^m$ and extend ϕ to a degree d rational map $\Phi : \mathbb{P}^m \to \mathbb{P}^m$. More precisely, let $\mathbf{X} := [X_m : X_{m-1} : \cdots : X_0]$ be a homogeneous set of coordinates on $\mathbb{P}^m$ and let $\Phi_i(\mathbf{X}) = X_0^d \mathbf{f_t}(X_i/X_0)$ for $i = m, \dots, 1$. Then, with respect to the homogeneous coordinates $\mathbf{X}$ we have $\Phi(\mathbf{X}) = [\Phi_m(\mathbf{X}) : \cdots : \Phi_1(\mathbf{X}) : X_0^d]$. It is easy to verify that Φ is actually a morphism on $\mathbb{P}^m$. In the following result, when we specialize our parameter $\mathbf{t} = (t_1, \dots, t_m)$ to $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, we write $\Phi_{\boldsymbol{\lambda}} : \mathbb{P}^m \to \mathbb{P}^m$ for the corresponding (specialized) morphism on $\mathbb{P}^m$.

Theorem 2.1. *Let $d > m \geq 2$ be integers, let K be a number field or a function field of finite transcendence degree over another field, and let $\Phi : \mathbb{P}^m \longrightarrow \mathbb{P}^m$ be the m-parameter family of endomorphisms defined as above. Let $c_1, \dots, c_m \in \overline{K}$ be distinct elements, and let $P := [c_m : \cdots : c_1 : 1] \in \mathbb{P}^m(\overline{K})$. Then for each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, we have the canonical height $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(P)$ constructed with respect to the endomorphism $\Phi_{\boldsymbol{\lambda}}$ of $\mathbb{P}^m$ defined over $\overline{K}$, and also we have the canonical height $\widehat{\mathrm{h}}_{\Phi}(P)$ constructed with respect to the endomorphism Φ of $\mathbb{P}^m$ defined over $\overline{K}(t_1, \dots, t_m)$. Then*

$$\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(P) = \widehat{\mathrm{h}}_{\Phi}(P) \cdot h\left((\lambda_1, \dots, \lambda_m)\right) + O(1), \tag{2.1.1}$$

where $h((\lambda_1,\dots,\lambda_m))$ *is the Weil height of the point* $(\lambda_1,\dots,\lambda_m)\in\mathbb{A}^m(\overline{K})$ *and the constant in* $O(1)$ *depends on* $c_1,\dots,c_m$ *only.*

It is essential in Theorem 2.1 that the c_i's are distinct. Indeed, assume $m=2$ and $c_1=c_2=c\in K$. Then for each $\lambda_1,\lambda_2\in\overline{K}$ satisfying

$$c^d+\lambda_1 c+\lambda_2=c, \tag{2.1.2}$$

we have that the point $P:=[c:c:1]$ is preperiodic under the action of $\Phi_{\lambda_1,\lambda_2}$ and thus $\widehat{\mathrm{h}}_{\Phi_{\lambda_1,\lambda_2}}(P)=0$. On the other hand, $\widehat{\mathrm{h}}_{\Phi}(P)=1/d$ (after an easy computation using degrees on the generic fiber of Φ) and thus (2.1.1) cannot hold because there are points $(\lambda_1,\lambda_2)\in\mathbb{A}^2(\overline{K})$ satisfying (2.1.2) of arbitrarily large height.

Theorem 2.1 is an improvement of a special case of Call-Silverman's general result [CS93] for the variation of the canonical height in arbitrary families of polarizable endomorphisms Φ_t of projective varieties X parametrized by $t\in T$ (for some base scheme T). In the case where the base variety T is a curve, Call and Silverman [CS93, Theorem 4.1] have shown that for $P\in X(\overline{\mathbb{Q}})$, then

$$\widehat{\mathrm{h}}_{\Phi_t}(P)=\widehat{\mathrm{h}}_{\Phi}(P)h(t)+o(h(t)) \tag{2.1.3}$$

as we vary $t\in T(\overline{\mathbb{Q}})$ where $h(\cdot)$ is a height function associated to a degree one divisor on T. Their result generalizes a result of Silverman [Sil83] on heights of families of abelian varieties. In a recent paper [Ing13], Ingram improves the error term to $O(1)$ when T is a curve, X is $\mathbb{P}^1$, and the family of endomorphisms Φ is totally ramified at infinity (i.e., Φ is a polynomial mapping). This result is an analogue of Tate's theorem [Tat83] in the setting of arithmetic dynamics.

In order to use Yuan's equidistribution theorem [Yua08] (and same for Gubler's extension [Gub08] to the function field setting) for points of small height to our situation, the error term in (2.1.3) needs to be controlled within $O(1)$ when Φ is an endomorphism of $\mathbb{P}^m$ as in Theorem 2.1. There are only a few results in the literature when the error term in (2.1.3) is known to be $O(1)$. Besides Tate's [Tat83] and Silverman's [Sil83] in the context of elliptic curves and more generally abelian varieties (see also the further improvements of Silverman [Sil94a, Sil94b] in the case of elliptic curves), there are only a few known results, all valid for 1-parameter families (see [Ing13, Ing, GHT15, GM13]). To our knowledge, Theorem 2.1 is the *first* result in the literature where one improves the error term in (2.1.3) to $O(1)$ for a higher dimensional parameter family of endomorphisms of $\mathbb{P}^m$.

3. Notation

In this section we setup the notation used in our paper.

3.1. **Maps and preperiodic points.** Let $\Phi : X \longrightarrow X$ be a self-map on some set X. As always in dynamics, we denote by Φ^n the n-th compositional iterate of Φ with itself. We denote by $\mathrm{id} := \mathrm{id}|_X$ the identity map on X.

For any quasiprojective variety X endowed with an endomorphism Φ, we call a point $x \in X$ *preperiodic* if there exist two distinct nonnegative integers m and n such that $\Phi^m(x) = \Phi^n(x)$. If $x = \Phi^n(x)$ for some positive integer n, then x is a *periodic* point of *period* n. For more details, we refer the reader to the comprehensive book [Sil07] of Silverman on arithmetic dynamics.

3.2. **Absolute values on product formula fields.** A product formula field K comes equipped with a standard set Ω_K of absolute values $|\cdot|_v$ which satisfy a product formula, i.e,

$$\prod_{v\in\Omega_K} |x|_v^{N_v} = 1 \quad \text{for every } x \in K^*, \tag{3.0.4}$$

where $N\colon \Omega_K \to \mathbb{N}$ and $N_v := N(v)$ (see [Lan83] for more details).

The typical examples of product formula fields are

(1) number fields; and
(2) function fields K of finite transcendence degree over some field F.

In the case of function fields K, one associates the absolute values in Ω_K to the irreducible divisors of a smooth, projective variety $\mathcal{V}$ defined over the constant field F such that K is the function field of $\mathcal{V}$; for more details, see [Lan83] and [BG06]. In the special case $K = F(t_1, \dots, t_m)$, we may take $\mathcal{V} = \mathbb{P}^m$.

As a convention, in order to simplify the notation in this paper, a product formula field is always either a number field or a function field over a constant field.

Let K be a product formula field. We fix an algebraic closure $\overline{K}$ of K; if K is a function field of finite transcendence degree over another field F (which we call the constant field), then we also fix an algebraic closure $\overline{F}$ of F inside $\overline{K}$. Let $v \in \Omega_K$. Let $\mathbb{C}_v$ be the completion of a fixed algebraic closure of the completion of $(K, |\cdot|_v)$. When v is an archimedean valuation, then $\mathbb{C}_v = \mathbb{C}$. We use the same notation $|\cdot|_v$ to denote the extension of the absolute value of $(K, |\cdot|_v)$ to $\mathbb{C}_v$ and we also fix an embedding of $\overline{K}$ into $\mathbb{C}_v$.

3.3. **The Weil height.** Let $m \geq 1$, and let L be a finite extension of the product formula field K. The (naive) Weil height $h(\cdot)$ of any point $P := [x_m : \cdots : x_0] \in \mathbb{P}^m(L)$ is defined as

$$h(P) = \frac{1}{[L:K]} \sum_{v\in\Omega_K} N_v \cdot \sum_{\substack{\sigma: L \hookrightarrow \overline{K} \\ \sigma|_K = \mathrm{id}}} \log\left(\max\{|x_m|_v, \cdots, |x_0|_v\}\right).$$

So, the above inner sum is over all possible embeddings of L into $\overline{K}$ which fix K pointwise, *counted appropriately.* Indeed, here and also later in our proof, a sum similar with the inner sum above counts each embedding $\sigma : L \hookrightarrow \overline{K}$

(fixing K pointwise) with the multiplicity $[L : L_0]$, where L_0 is the separable closure of K inside L. Also, one can check that the above definition of height does not depend on the particular choice of the field L containing each x_i. We also use the notation $h((x_m, \dots, x_1)) := h([x_m : \cdots : x_1 : 1])$ to denote the height of the point $(x_m, \dots, x_1)$ in the affine space $\mathbb{A}^m$ embedded in the usual way in $\mathbb{P}^m$.

In the special case of the function field $K = F(t_1, \dots, t_\ell)$, for a point $P = [x_m : \cdots : x_0] \in \mathbb{P}^m(K)$, assuming each $x_i \in F[t_1, \dots, t_\ell]$ and moreover, the polynomials x_i are coprime, then $h(P) = \max_{i=0}^m \deg(x_i)$, where $\deg(\cdot)$ is the total degree function on $F[t_1, \dots, t_\ell]$.

3.4. **Canonical heights.** Let $m \geq 1$, and let $f : \mathbb{P}^m \longrightarrow \mathbb{P}^m$ be an endomorphism of degree $d \geq 2$. In [CS93], Call and Silverman defined the *global canonical height* $\widehat{\mathrm{h}}_f(x)$ for each $x \in \mathbb{P}^m(\overline{K})$ as

$$\widehat{\mathrm{h}}_f(x) = \lim_{n \to \infty} \frac{h(f^n(x))}{d^n}. \tag{3.0.5}$$

If K is a number field, then using Northcott's Theorem one deduces that x is preperiodic for f if and only if $\widehat{\mathrm{h}}_f(x) = 0$. This statement does not hold if K is a function field over a constant field F (which is not a subfield of some $\overline{\mathbb{F}_p}$) since $\widehat{\mathrm{h}}_f(x) = 0$ for all $x \in F$ if f is defined over F. However, as proven by Benedetto [Ben05] and Baker [Bak09], this is essentially the only counterexample.

3.5. **Canonical height over function fields.** In order to state the results of Baker and Benedetto, we first define isotrivial polynomials.

Definition 3.1. *We say a polynomial $f \in K[z]$ is* isotrivial *over F if there exists a linear $\ell \in \overline{K}[z]$ such that $\ell \circ f \circ \ell^{-1} \in \overline{F}[z]$.*

Benedetto proved that a non-isotrivial polynomial has nonzero canonical height at its non-preperiodic points [Ben05, Thm. B]. As stated, Benedetto's result applies only to function fields of transcendence dimension one, but the proof extends easily to function fields of any transcendence dimension. Baker [Bak09] later generalized the result to the case of rational functions over arbitrary product formula fields.

Lemma 3.2 (Benedetto [Ben05], Baker [Bak09])**.** *Let $f \in K[z]$ with $\deg(f) \geq 2$, and let $x \in \overline{K}$. If f is non-isotrivial over F, then $\widehat{\mathrm{h}}_f(x) = 0$ if and only if x is preperiodic for f.*

A crucial observation for our paper is that a polynomial in normal form is isotrivial if and only if it is defined over the constant field; the following result is proven in [GHT13, Lemma 10.2].

Proposition 3.3. *Let $f \in K[z]$ be a polynomial in normal form. Then f is isotrivial over F if and only if $f \in \overline{F}[z]$.*

4. Proof of the specialization theorem

In this section we prove Theorem 2.1. So, we work with the following setup:

- $d > m \geq 2$ are integers.
- K is a product formula field.
- For algebraically independent variables $t_1, \dots, t_m$ we define
$$\mathbf{f}(z) := z^d + t_1 z^{m-1} + \cdots + t_{m-1} z + t_m.$$
Let $\Phi : \mathbb{P}^m \longrightarrow \mathbb{P}^m$ be the map on $\mathbb{P}^m$ defined by
$$\Phi([X_m : \cdots : X_1 : X_0]) = \left[X_0^d \mathbf{f}\left(\frac{X_m}{X_0}\right) : \cdots : X_0^d \mathbf{f}\left(\frac{X_1}{X_0}\right) : X_0^d\right].$$
It is straightforward to verify that Φ is a morphism on $\mathbb{P}^m$ over $K(t_1, \dots, t_m)$.
- When we specialize each t_i to some $\lambda_i \in \overline{K}$, we use the notation $\mathbf{f}_{\boldsymbol{\lambda}}(z) := z^d + \lambda_1 z^{m-1} + \cdots + \lambda_{m-1} z + \lambda_m$ and $\Phi_{\boldsymbol{\lambda}}$ to denote the corresponding specialized polynomial and endomorphism of $\mathbb{P}^m$ respectively, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$.
- Let $\mathbf{c} := (c_m, \dots, c_1) \in \mathbb{A}^m(K)$ where the c_i's are distinct. We denote the point $[c_m : \dots : c_1 : 1] \in \mathbb{P}^m(K)$ by $\tilde{\mathbf{c}}$.
- For each $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, we let $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}$ be the canonical height corresponding to the endomorphism $\Phi_{\boldsymbol{\lambda}}$ defined over $\overline{K}$; also we let $\widehat{\mathrm{h}}_{\Phi}$ be the canonical height corresponding to the endomorphism Φ defined over the function field $K(t_1, \dots, t_m)$.
- Let $v \in \Omega_K$. The (v-adic) *norm* of a point $P = [x_m : \cdots : x_1 : x_0] \in \mathbb{P}^m(\mathbb{C}_v)$ with $x_0 \neq 0$ is defined by $\|P\| := \max\{|x_m/x_0|_v, \dots, |x_0/x_0|_v\}$. Also, for a point $Q = (a_m, \dots, a_1) \in \mathbb{A}^m(\mathbb{C}_v)$, we define its norm $\|Q\|_v := \max\{|a_m|_v, \dots, |a_1|_v, 1\}$; it is clear that $\|Q\|_v = \|\tilde{Q}\|_v$, where $\tilde{Q} := [a_m : \cdots : a_1 : 1]$. For a polynomial
$$g(x_1, \dots, x_m) = \sum a_{i_1, \dots, i_m} x_1^{i_1} \cdots x_m^{i_m} \in \mathbb{C}_v[x_1, \dots, x_m],$$
the norm of g is defined by $\|g\|_v := \max_{i_1, \dots, i_m}\{|a_{i_1, \dots, i_m}|_v\}$. Similarly, for a morphism $\Psi = [\psi_m : \dots : \psi_1 : \psi_0] : \mathbb{P}^m \to \mathbb{P}^m$ over $\mathbb{C}_v$, we set $\|\Psi\|_v := \max_i\{\|\psi_i\|_v\}/\|\psi_0\|_v$.
- By abuse of the notation, we simply write $\Phi^n(\mathbf{c})$ for $\Phi^n(\tilde{\mathbf{c}})$ and similarly, we let $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c})$ denote the canonical height of the point $\tilde{\mathbf{c}}$ etc.

For each $n \geq 0$ and each $i = 1, \dots, m$ we define $A_{n,i}(t_1, \dots, t_m)$ such that
$$\Phi^n(\mathbf{c}) := [A_{n,m} : \cdots : A_{n,1} : 1].$$
Then $A_{0,i} = c_i$ for each $i = 1, \dots, m$ and for general $n \geq 0$:
$$A_{n+1,i}(t_1, \dots, t_m) := \mathbf{f}(A_{n,i}(t_1, \dots, t_m)).$$

By induction, we see that the total degree $\deg(A_{n,i})$ in the variables $t_1, \dots, t_m$ equals d^{n-1}; so $\widehat{\mathrm{h}}_\Phi(\mathbf{c}) = \frac{1}{d}$ (see (3.0.5)). Therefore (2.1.1) reduces to proving

$$\widehat{\mathrm{h}}_{\Phi_{\boldsymbol\lambda}}(\mathbf{c}) = \frac{h(\boldsymbol\lambda)}{d} + O(1), \quad \boldsymbol\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K}). \tag{4.0.1}$$

Note that by our convention mentioned above, we have

$$\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v := \max\{1, |A_{n,1}(\lambda_1, \dots, \lambda_m)|_v, \cdots, |A_{n,m}(\lambda_1, \dots, \lambda_m)|_v\}.$$

To ease the notation, in the following discussion we simply denote $A_{n,i}(\lambda_1, \dots, \lambda_m)$ by $A_{n,i}$ when $\lambda_1, \dots, \lambda_m$ are fixed.

Using the definition of $\Phi^n(\mathbf{c})$ which yields that each $A_{n,i}$ has total degree d^{n-1} in $\boldsymbol\lambda$ and also degree at most d^n in $\mathbf{c}$, we obtain an upper bound for $\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v$ when v is a nonarchimedean place of K.

$$\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v \le \|\boldsymbol\lambda\|_v^{d^{n-1}} \|\mathbf{c}\|_v^{d^n} \tag{4.0.2}$$

We first observe that from the definition of the v-adic norm of a point, we have that always the v-adic norm of a point is at least equal to 1, i.e.,

$$\|\mathbf{c}\|_v \ge 1 \text{ and } \|\boldsymbol\lambda\|_v \ge 1. \tag{4.0.3}$$

Next we prove a couple of easy lemmas.

Lemma 4.1. *Let $\boldsymbol\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, and let $|\cdot|_v$ be a nonarchimedean absolute value such that $\|\mathbf{c}\|_v = 1$ and $\|\boldsymbol\lambda\|_v = 1$. Then $\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v = 1$ for each $n \ge 0$.*

Proof. The result follows using (4.0.2) and also that (just as for any point; see for example, (4.0.3)) $\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v \ge 1$ for every $n \ge 0$. □

Lemma 4.2. *Let $|\cdot|_v$ be a nonarchimedean absolute value such that*

(i) $\|\mathbf{c}\|_v = 1$*; and*
(ii) $|c_i - c_j|_v = 1$ *for each* $1 \le i < j \le m$.

Then for each $\boldsymbol\lambda \in \mathbb{A}^m(\overline{K})$, and for each $n \ge 1$ we have

$$\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v = \|\boldsymbol\lambda\|_v^{d^{n-1}}. \tag{4.2.1}$$

Proof. First, we note by Lemma 4.1 (see condition (i) above) that if $\|\boldsymbol\lambda\|_v = 1$, then $\|\Phi^n_{\boldsymbol\lambda}(\mathbf{c})\|_v = 1$ as claimed in the above conclusion. So, from now on, we assume that $\|\boldsymbol\lambda\|_v > 1$. Hence $|\lambda_i|_v > 1$ for some $i = 1, \dots, m$.

For $n = 1$, we note that

$$\begin{cases} c_m^{m-1}\lambda_1 + \cdots + c_m\lambda_{m-1} + \lambda_m & = & A_{1,m} - c_m^d \\ \cdots\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots \\ c_1^{m-1}\lambda_1 + \cdots + c_1\lambda_{m-1} + \lambda_m & = & A_{1,1} - c_1^d \end{cases} \tag{4.2.2}$$

seen as a system with unknowns $\lambda_1, \dots, \lambda_m$ has the determinant equal with a van der Monde determinant which is a v-adic unit (see condition (ii) above). Therefore, using also that $|c_i|_v \le 1$, we get that

$$|\lambda_i|_v \le \max\{1, |A_{1,1}|_v, \dots, |A_{1,m}|_v\} \quad i = 1, \dots, m.$$

Thus, $\|\boldsymbol{\lambda}\|_v \le \|\Phi_{\boldsymbol{\lambda}}(\mathbf{c})\|_v$. Combining this last inequality with (4.0.2) for $n = 1$, we conclude that $\|\Phi_{\boldsymbol{\lambda}}(\mathbf{c})\|_v = \|\boldsymbol{\lambda}\|_v$.

Now, for $n > 1$, we argue by induction on n. So, assume (4.2.1) holds for $n = k \ge 1$, and we prove the same equality holds for $n = k + 1$. By induction hypothesis and using the fact that $d > m$, we have that for each $j = 0, \dots, m-1$,

$$|A_{k,i}^j \lambda_{m-j}|_v \le \|\boldsymbol{\lambda}\|_v^{j \cdot d^{k-1}} \|\boldsymbol{\lambda}\|_v < \|\boldsymbol{\lambda}\|_v^{d^k}. \tag{4.2.3}$$

For the last inequality we also use the fact that $d > m > j$ and that $\|\boldsymbol{\lambda}\|_v > 1$. So, as $A_{k+1,i} = \mathbf{f}(A_{k,i})$ for $k \ge 0$, we obtain that

$$|A_{k+1,i}|_v \le \max\left\{|A_{k,i}|_v^d, \max_{0 \le j \le m-1} |A_{k,i}^j \lambda_{m-j}|_v\right\} \le \|\boldsymbol{\lambda}\|_v^{d^k}.$$

On the other hand, since $\|\Phi_{\boldsymbol{\lambda}}^k(\mathbf{c})\|_v = \|\boldsymbol{\lambda}\|_v^{d^{k-1}} > 1$ by the induction hypothesis, there exists some $i = 1, \dots, m$ such that $|A_{k,i}|_v = \|\boldsymbol{\lambda}\|_v^{d^{k-1}}$ and so, for that index i (using (4.2.3)), we have $|A_{k+1,i}|_v = |A_{k,i}|_v^d = \|\boldsymbol{\lambda}\|_v^{d^k}$, as claimed. □

Let $S \subset \Omega_K$ consist of all the archimedean places of K and all the places v which do *not* satisfy at least one of the two conditions (i) and (ii) from Lemma 4.2. It is clear that the set S is finite (note that $c_i \ne c_j$ for $i \ne j$ and thus condition (ii) from Lemma 4.2 is satisfied by all but finitely many places v).

Lemma 4.3. *Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, and let L be a finite extension of K containing $\lambda_1, \dots, \lambda_m$. Then we have*

$$\begin{aligned} &\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}) - \widehat{\mathrm{h}}_{\Phi}(\mathbf{c}) \cdot h(\Phi_{\boldsymbol{\lambda}}(\mathbf{c})) \\ &= \sum_{v \in S} \frac{N_v}{[L:K]} \cdot \sum_{\substack{\sigma: L \hookrightarrow \overline{K} \\ \sigma|_K = \mathrm{id}}} \left(\lim_{n \to \infty} \frac{\log \|\Phi_{\sigma(\boldsymbol{\lambda})}^n(\mathbf{c})\|_v}{d^n} - \frac{\log \|\Phi_{\sigma(\boldsymbol{\lambda})}(\mathbf{c})\|_v}{d} \right). \end{aligned}$$

Proof. We have

$$\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}) = \lim_{n \to \infty} \sum_{v \in \Omega_K} \frac{N_v}{[L:K]} \cdot \sum_{\substack{\sigma: L \hookrightarrow \overline{K} \\ \sigma|_K = \mathrm{id}}} \frac{\log \|\Phi_{\sigma(\boldsymbol{\lambda})}^n(\mathbf{c})\|_v}{d^n}.$$

Lemma 4.1 yields that for all but finitely many absolute values $|\cdot|_v$ we have that $\|\Phi_{\sigma(\boldsymbol{\lambda})}^n(\mathbf{c})\|_v = 1$ for each $\sigma \in \mathrm{Gal}(L/K)$. So, we can interchange the above limit with the sum formula and get

$$\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}) = \sum_{v \in \Omega_K} \frac{N_v}{[L:K]} \cdot \sum_{\substack{\sigma: L \hookrightarrow \overline{K} \\ \sigma|_K = \mathrm{id}}} \lim_{n \to \infty} \frac{\log \|\Phi_{\sigma(\boldsymbol{\lambda})}^n(\mathbf{c})\|_v}{d^n}.$$

Lemma 4.2 finishes the proof of Lemma 4.3 □

The next result is the key technical step which allows us to deal with the *potentially bad* places $v \in S$ for proving (4.0.1).

Lemma 4.4. *Let $v \in S$. There exists a constant $C(v, \mathbf{c})$ depending only on the absolute value $|\cdot|_v$ and on the point $\mathbf{c}$ such that for each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, and for each positive integers $n_2 > n_1$, we have*

$$\left| \frac{\log \|\Phi_{\boldsymbol{\lambda}}^{n_2}(\mathbf{c})\|_v}{d^{n_2}} - \frac{\log \|\Phi_{\boldsymbol{\lambda}}^{n_1}(\mathbf{c})\|_v}{d^{n_1}} \right| < \frac{C(v, \mathbf{c})}{d^{n_1}}.$$

Proof. Since we will fix the place v and $\lambda_1, \dots, \lambda_m$ and $|\cdot|_v$, we simply denote $M_n := \|\Phi_{\boldsymbol{\lambda}}^n(\mathbf{c})\|_v$. Similarly, as stated above, we will use the notation $A_{n,i} := A_{n,i}(\lambda_1, \dots, \lambda_m)$ for $i = 1, \dots, m$. We split the analysis based on whether there exists at least one λ_i with large absolute value or not.

Claim 4.5. *Let L be any real number larger than 1. If $\|\boldsymbol{\lambda}\|_v \le L$, then*

$$\frac{1}{2^d L^d} \le \frac{M_{n+1}}{M_n^d} \le (m+1)L,$$

for all $n \ge 1$.

Proof of Claim 4.5. Now, by definition, $M_n \ge 1$; so, using also that $L > 1$, we get that for each $i = 1, \dots, m$ we have

$$|A_{n+1,i}|_v \le |A_n|_v^d + \sum_{j=1}^m |\lambda_j|_v \cdot |A_n|_v^{m-j} \le (m+1)L \cdot M_n^d.$$

Thus $M_{n+1} = \max\{1, |A_{n+1,1}|_v, \dots, |A_{n+1,m}|_v\} \le (m+1)L \cdot M_n^d$. This proves the existence of the upper bound in Claim 4.5.

For the proof of the existence of the lower bound, we split our analysis into two cases:

Case 1. $M_n \le 2L$.

In this case, using that $M_{n+1} \ge 1$, we immediately obtain that

$$\frac{M_{n+1}}{M_n^d} \ge \frac{1}{2^d L^d}. \tag{4.5.1}$$

Case 2. $M_n > 2L$.

Let $j \in \{1, \dots, m\}$ such that $|A_{n,j}|_v = M_n$; then

$$\begin{aligned}
|A_{n+1,j}|_v &= |A_{n,j}^d + \sum_{i=1}^m A_{n,j}^{m-i} \lambda_i|_v \\
&\ge |A_{n,j}|_v^d - \sum_{i=1}^m |A_{n,j}|_v^{m-i} \cdot |\lambda_i|_v \\
&\ge |A_{n,j}|_v^d \cdot \left(1 - \sum_{i=1}^m \frac{|\lambda_i|_v}{|A_{n,j}|_v^{d-m+i}}\right) \\
&\ge M_n^d \cdot \left(1 - \sum_{i=1}^m \frac{L}{M_n^{i+1}}\right) \text{ since } \|\boldsymbol{\lambda}\|_v \le L, |A_{n,j}|_v = M_n \text{ and } m < d,
\end{aligned}$$

$$\geq M_n^d \cdot \left(1 - \frac{1}{2}\right) \text{ since } M_n > 2L,$$
$$\geq \frac{1}{2} \cdot M_n^d.$$

Since $M_{n+1} \geq |A_{n+1,j}|_v$, the above inequality coupled with inequality (4.5.1) yields the lower bound from the conclusion of Claim 4.5. □

We continue the proof of Lemma 4.4. We solve for the λ_i's in terms of the $A_{1,i}$'s from the system (4.2.2) and obtain that for each $k = 1, \dots, m$ we have

$$\lambda_k = \frac{Q_{k,0}(c_1, \dots, c_m) + \sum_{i=1}^m Q_{k,i}(c_1, \dots, c_m) \cdot A_{1,i}}{\prod_{1 \leq i < j \leq m}(c_i - c_j)}, \tag{4.5.2}$$

where each $Q_{k,i}(X_1, \dots, X_m)$ is a polynomial of degree at most d in each variable X_i.

Let L_0 be a real number satisfying the following inequalities:

(1) $L_0 \geq (m+1)(d+1)^m \cdot \|\mathbf{c}\|_v^{dm}$;
(2) L_0 is larger than the v-adic absolute value of each coefficient of each $Q_{k,i}$, for $k = 1, \dots, m$ and $i = 0, \dots, m$;
(3) $L_0 \geq \dfrac{1}{\prod\limits_{1 \leq i < j \leq m} |c_i - c_j|_v}$.

Claim 4.6. *Let $L \geq 4L_0^6$ be a real number. Then for each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\mathbb{C}_v)$ such that $\|\boldsymbol{\lambda}\|_v > L$, we have*

$$\frac{1}{2} \leq \frac{M_{n+1}}{M_n^d} \leq 2,$$

for each $n \geq 1$.

Proof of Claim 4.6. First we prove that $M_1 \geq \frac{\|\boldsymbol{\lambda}\|_v}{L_0^3}$. Note that by our choice of L_0, the triangle inequality gives

$$|Q_{k,i}(c_1, \dots, c_m)|_v \leq L_0(d+1)^m \|\mathbf{c}\|_v^{dm}.$$

Using (4.5.2) (coupled with inequalities (1)-(3) for L_0), we get that

$$|\lambda_k|_v \leq \frac{(m+1)L_0 \cdot (d+1)^m \cdot \|\mathbf{c}\|^{dm} \cdot M_1}{\prod_{1 \leq i < j \leq m} |c_i - c_j|_v} \leq L_0^3 M_1. \tag{4.6.1}$$

Thus, $\|\boldsymbol{\lambda}\|_v = \max\{1, |\lambda_1|_v, \dots, |\lambda_m|_v\} \leq L_0^3 M_1$ as first claimed.

We prove by induction that for each $n \geq 1$, we have $M_n \geq \frac{\|\boldsymbol{\lambda}\|_v}{L_0^3}$. We already established the inequality for $n = 1$. Now, assume $M_n \geq \frac{\|\boldsymbol{\lambda}\|_v}{L_0^3}$ and we show next that

$$M_{n+1} \geq \frac{M_n^d}{2} \geq \frac{\|\boldsymbol{\lambda}\|_v}{L_0^3} > 2. \tag{4.6.2}$$

Note that the last inequality from (4.6.2) follows from the fact that $\|\boldsymbol{\lambda}\|_v > L \ge 4L_0^6 > 2L_0^3$. Without loss of generality, we may assume $|A_{n,i}|_v = M_n$ for some i, and so by the induction hypothesis $|A_{n,i}|_v = M_n \ge \frac{\|\boldsymbol{\lambda}\|_v}{L_0^3} \ge 2$. Now,

$$\begin{aligned}
M_{n+1} &\ge |A_{n+1,i}|_v \\
&\ge |A_{n,i}|_v^d - \sum_{j=1}^m |\lambda_j|_v \cdot |A_{n,i}|_v^{m-j} \\
&\ge |A_{n,i}|_v^d \cdot \left(1 - \sum_{j=1}^m \frac{|\lambda_j|_v}{|A_{n,i}|_v^{d-m+j}}\right) \\
&\ge |A_{n,i}|_v^d \cdot \left(1 - \frac{2\|\boldsymbol{\lambda}\|_v}{|A_{n,i}|_v^2} \sum_{j=1}^m \frac{|A_{n,i}|_v^2}{2|A_{n,i}|_v^{d-m+j}}\right) \text{ since } |\lambda_i|_v \le \|\boldsymbol{\lambda}\|_v, \\
&\ge |A_{n,i}|_v^d \cdot \left(1 - \frac{2\|\boldsymbol{\lambda}\|_v}{|A_{n,i}|_v^2}\right) \text{ since } |A_{n,i}|_v \ge 2, \\
&\ge |A_{n,i}|_v^d \cdot \left(1 - \frac{2L_0^6}{\|\boldsymbol{\lambda}\|_v}\right) \text{ by the induction hypothesis,} \\
&\ge |A_{n,i}|_v^d \cdot \left(1 - \frac{2L_0^6}{L}\right) \text{ by the hypothesis of Claim 4.6} \\
&\ge \frac{|A_{n,i}|_v^d}{2} = \frac{M_n^d}{2} \\
&\ge \frac{\|\boldsymbol{\lambda}\|_v^d}{2L_0^{3d}} \\
&\ge \frac{\|\boldsymbol{\lambda}\|_v}{L_0^3} \text{ because } \|\boldsymbol{\lambda}\|_v > L \ge 4L_0^6.
\end{aligned}$$

We note that the first inequality from (4.6.2) already yields the lower bound from the conclusion of Claim 4.6.

Next we prove that for all $n \ge 1$, we have

$$\frac{M_{n+1}}{M_n^d} \le 2. \tag{4.6.3}$$

Again, without loss of generality, we may assume $|A_{n+1,i}|_v = M_{n+1}$. Then using inequality (4.6.2), we get

$$\begin{aligned}
M_{n+1} = |A_{n+1,i}|_v &\le |A_{n,i}|_v^d + \sum_{j=1}^m |\lambda_j|_v \cdot |A_{n,i}|_v^{m-j} \\
&\le M_n^d + \sum_{j=1}^m |\lambda_j|_v \cdot M_n^{m-j}
\end{aligned}$$

$$
\begin{aligned}
&\le M_n^d \cdot \left(1 + \sum_{j=1}^{m} \frac{|\lambda_j|_v}{M_n^{d-m+j}}\right) \\
&\le M_n^d \cdot \left(1 + \frac{2\|\boldsymbol{\lambda}\|_v}{M_n^2}\right) \quad \text{because } M_n \ge 2 \\
&\le M_n^d \cdot \left(1 + \frac{2L_0^6}{\|\boldsymbol{\lambda}\|_v}\right) \quad \text{by induction hypothesis} \\
&\le M_n^d \cdot \left(1 + \frac{2L_0^6}{L}\right) \quad \text{since } \|\boldsymbol{\lambda}\|_v > L \\
&\le 2M_n^d \quad \text{since } L \ge 4L_0^6.
\end{aligned}
$$

This concludes the proof of Claim 4.6. □

We take $L = 4L_0^6$ in Claim 4.6. It follows from the definition of L_0 that the constant L now depends only on v and $\mathbf{c}$. Then Claims 4.5 and 4.6 yield that there exists a constant $C > 1$ (depending on v and $\mathbf{c}$) such that

$$\frac{1}{C} \le \frac{M_{n+1}}{M_n^d} \le C, \tag{4.6.4}$$

for each $n \ge 1$. An easy telescoping sum after taking the logarithm of the inequalities from (4.6.4) finishes the proof of Lemma 4.4. □

An immediate corollary of Lemma 4.4 (for $n_1 = 1$) is the following result.

Lemma 4.7. *Let $v \in S$. There exists a constant $C(v, \mathbf{c})$ depending only on the absolute value $|\cdot|_v$ and on the point $\mathbf{c}$ such that for each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, and for each positive integer n, we have*

$$\left|\lim_{n\to\infty} \frac{\log \|\Phi_{\boldsymbol{\lambda}}^n(\mathbf{c})\|_v}{d^n} - \frac{\log \|\Phi_{\boldsymbol{\lambda}}(\mathbf{c})\|_v}{d}\right| < \frac{C(v, \mathbf{c})}{d}.$$

The next result is an easy consequence of the height machine.

Lemma 4.8. *There exists a constant $C(\mathbf{c})$ depending only on the point $\mathbf{c}$ such that*

$$|h\left(\Phi_{\boldsymbol{\lambda}}(\mathbf{c})\right) - h(\boldsymbol{\lambda})| \le C(\mathbf{c}),$$

for each $\boldsymbol{\lambda} \in \mathbb{A}^m(\overline{K})$.

Proof. Recall that $\mathbf{f}(x) = x^d + t_1 x^{m-1} + \cdots + t_{m-1}x + t_m$. We consider the linear transformation $\Psi : \mathbb{P}^m \longrightarrow \mathbb{P}^m$ defined by

$$\Psi(P) = [T_0\mathbf{f}(c_m) : \ldots : T_0\mathbf{f}(c_1) : T_0]$$

where $P = [T_m : \ldots : T_1 : T_0]$ and $t_i = T_i/T_0$ for $i = 1, \dots, m$. Since the c_i's are distinct, the map Ψ is an automorphism of $\mathbb{P}^m$. So, there exists a constant $C(\mathbf{c})$ (see [BG06]) depending only on the constants c_i such that

$$|h(\Psi([\lambda_m : \cdots : \lambda_1 : 1])) - h([\lambda_m : \cdots : \lambda_1 : 1])| \le C(\mathbf{c}). \tag{4.8.1}$$

Because $\Phi_{\boldsymbol{\lambda}}(\mathbf{c}) = \Psi([\lambda_m : \cdots : \lambda_1 : 1])$ and $h(\boldsymbol{\lambda}) = h([\lambda_m : \cdots : \lambda_1 : 1])$, we obtain the conclusion of Lemma 4.8. □

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let L be a finite extension of K containing each λ_i (for $i = 1, \dots, m$). Recall that we are given the point $P = \tilde{\mathbf{c}} = [c_m : \cdots : c_1 : 1]$. We need to show that $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}) - \widehat{\mathrm{h}}_{\Phi}(P) \cdot h(\boldsymbol{\lambda})$ is bounded by a constant independent of $\boldsymbol{\lambda}$. Note that $\widehat{\mathrm{h}}_{\Phi}(P) = \widehat{\mathrm{h}}_{\Phi}(\mathbf{c}) = 1/d$. Combining Lemmas 4.3, 4.7 and 4.8 yields that

$$\begin{aligned}
\left|\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}) - \widehat{\mathrm{h}}_{\Phi}(P) \cdot h(\boldsymbol{\lambda})\right| &\le \frac{1}{d} \cdot \left|h\left(\Phi_{\boldsymbol{\lambda}}(\mathbf{c})\right) - h(\boldsymbol{\lambda})\right| + \left|\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}) - \frac{h\left(\Phi_{\boldsymbol{\lambda}}(\mathbf{c})\right)}{d}\right| \\
&\le \frac{C(\mathbf{c})}{d} + \sum_{v\in S} \frac{N_v}{[L:K]} \cdot \sum_{\substack{\sigma: L \hookrightarrow \overline{K} \\ \sigma|_K = \mathrm{id}}} \left| \lim_{n\to\infty} \frac{\log \|\Phi^n_{\sigma(\boldsymbol{\lambda})}(\mathbf{c})\|_v}{d^n} - \frac{\log \|\Phi_{\sigma(\boldsymbol{\lambda})}(\mathbf{c})\|_v}{d} \right| \\
&\le \frac{C(\mathbf{c})}{d} + \sum_{v\in S} \frac{N_v}{[L:K]} \cdot [L:K] \cdot \frac{C(v,\mathbf{c})}{d} \\
&\le \frac{1}{d}\left(C(\mathbf{c}) + \sum_{v\in S} N_v \cdot C(v,\mathbf{c}) \right)
\end{aligned}$$

as desired.

□

5. Simultaneously preperiodic points for polynomials of arbitrary degree

We retain the notation used in Section 4. In this section we prove the following result.

Theorem 5.1. *Let K be a number field, or a function field of finite transcendence degree over $\overline{\mathbb{Q}}$, let $d > m \ge 2$ be integers, and let*

$$\mathbf{f}(z) := z^d + t_1 z^{m-1} + \cdots + t_{m-1} z + t_m$$

be an m-parameter family of polynomials of degree d. For each point $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ of $\mathbb{A}^m(\overline{K})$ we let $\mathbf{f}_{\boldsymbol{\lambda}}$ be the corresponding polynomial defined over $\overline{K}$ obtained by specializing each t_i to λ_i for $i = 1, \dots, m$.

Let $c_1, \dots, c_{m+1} \in K$ be distinct elements. Let $\mathrm{Prep}(c_1, \dots, c_{m+1})$ *be the set consisting of parameters $\boldsymbol{\lambda} \in \mathbb{A}^m(\overline{K})$ such that c_i is preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$ for each $i = 1, \dots, m+1$. If* $\mathrm{Prep}(c_1, \dots, c_{m+1})$ *is Zariski dense in $\mathbb{A}^m(\overline{K})$ then the following holds: for each $\boldsymbol{\lambda} \in \mathbb{A}^m(\overline{K})$, if m of the points $c_1, \dots, c_{m+1}$ are preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$, then all $(m+1)$ points are preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$.*

Remark 5.2. We note that Theorem 5.1 does not hold if the c_i's are not all distinct. This can be seen for example when $d = 3$ and $m = 2$ by considering starting points $c_1 \ne c_2 = c_3$. One can show that in this case, $\mathrm{Prep}(c_1, c_2, c_3) = \mathrm{Prep}(c_1, c_2)$ is Zariski dense in $\mathbb{A}^2$. Indeed, otherwise there are finitely many irreducible plane curves C_i (for $i = 1, \dots, \ell$) containing all points from $\mathrm{Prep}(c_1, c_2)$. Then consider a preperiodicity portrait (m_1, n_1) for the point c_1 which is not identically realized along any of the curves C_i; the existence of such a portrait is guaranteed by [GNT, Theorem 1.3]. Then there exists a curve $C := C_{(m_1,n_1)} \subset \mathbb{A}^2$ such that for each $(a_1, a_0) \in C(\overline{K})$, the preperiodicity portrait of c_1 under $\mathbf{f}_{\mathbf{a}}(z) := z^3 + a_1 z + a_0$ is (m_1, n_1). Another application of [GNT, Theorem 1.3] yields the existence of infinitely many points $(a_1, a_0) \in C(\overline{K})$ such that c_2 is preperiodic under the action of $\mathbf{f}_{\mathbf{a}}$. But this means that C must be contained in the Zariski closure of $\mathrm{Prep}(c_1, c_2)$ contradicting the fact that C is not one of the curves C_i for $i = 1, \dots, \ell$.

So, Theorem 5.1 yields that if there exists a Zariski dense set of m-tuples $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$ such that each c_i is preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$, then something *quite unlikely* holds: for any specialization polynomial $\mathbf{f}_{\boldsymbol{\lambda}}$, if m points c_i are preperiodic, then *all* $(m+1)$ points c_i are preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$. As discussed in Section 1 (see also Remark 5.2 and [GNT]), it is expected that there are *many* specializations $\mathbf{f}_{\boldsymbol{\lambda}}$ such that $c_1, \dots, c_m$ are preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$. So, Theorem 5.1 yields that under the given conclusion, for each of these *many* specializations, *all* $(m+1)$ points c_i are preperiodic. We expect that such a conclusion should actually yield a contradiction, and in the next Section we are able to prove this in the case $m = 2$.

The main ingredient in proving Theorem 5.1 is the powerful equidistribution theorem for points of small height with respect to metrized adélic line bundles (see [Yua08] and also [Gub08] for the function field version), which can be applied due to our Theorem 2.1.

Proof of Theorem 5.1. By assumption, we know $\mathrm{Prep}(c_1, \dots, c_{m+1})$ is a Zariski dense set of $\mathbb{A}^m(\overline{K})$. Without loss of generality, it suffices to prove that for each $(\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, if c_i is preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$ for each $i = 1, \dots, m$, then also c_{m+1} is preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$.

Recall from Section 4 the family Φ of endomorphisms of $\mathbb{P}^m$ defined by

$$\Phi([X_m : \cdots : X_1 : X_0]) = \left[X_0^d \mathbf{f}\left(\frac{X_m}{X_0}\right) : \cdots : X_0^d \mathbf{f}\left(\frac{X_1}{X_0}\right) : X_0^d\right].$$

As before, for each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m(\overline{K})$, we denote by $\Phi_{\boldsymbol{\lambda}}$ the corresponding endomorphism of $\mathbb{P}^m$ obtained by specializing each t_i to λ_i. For each $j = 1, 2$ we let

$$\mathbf{c}^{(j)} := [c_{m-1+j} : c_{m-1} : \cdots : c_1 : 1]$$

and for each $n \geq 0$ we define polynomials $A_{n,i}^{(j)}(t_1, \dots, t_m) \in \overline{K}[t_1, \dots, t_m]$ (for $i = 1, \dots, m$) such that

$$\Phi^n(\mathbf{c}^{(j)}) = [A_{n,m}^{(j)} : \cdots : A_{n,1}^{(j)} : 1].$$

More precisely, $[A_{0,m}^{(j)} : \cdots : A_{0,1}^{(j)} : 1] = \mathbf{c}^{(j)}$, while for each $n \geq 0$, we have $A_{n+1,i}^{(j)} = \mathbf{f}(A_{n,i}^{(j)})$. It is easy to check that the total degree in $t_1, \ldots, t_m$ is $\deg A_{n,i}^{(j)} = d^{n-1}$ for all $n \geq 1$ (and each $i = 1, \ldots, m$ and each $j = 1, 2$).

We note that if we let

$$\tilde{A}_{n,i}^{(j)}(u_1, \ldots, u_{m+1}) := u_{m+1}^{d^{n-1}} \cdot A_{n,i}^{(j)}\left(\frac{u_1}{u_{m+1}}, \ldots, \frac{u_m}{u_{m+1}}\right)$$

(for each $j = 1, 2$, each $i = 1, \ldots, m$ and each $n \in \mathbb{N}$), then the map

$$\theta_n^{(j)} : \mathbb{P}^m \longrightarrow \mathbb{P}^m \quad \text{given by}$$
$$\theta_n^{(j)}(\mathbf{u}) = \left[\tilde{A}_{n,m}^{(j)}(\mathbf{u}) : \cdots : \tilde{A}_{n,1}^{(j)}(\mathbf{u}) : u_{m+1}^{d^{n-1}}\right],\ \mathbf{u} = [u_1 : \cdots : u_{m+1}],$$

is a morphism defined over K. Indeed, if $u_{m+1} = 0$ and $1 \leq i \leq m-1$, then

$$\tilde{A}_{1,i}^{(j)}(u_1, \ldots, u_m, 0) = \sum_{k=1}^{m} c_i^{m-k} u_k,$$

while $\tilde{A}_{1,m}^{(j)}(u_1, \ldots, u_m, 0) = \sum_{k=1}^{m} c_{m-1+j}^{m-k} u_k$. An easy induction coupled with the observation that the leading term in $\tilde{A}_{n,i}^{(j)}$ (in terms of degree) is $\left(\tilde{A}_{1,i}^{(j)}\right)^{d^{n-1}}$ yields the formulas:

$$\tilde{A}_{n,i}^{(j)}(u_1, \ldots, u_m, 0) = \left(\sum_{k=1}^{m} c_i^{m-k} u_k\right)^{d^{n-1}}$$

for $i = 1, \ldots, m-1$, and

$$\tilde{A}_{n,m}^{(j)}(u_1, \ldots, u_m, 0) = \left(\sum_{k=1}^{m} c_{m-1+j}^{m-k} u_k\right)^{d^{n-1}}.$$

Now the assumption that the c_i's are distinct ensures that the above map $\theta_n^{(j)}$ is well-defined on $\mathbb{P}^m$. Thus, we have an isomorphism

$$\tau_n^{(j)} : \mathcal{O}_{\mathbb{P}^m}(d^{n-1}) \xrightarrow{\sim} \left(\theta_n^{(j)}\right)^* \mathcal{O}_{\mathbb{P}^m}(1),$$

given by

$$\tau_n^{(j)}\left(u_i^{d^{n-1}}\right) = \tilde{A}_{n,i}^{(j)}(u_1, \ldots, u_{m+1})$$

for $i = 1, \ldots, m$, and also $\tau_n^{(j)}\left(u_{m+1}^{d^{n-1}}\right) = u_{m+1}^{d^{n-1}}$.

We consider the following two families of metrics corresponding to any section $s := a_1 u_1 + \cdots + a_{m+1} u_{m+1}$ (with scalars a_i) of the line bundle $\mathcal{O}_{\mathbb{P}^m}(1)$ of $\mathbb{P}^m$. Using the coordinates $t_i = \frac{u_i}{u_{m+1}}$ (for $i = 1, \ldots, m$) on the affine subset of $\mathbb{P}^m$ corresponding to $u_{m+1} \neq 0$, then for each $v \in \Omega_K$, for each $n \in \mathbb{N}$ (and each $j = 1, 2$) we get that the metrics $\|s(\cdot)\|_{v,n}^{(j)}$ are defined as follows:

$$\|s([u_1 : \cdots : u_{m+1}])\|_{v,n}^{(j)} = \begin{cases} \frac{\left|\sum_{k=1}^{m} a_k u_k\right|_v}{\max_{i=1}^{m}\{|\tilde{A}_{1,i}^{(j)}(u_1,\ldots,u_m,0)|_v\}} & \text{if } u_{m+1} = 0, \\ \frac{\left|a_{m+1}+\sum_{k=1}^{m} a_k t_k\right|_v}{\sqrt[d^{n-1}]{\|\Phi^n(\mathbf{c}^{(j)})\|_v}} & \text{if } u_{m+1} \neq 0. \end{cases} \tag{5.2.1}$$

Let $\|\cdot\|_v'$ be the metric on $\mathcal{O}_{\mathbb{P}^m}(1)$ corresponding to the section $s = a_1u_1 + \cdots + a_{m+1}u_{m+1}$ given by

$$\|s([b_1 : \cdots : b_{m+1}])\|_v' = \frac{\left|\sum_{k=1}^{m+1} a_k b_k\right|_v}{\max\{|b_1|_v, \ldots, |b_{m+1}|_v\}}.$$

We see then that $\|s\|_{v,n}^{(j)}$ is simply the d^{n-1}-th root of $\left(\tau_n^{(j)}\right)^*\left(\theta_n^{(j)}\right)^*\|\cdot\|_v'$. Note that the degree of $\theta_n^{(j)}$ is the same as the total degree of the polynomials $A_{n,i}$, and thus $\deg\theta_n^{(j)} = d^{n-1}$. Hence, for each n, we have that $\|s\|_{v,n}^{(j)}$ are semipositive metrics on $\mathcal{L} = \mathcal{O}_{\mathbb{P}^m}(1)$. Following [Yua08] (in the case of number fields) and [Gub08] (in the case of function fields), we let $\overline{\mathcal{L}}_n^{(j)}$ denote the algebraic adelic metrized line bundle corresponding to the collection of metrics $\|s\|_{v,n}^{(j)}$.

Let $j = 1, 2$. Clearly, $\left\{\log\|s\|_{v,n}^{(j)}\right\}_n$ converges uniformly on the hyperplane at infinity (defined by $u_{m+1} = 0$) from $\mathbb{P}^m$ since there is no dependence on n in this case. Lemmas 4.3 and 4.4 yield that $\left\{\log\|s\|_{v,n}^{(j)}\right\}_n$ converges uniformly also when $u_{m+1} \neq 0$. Furthermore (as shown by Lemma 4.3), for all but finitely many places of K, the metrics $\|s\|_{v,n}^{(j)}$ do not vary with n. For each $v \in \Omega_K$, we let $\|s\|_v^{(j)}$ be the metric which is the limit of the metrics $\|s\|_{v,n}^{(j)}$. We denote by $\overline{\mathcal{L}}^{(j)}$ the corresponding adélic metrized line bundles $\left(\mathcal{O}_{\mathbb{P}^m}(1), \{\|s\|_v^{(j)}\}\right)$.

Let $Q \in \mathbb{P}^m(\overline{K})$. Let s be a section of $\mathcal{L}$ as above such that $s(Q) \neq 0$; we define the height $\widehat{\mathrm{h}}_{\overline{\mathcal{L}}^{(j)}}(Q)$ associated to the metrized line bundle $\overline{\mathcal{L}}^{(j)}$ as follows. We let L be a normal, finite extension of K such that $Q \in \mathbb{P}^m(K)$ and then define:

$$\widehat{\mathrm{h}}_{\overline{\mathcal{L}}^{(j)}}(Q) := \sum_{v\in\Omega_K} \frac{N_v}{[L:K]} \sum_{\sigma\in\mathrm{Gal}(L/K)} -\log\|s(\sigma(Q))\|_v^{(j)}. \tag{5.2.2}$$

By the definition of the above adélic metrics, we have (see also [GHT15, (9.0.8)]) for each $j = 1, 2$:

$$\widehat{\mathrm{h}}_{\overline{\mathcal{L}}^{(j)}}([\lambda_m : \cdots : \lambda_1 : 1]) = d \cdot \widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}^{(j)}). \tag{5.2.3}$$

By our assumption, there exists a Zariski dense set of points $\boldsymbol{\lambda} \in \mathbb{A}^m(\overline{K})$ such that

$$\widehat{\mathrm{h}}_{\overline{\mathcal{L}}^{(1)}}(\boldsymbol{\lambda}) = \widehat{\mathrm{h}}_{\overline{\mathcal{L}}^{(2)}}(\boldsymbol{\lambda}) = 0. \tag{5.2.4}$$

On the other hand, we note that
(5.2.5)

$$\|\theta_1^{(j)}([u_1 : \cdots : u_m : 0])\|_v = \max\left\{\left|\sum_{k=1}^{m} c_{m-1+j}^{m-k} u_k\right|_v, \max_{i=1}^{m-1}\left|\sum_{k=1}^{m} c_i^{m-k} u_k\right|_v\right\}$$

for each place $v \in \Omega_K$. Then for each point $[u_1 : \cdots : u_m : 0]$ on the hyperplane at infinity of $\mathbb{P}^m$ such that

$$\sum_{k=1}^{m} c_m^{m-k} u_k = \sum_{k=1}^{m} c_{m+1}^{m-k} u_k, \tag{5.2.6}$$

(5.2.5) yields that

$$\|\theta_1^{(1)}([u_1 : \cdots : u_m : 0])\|_v = \|\theta_1^{(2)}([u_1 : \cdots : u_m : 0])\|_v. \tag{5.2.7}$$

On the other hand, the definition (5.2.1) of the metric $\|\cdot\|_v^{(j)}$ at any point $[u_1 : \cdots : u_m : 0]$ on the hyperplane at infinity and for any section $s := a_1u_1 + \cdots + a_{m+1}u_{m+1}$ gives

$$\begin{aligned}\|s([u_1 : \cdots : u_m : 0])\|_v^{(j)} &= \frac{|\sum_{k=1}^m a_k u_k|_v}{\max_{i=1}^m\{|\tilde{A}_{1,i}^{(j)}(u_1, \ldots, u_m, 0)|_v\}} \\ &= \frac{|\sum_{k=1}^m a_k u_k|_v}{\|\theta_1^{(j)}([u_1 : \cdots : u_m : 0])\|_v}\end{aligned}$$

for each $j = 1, 2$. So, for a point $[u_1 : \cdots : u_m : 0]$ satisfying (5.2.6), equality (5.2.7) yields

$$\|s([u_1 : \cdots : u_m : 0])\|_v^{(1)} = \|s([u_1 : \cdots : u_m : 0])\|_v^{(2)}. \tag{5.2.8}$$

Combining (5.2.4) and (5.2.8) allows us to use [GHT15, Corollary 4.3] and conclude that for *all* $\lambda_1, \ldots, \lambda_m \in \overline{K}$ we have

$$h_{\overline{\mathcal{L}}^{(1)}}(\boldsymbol{\lambda}) = h_{\overline{\mathcal{L}}^{(2)}}(\boldsymbol{\lambda}). \tag{5.2.9}$$

Strictly speaking, [GHT15, Corollary 4.3] was stated only for metrized line bundles defined over $\overline{\mathbb{Q}}$ since the authors employed in that paper Yuan's equidistribution theorem from [Yua08]. However, using [Gub08, Theorem 1.1] and arguing identically as in the proof of [GHT15, Corollary 4.3] one can extend the result from number fields to any function field of characteristic 0.

Using (5.2.9) coupled with (5.2.3), we obtain that $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}^{(1)}) = 0$ if and only if $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}^{(2)}) = 0$.

Assume now that K is a number field, i.e., that each $c_i \in \overline{\mathbb{Q}}$. Then, as shown in [CS93], a point is preperiodic under the action of $\Phi_{\boldsymbol{\lambda}}$ if and only if its canonical height equals 0. On the other hand, in general, a point

$[a_1 : \cdots : a_m : 1] \in \mathbb{P}^m(\overline{\mathbb{Q}})$ is preperiodic under the action of $\Phi_{\boldsymbol{\lambda}}$ if and only if each a_i is preperiodic for the action of $\mathbf{f}_{\boldsymbol{\lambda}}$. Therefore, we obtain that for each $\boldsymbol{\lambda} \in \mathbb{A}^m(\overline{\mathbb{Q}})$, if each c_i (for $i = 1, \ldots, m$) is preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$, then also c_{m+1} is preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$.

So, from now on, assume that not all c_i are contained in $\overline{\mathbb{Q}}$. It is still true that $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}^{(1)}) = 0$ if and only if $\widehat{\mathrm{h}}_{\mathbf{f}_{\boldsymbol{\lambda}}}(c_i) = 0$ for $i = 1, \ldots, m$. Arguing similarly for $\mathbf{c}^{(2)}$, we get that for a $\boldsymbol{\lambda} \in \mathbb{A}^M(\overline{K})$ such that $\widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}^{(1)}) = \widehat{\mathrm{h}}_{\Phi_{\boldsymbol{\lambda}}}(\mathbf{c}^{(2)}) = 0$, we have that each $\widehat{\mathrm{h}}_{\mathbf{f}_{\boldsymbol{\lambda}}}(c_i) = 0$ for $i = 1, \ldots, m+1$. But $\mathbf{f}_{\boldsymbol{\lambda}}$ is a polynomial in normal form and therefore, it is isotrivial if and only if each one of its coefficients are in the constant field, i.e. they are contained in $\overline{\mathbb{Q}}$ (see Proposition 3.3). But if this happens then we cannot have that each $\widehat{\mathrm{h}}_{\mathbf{f}_{\boldsymbol{\lambda}}}(c_i) = 0$ since not all c_i are in the constant field. In conclusion, $\mathbf{f}_{\boldsymbol{\lambda}}$ is not isotrivial, and then by Lemma 3.2, we conclude that $\widehat{\mathrm{h}}_{\mathbf{f}_{\boldsymbol{\lambda}}}(c_i) = 0$ if and only if c_i is preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$. Hence, if c_i is preperiodic for each $i = 1, \ldots, m$, then also c_{m+1} is preperiodic under the action of $\mathbf{f}_{\boldsymbol{\lambda}}$. $\square$

6. Proof of Theorem 1.4

We work under the hypotheses of Theorem 1.4. We prove the theorem by contradiction under the assumption that we have a Zariski dense set of parameters $\boldsymbol{\lambda}$ such that c_1, c_2 and c_3 are preperiodic for $\mathbf{f}_{\boldsymbol{\lambda}}$.

Because the starting points c_i are all distinct, we may assume $c_1 \neq 0$. Let now $V \subset \mathbb{P}^2$ be the line which is the Zariski closure in $\mathbb{P}^2$ of the affine line containing all $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{A}^2(\overline{K})$ such that c_1 is a fixed point for $\mathbf{f}_{\boldsymbol{\lambda}}(x) = x^d + \lambda_1 x + \lambda_2$. This last condition is equivalent with asking that $c_1^d + c_1\lambda_1 + \lambda_2 = c_1$, or in other words, V is the line in $\mathbb{P}^2$ whose intersection with the affine plane containing all points in $\mathbb{P}^2$ with a nonzero last coordinate is the line $(t, \alpha + \beta t)$, where $\alpha := c_1 - c_1^d$ and $\beta := -c_1 \neq 0$.

Let $\mathbf{g}_t(x) := x^d + tx + (\alpha + \beta t)$ be a 1-parameter family of degree d polynomials. Note that $\alpha + \beta t \neq 0$ (because $\beta \neq 0$). By Theorem 5.1, we know that for each $t \in \overline{K}$, c_2 is preperiodic for $\mathbf{g}_t$ if and only if c_3 is preperiodic for $\mathbf{g}_t$. It follows from [BD13, Theorem 1.3] that there exists a polynomial $\mathbf{h}_t$ commuting with an iterate of $\mathbf{g}_t$, and there exist $m, n \in \mathbb{N}$ such that

$$\mathbf{g}_t^m(c_2) = \mathbf{h}_t\left(\mathbf{g}_t^n(c_3)\right). \tag{6.0.10}$$

We claim that if $\mathbf{h}_t \in \overline{K}[t, x]$ is any (non-constant) polynomial such that $\mathbf{h}_t$ commutes with an iterate of $\mathbf{g}_t$, then $\mathbf{h}_t = \mathbf{g}_t^\ell$ for some $\ell \geq 0$. This statement follows from [Ngu15, Proposition 2.3]. First, since $\mathbf{g}_t$ is in normal form, the only linear polynomials commuting with $\mathbf{g}_t$ are of the form γx, and because $\alpha + \beta t \neq 0$, then $\gamma = 1$. Secondly, according to [Ngu15, Proposition 2.3], the non-linear polynomial $\mathbf{h}_t$ of smallest degree commuting with $\mathbf{g}_t$ must satisfy the condition that $\mathbf{h}_t^e = \mathbf{g}_t$ for some positive integer e. Write $\mathbf{h}_t(x) = a(t)x^\ell + b(t)x^{\ell - i} + \text{lower degree terms}$, where $\ell \geq 2, 1 \leq i \leq \ell$ and both $a(t)$ and $b(t)$ are nonzero polynomials in t. A straightforward induction shows that $\mathbf{h}_t^e(x) = a(t)^{d_e}x^{\ell e} + \beta_e(t)x^{\ell e - i}$ + lower degree terms, where d_e is some positive integer and $\beta_e(t)$ is a monomial in $a(t)$ and $b(t)$ which is nonzero. Due to the shape of $\mathbf{g}_t$, by comparing degrees in x, we see that the only possibility is $e = 1$, which yields our claim that the only polynomials commuting with $\mathbf{g}_t$ are of the form $\mathbf{g}_t^\ell$ for $\ell \geq 0$.

Therefore, in (6.0.10) we can take $\mathbf{h}_t(x) = x$; hence

$$\mathbf{g}_t^m(c_2) = \mathbf{g}_t^n(c_3). \tag{6.0.11}$$

Now, for each $c \in \overline{K}$, if $c \neq c_1$ then $\deg_t(\mathbf{g}_t(c)) = 1$ and then a simple induction proves that $\deg_t(\mathbf{g}_t^\ell(c)) = d^{\ell-1}$ for all $\ell \in \mathbb{N}$. Therefore, (6.0.11) yields that $m = n$. The equality $\mathbf{g}_t^m(c_2) = \mathbf{g}_t^m(c_3)$ yields in particular that the leading coefficients of the polynomials $\mathbf{g}_t^m(c_2)$ and $\mathbf{g}_t^m(c_3)$ are the same, i.e.

$$(c_2 - c_1)^{d^{m-1}} = (c_3 - c_1)^{d^{m-1}}. \tag{6.0.12}$$

It is immediate to see that for $m \geq 2$ we have

$$\mathbf{g}_t^{m-1}(x) = x^{d^{m-1}} + d^{m-2}t \cdot x^{d^{m-1}-d+1} + \text{ lower order terms in } x.$$

The rest of the argument is split into two cases depending on whether d is even or not. If d is odd, then we can easily derive the conclusion, as follows.

Assume d is odd.

Without loss of generality, we may also assume $c_2 \neq 0$ (because all three numbers c_1, c_2, c_3 are distinct). Then the exact same argument as above used for deriving the equation (6.0.12) (applied this time to the line in the parameter space along which c_2 is a fixed point) yields that

$$(c_1 - c_2)^{d^{\ell-1}} = (c_3 - c_2)^{d^{\ell-1}}, \tag{6.0.13}$$

for some $\ell \in \mathbb{N}$. Using (6.0.12) and (6.0.13), at the expense of replacing ℓ by a larger number, we may assume

$$(c_2 - c_1)^{d^\ell} = (c_3 - c_1)^{d^\ell} \text{ and } (c_1 - c_2)^{d^\ell} = (c_3 - c_2)^{d^\ell}. \tag{6.0.14}$$

We split now the analysis depending on whether c_3 is also nonzero, or $c_3 = 0$.

Case 1. $c_3 \neq 0$.

In this case, we can apply (a third time) the above argument, this time for the curve in the parameter space along which c_3 is a fixed point, and therefore conclude (at the expense of replacing ℓ by a larger integer) that

$$(c_2-c_1)^{d^\ell} = (c_3-c_1)^{d^\ell},\ (c_1-c_2)^{d^\ell} = (c_3-c_2)^{d^\ell} \text{ and } (c_1-c_3)^{d^\ell} = (c_2-c_3)^{d^\ell}. \tag{6.0.15}$$

But then $(c_2-c_3)^{d^\ell} = (c_3-c_2)^{d^\ell}$, and since d is odd, this yields that $c_2 = c_3$, contradiction.

Case 2. $c_3 = 0$.

Under this assumption, we rewrite (6.0.14) as follows:

$$(c_2 - c_1)^{d^\ell} = (-c_1)^{d^\ell} \text{ and } (c_1 - c_2)^{d^\ell} = (-c_2)^{d^\ell}. \tag{6.0.16}$$

Hence there exist d^ℓ-th roots of unity ζ_1 and ζ_2 such that $c_1 - c_2 = -\zeta_1 c_2$ and $c_2 - c_1 = -\zeta_2 c_1$. So, $c_1 = (1-\zeta_1)c_2$ and $c_2 = (1-\zeta_2)c_1$ and because $c_1 \neq 0$, we conclude that $(1-\zeta_1)(1-\zeta_2) = 1$, i.e. $-\zeta_1 - \zeta_2 + \zeta_1\zeta_2 = 0$. Hence $\zeta_2(\zeta_1 - 1) = \zeta_1$ and thus also $\zeta_1 - 1$ is a d^ℓ-th root of unity. Now, the only roots of unity ζ with the property that also $\zeta - 1$ is a root of unity are $\zeta = \frac{1\pm\sqrt{-3}}{2}$. However, $\frac{1\pm\sqrt{-3}}{2}$ is a primitive 6-th root of unity and not a d^ℓ-th root of unity when d is odd, contradiction.

Therefore, Theorem 1.4 holds whenever d is odd; in particular, it holds when $d = 3$. Next we prove that Theorem 1.4 holds for all $d \geq 4$. So, actually it is only the case $d = 3$ that requires a different argument than the general case (and luckily the case $d = 3$ is covered by the above analysis which works for any *odd* d); see Remark 6.1 for a technical explanation of why the case $d = 3$ is not covered by the argument we provide next for $d \geq 4$.

Assume now that $d \geq 4$.

Noting that $\mathbf{g}_t(c_2) = t(c_2 - c_1) + c_2^d + c_1 - c_1^d$, we will show further down that for $m \geq 2$ we get

$$\begin{aligned}
\mathbf{g}_t^m(c_2) &= \mathbf{g}_t(c_2)^{d^{m-1}} + d^{m-2}t \cdot \mathbf{g}_t(c_2)^{d^{m-1}-d+1} + R_m(t) \\
&= t^{d^{m-1}} \cdot (c_2 - c_1)^{d^{m-1}} \\
&+ d^{m-1} \cdot (c_2 - c_1)^{d^{m-1}-1}(c_2^d + c_1 - c_1^d) \cdot t^{d^{m-1}-1} + O\left(t^{d^{m-1}-2}\right)
\end{aligned} \tag{6.0.17}$$

where $R_m(t)$ is a polynomial in t of degree at most $d^{m-1} - d + 1$. In the above computation we used the fact that $d > 3$ and therefore the second leading term of $\mathbf{g}_t^m(c_2)$ is indeed $d^{m-1} \cdot (c_2 - c_1)^{d^{m-1}-1}(c_2^d + c_1 - c_1^d) \cdot t^{d^{m-1}-1}$. Also, the first equality in the above expansion of $\mathbf{g}_t^m(c_2)$ follows by induction on m. This follows immediately since $\mathbf{g}_t(x) = x^d + tx + (c_1 - c_1^d - tc_1)$ and so,

$$\begin{aligned}
\mathbf{g}_t^{m+1}(c_2) &= \mathbf{g}_t(\mathbf{g}_t^m(c_2)) \\
&= \mathbf{g}_t^m(c_2)^d + t\mathbf{g}_t^m(c_2) + (c_1 - c_1^d - tc_1) \\
&= \left(\mathbf{g}_t(c_2)^{d^{m-1}} + d^{m-2}t \cdot \mathbf{g}_t(c_2)^{d^{m-1}-d+1} + R_m(t)\right)^d \\
&+ t \cdot \left(\mathbf{g}_t(c_2)^{d^{m-1}} + d^{m-2}t\mathbf{g}_t(c_2)^{d^{m-1}-d+1} + R_m(t)\right) + (c_1 - c_1^d - c_1 t)
\end{aligned}$$

Using the induction hypothesis (and that $d \geq 3$), we know that

$$\deg_t \mathbf{g}_t(c_2)^{d^{m-1}} > \deg_t t \cdot \mathbf{g}_t(c_2)^{d^{m-1}-d+1} > \deg_t R_m(t).$$

We thus conclude that

$$\begin{aligned}
\mathbf{g}_t^{m+1}(c_2) &= \mathbf{g}_t(c_2)^{d^m} + d^{m-1}t \cdot \mathbf{g}_t(c_2)^{d^m-d+1} + O(t^{d^m-d+1}) \\
&+ t \cdot \mathbf{g}_t(c_2)^{d^{m-1}} + O(t^{d^{m-1}})
\end{aligned}$$

Clearly, $d^m - d + 1 > d^{m-1}$ for $m \geq 2$ (because $d \geq 3$), which yields the desired claim (6.0.17). Hence $\mathbf{g}_t^m(c_2) = \mathbf{g}_t^m(c_3)$ yields not only (6.0.12) but also that

$$(c_2-c_1)^{d^{m-1}-1}\cdot(c_2^d+c_1-c_1^d)=(c_3-c_1)^{d^{m-1}-1}\cdot(c_3^d+c_1-c_1^d). \tag{6.0.18}$$

Equations (6.0.12) and (6.0.18) yield that there exists $u\in K$ such that

$$u:=\frac{c_2^d+c_1-c_1^d}{c_2-c_1}=\frac{c_3^d+c_1-c_1^d}{c_3-c_1}. \tag{6.0.19}$$

Combining (6.0.12) and (6.0.19), we get that

$$\mathbf{g}_t(c_2)^{d^{m-1}}=\mathbf{g}_t(c_3)^{d^{m-1}}. \tag{6.0.20}$$

Then using (6.0.20) and the expansion of $\mathbf{g}_t^m(c_2)=\mathbf{g}_t^m(c_3)$ in terms of powers of t, we get that

$$\begin{aligned}&d^{m-2}t\cdot\Big(t(c_2-c_1)+(c_2^d+c_1-c_1^d)\Big)^{d^{m-1}-d+1}\\&=d^{m-2}t\cdot\Big(t(c_3-c_1)+(c_3^d+c_1-c_1^d)\Big)^{d^{m-1}-d+1}+O(t^{d^{m-1}-d+1}).\end{aligned}$$

This yields that

$$(c_2-c_1)^{d^{m-1}-d+1}=(c_3-c_1)^{d^{m-1}-d+1}. \tag{6.0.21}$$

Since $\gcd\big(d^{m-1},d^{m-1}-d+1\big)=1$, (6.0.12) and (6.0.21) yield that

$$c_2-c_1=c_3-c_1,$$

i.e., that $c_2=c_3$, contradiction. This concludes our proof when $d\geq 4$, and in turn, it finishes the proof of Theorem 1.4.

Remark 6.1. In the special case $d=3$ we cannot employ the same argument as is in the case $d\geq 4$ since the second leading term in $\mathbf{g}_t^m(c_2)$ involves also contribution from

$$d^{m-2}t\cdot\Big(t(c_2-c_1)+c_2^d+c_1-c_1^d\Big)^{d^{m-1}-d+1}.$$

Instead we had to use the additional specialization of $\mathbf{f}(x)=x^d+t_1x+t_2$ along other lines in the moduli space $\mathbb{A}^2$ with the property that c_2 (and c_3) are fixed along these other lines. This allowed us to derive additional relations between the c_i's similar to (6.0.12) and as long as d is odd (which is the case when $d=3$) suffices for deriving a contradiction.

The above analysis from the proof of Theorem 1.4 which is quite delicate gives an insight into the difficulty of the general case of Question 1.1 when one deals with $d-1$ parameters a_i and d starting points c_i.

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