PROOF OF A DYNAMICAL BOGOMOLOV CONJECTURE FOR LINES UNDER POLYNOMIAL ACTIONS

DRAGOS GHIOCA AND THOMAS J. TUCKER

Abstract. We prove a dynamical version of the Bogomolov conjecture in the special case of lines in $\mathbb{A}^m$ under the action of a map $(f_1, \dots, f_m)$ where each f_i is a polynomial in $\overline{\mathbb{Q}}[X]$ of the same degree.

1. Introduction

In 1998, Ullmo [Ull98] and Zhang [Zha98] proved the following conjecture of Bogomolov [Bog91].

Theorem 1.1. *Let A be an abelian variety defined over a number field with Néron-Tate height $\widehat{h}_{\mathrm{nt}}$ and let W be an irreducible subvariety of A that is not a torsion translate of an abelian subvariety of A. Then there exists an $\epsilon > 0$ such that the set*

$$\{x \in A(\overline{\mathbb{Q}}) \mid \widehat{h}_{\mathrm{nt}}(x) \leq \epsilon\}$$

is not Zariski dense in W.

Earlier, Zhang [Zha95a] had proved a similar result for the multiplicative group $\mathbb{G}_m^n$. Zhang [Zha95b, Zha06] also proposed a more general conjecture for what he called *polarizable* morphisms; a morphism $\Phi : X \longrightarrow X$ on a projective variety X is said to be polarizable if there is an ample line bundle $\mathcal{L}$ on X such that $\Phi^*\mathcal{L} \cong q\mathcal{L}$ for some integer $q > 1$. When a polarizable map Φ is defined over a number field, it gives rise to a canonical height $\widehat{h}_\Phi$ with the property that $\widehat{h}_\Phi(\Phi(\alpha)) = q\widehat{h}_\Phi(\alpha)$ for all $\alpha \in X(\overline{\mathbb{Q}})$. Zhang makes the following Bogomolov-type conjecture in this more general context.

Conjecture 1.2. *(Zhang) Let $\Phi : X \longrightarrow X$ be a polarizable morphism of a projective variety defined over a number field and let W be a subvariety of X that is not preperiodic under Φ. Then there exists an $\epsilon > 0$ such that the set*

$$\{x \in W(\overline{\mathbb{Q}}) \mid h_\Phi(x) \leq \epsilon\}$$

is not Zariski dense in W.

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The definition of preperiodicity for varieties here is the same as the usual definition of preperiodicity for points. More precisely, for any quasiprojective variety X, any endomorphism $\Phi : X \longrightarrow X$, and any subvariety $V \subset X$, we say that V is *Φ-preperiodic* if there exists $N \geq 0$, and $k \geq 1$ such that $\Phi^N(V) = \Phi^{N+k}(V)$. Note that when A is an abelian variety and Φ is a multiplication-by-n map (for $n \geq 2$), an irreducible subvariety W is preperiodic if and only if it is a torsion translate of an abelian subvariety of A.

We note that if $X = \left(\mathbb{P}^1\right)^n$, and if Φ is given by the coordinatewise action of z^d (for some $d \geq 2$) on X, then Conjecture 1.2 reduces to the result proved by Zhang in [Zha95a].

In this paper, we prove the following special case of Conjecture 1.2.

Theorem 1.3. *Let $f_1, \dots, f_m \in \overline{\mathbb{Q}}[X]$ be polynomials of degree $d > 1$, let $\Phi := (f_1, \dots, f_m)$ be their coordinatewise action on $\mathbb{A}^m$, and let L be a line in $\mathbb{A}^m$ defined over $\overline{\mathbb{Q}}$. If L is not Φ-preperiodic, then there exists an $\epsilon > 0$ such that*

$$S_{L,\Phi,\epsilon} := \{x \in L(\overline{\mathbb{Q}}) \mid \widehat{h}_\Phi(x) \leq \epsilon\}$$

is finite (see Section 2 for the definition of $\widehat{h}_\Phi$).

Baker and Hsia [BH05, Theorem 8.10] previously proved Theorem 1.3 in the special case where $f_1 = f_2$ and $m = 2$.

In Section 2 we introduce our notation for canonical heights associated to polynomials, then in Section 3 we present some general results regarding polynomials which share the same Julia set. In Section 4 we prove Theorem 1.3, and then describe few examples of preperiodic lines under the coordinatewise action of polynomials of same degree.

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2. Preliminaries

Heights. Let $\mathbb{M}_{\mathbb{Q}}$ be the usual set of absolute values on $\mathbb{Q}$, normalized so that the archimedean absolute value is simply the absolute value $|\cdot|$ and $|p|_p = 1/p$ for each p-adic absolute value $|\cdot|_p$. For any finite extension K of $\mathbb{Q}$ we define $\mathbb{M}_K$ to be the set of absolute values on K that extend elements of $\mathbb{M}_{\mathbb{Q}}$. Then, for any $x \in \overline{\mathbb{Q}}$ we define the Weil height of x to be

$$h(x) = \frac{1}{[\mathbb{Q}(x) : \mathbb{Q}]} \cdot \sum_{v \in \mathbb{M}_{\mathbb{Q}(x)}} \sum_{\substack{w|v \\ w \in \mathbb{M}_{\mathbb{Q}(x)}}} \log \max\{|x|_w^{[\mathbb{Q}(x)_w : \mathbb{Q}_v]}, 1\}$$

where $\mathbb{Q}_v$ and $\mathbb{Q}(x)_w$ are the completions of $\mathbb{Q}$ and $\mathbb{Q}(x)$ at v and w respectively (see [BG06, Chapter 1] for details).

For a polynomial $f \in \overline{\mathbb{Q}}[X]$ of degree greater than 1, define the f-canonical height $\widehat{h}_f : \overline{\mathbb{Q}} \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\widehat{h}_f(x) = \lim_{n\to\infty} \frac{h(f^n(x))}{(\deg f)^n}, \tag{2.1}$$

following Call-Silverman [CS93] (where f^n denotes the n-th iterate of f).

Let $f_1, \ldots, f_m \in \overline{\mathbb{Q}}[X]$ be polynomials of degree $d > 1$, and let $\Phi := (f_1, \ldots, f_m)$ be their coordinatewise action on $\mathbb{A}^m$; that is,

$$\Phi(x_1, \ldots, x_m) = (f_1(x_1), \ldots, f_m(x_m)).$$

We define the Φ-canonical height $\widehat{h}_\Phi : \mathbb{A}^m(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\widehat{h}_\Phi(x_1, \ldots, x_m) = \sum_{i=1}^m \widehat{h}_{f_i}(x_i).$$

Note that while $\mathbb{A}^m$ is not a projective variety, Φ extends to a map $\tilde{\Phi} : (\mathbb{P}^1)^m \longrightarrow (\mathbb{P}^1)^m$. Furthermore, $\tilde{\Phi}$ is polarizable, since

$$\tilde{\Phi}^* \bigotimes_{i=1}^m \mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \bigotimes_{i=1}^m \mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(d),$$

where pr_i is the projection of $(\mathbb{P}^1)^m$ onto its i-th coordinate.

Remark. Theorem 1.3 is *not* true if one allows the polynomials f_i to have different degrees. This is easily seen, for example, in the case where $m = 2$, the line L is the diagonal, and $f_2 = f_1^2$. The map $\Phi = (f_1, \ldots, f_m)$ is only polarizable when $\deg f_i = \deg f_j$, so this is not a counterexample to Conjecture 1.2.

3. Symmetries of the Julia set

In this section we recall the main results regarding polynomials which share the same Julia set. For a polynomial $f \in \mathbb{C}[X]$, we let $J(f)$ denote the Julia set of f (see [Bea91, Chapter 3] or [Mil99] for the definition of a Julia set of a rational function over the complex numbers). As proved by Beardon [Bea90, Bea92], any family of polynomials which have the same Julia set $\mathcal{J}$ is determined by the symmetries of $\mathcal{J}$.

Definition 3.1. *If $f \in \mathbb{C}[z]$ is a polynomial and $J(f)$ is its Julia set, then the symmetry group $\Sigma(f)$ of $J(f)$ is defined by*

$$\Sigma(f) = \{\sigma \in \mathfrak{C} : \sigma(J(f)) = J(f)\},$$

where $\mathfrak{C}$ is the group of conformal Euclidean isometries.

Beardon [Bea90, Lemma 3] computed $\Sigma(f)$ for any $f \in \mathbb{C}[z]$.

Lemma 3.2. *The isometry group $\Sigma(f)$ is a group of rotations about some point $\zeta \in \mathbb{C}$, and it is either trivial, or finite cyclic group, or the group of all rotations about ζ.*

To prove the result above, Beardon uses the fact that each polynomial f of degree $d \geq 2$ is conjugate to a monic polynomial $\tilde{f} \in \mathbb{C}[z]$ which has no term in z^{d-1}. Clearly, if $\tilde{f} = \gamma \circ f \circ \gamma^{-1}$ where $\gamma \in \mathbb{C}[z]$ is a linear polynomial, then $J(\tilde{f}) = \gamma(J(f))$; thus it suffices to prove Lemma 3.2 for $\tilde{f}$ instead of f. If $\tilde{f}(z) = z^d$, then $J(\tilde{f})$ is the unit circle, and $\Sigma(\tilde{f})$ is the group of all rotations about 0 (see also [Bea90, Lemma 4]). If $\tilde{f}$ is not a monomial, then we choose $b \geq 1$ maximal such that we can find $a \geq 0$ and $\tilde{f}_1 \in \mathbb{C}[z]$ satisfying $\tilde{f}(z) = z^a \tilde{f}_1(z^b)$. In this case, $J(\tilde{f})$ is the finite cyclic group of rotations generated by the multiplication by $\exp(2\pi i/b)$ on $\mathbb{C}$ (see [Bea90, Theorem 5]). Beardon also proves the following result (see [Bea90, Lemma 7]) which we will use later.

Lemma 3.3. *If $f \in \mathbb{C}[z]$ is a polynomial of degree $d \geq 2$ and $\sigma \in J(f)$, then $f \circ \sigma = \sigma^d \circ f$.*

The following classification of polynomials which have the same Julia set is proven by Beardon in [Bea92, Theorem 1].

Lemma 3.4. *If $f, g \in \mathbb{C}[z]$ have the same Julia set, then there exists $\sigma \in \Sigma(f)$ such that $g = \sigma \circ f$.*

4. Proof of our main result

Proof of Theorem 1.3. Suppose that for every $\epsilon > 0$, the set $S_{L,\Phi,\epsilon}$ is infinite. We will show that this implies that L must be Φ-preperiodic.

We first note that it suffices to prove the theorem for the line $L' = (\sigma_1, \dots, \sigma_m)(L)$ and the map

$$\Phi' = (\sigma_1 \circ f_1 \circ \sigma_1^{-1}, \dots, \sigma_m \circ f_m \circ \sigma_m^{-1})$$

for some linear automorphisms $\sigma_1, \dots, \sigma_m$ of $\mathbb{A}^1$. This follows from the fact that L is preperiodic for Φ if and only if L' is preperiodic for Φ' along with the equality

$$\widehat{h}_{\Phi'}(\sigma_1\alpha_1, \dots, \sigma_m\alpha_m) = \widehat{h}_{\Phi}(\alpha_1, \dots, \alpha_m). \tag{4.1}$$

Note that (4.1) is a simple consequence of Definition 2.1, since $|h(\sigma_i x) - h(x)|$ is bounded for all $x \in \overline{\mathbb{Q}}$.

We now proceed by induction on m; the case $m = 1$ is obvious.

If the projection of L on any of the coordinates consists of only one point, we are done by the inductive hypothesis. Indeed, without loss of generality, assume the projection of L on the first coordinate equals $\{z_1\}$, then $L = \{z_1\} \times L_1$, where $L_1 \subset \mathbb{A}^{m-1}$ is a line, and $\widehat{h}_{f_1}(z_1) = 0$. Since only preperiodic points have canonical height equal to 0 (see [CS93, Cor. 1.1.1]), we conclude that z_1 is f_1-preperiodic, and thus we are done by the induction hypothesis applied to L_1.

Suppose now that L projects dominantly onto each coordinate of $\mathbb{A}^m$. For each $i = 2, \dots, m$, we let L_i be the projection of L on the first and the i-th coordinates of $\mathbb{A}^m$. Then L_i is a line given by an equation $X_1 = \sigma_i(X_i)$, for some linear polynomial $\sigma_i \in \overline{\mathbb{Q}}[X]$. Clearly, it suffices to show that for each $i = 2, \ldots, m$, the line L_i is preperiodic under the action of (f_1, f_i) on the corresponding two coordinates of $\mathbb{A}^m$. Indeed, if for each $i = 2, \ldots, m$, we show that there exist $a_i, b_i \in \mathbb{N}$ (with $a_i < b_i$) such that $(f_1^{a_i}, f_i^{a_i})(L_i) = (f_1^{b_i}, f_i^{b_i})(L_i)$, then

$$(f_1^a, \ldots, f_m^a)(L) = (f_1^{a+b}, \ldots, f_m^{a+b})(L),$$

where $a := \max_{i=2}^m a_i$, and b is the least common multiple of all $(b_i - a_i)$ for $i = 2, \ldots, m$.

Let $\tilde{f}_i := \sigma_i \circ f_i \circ \sigma_i^{-1}$ and let $\Delta = (x, x) \in \mathbb{A}^2$ be the diagonal on $\mathbb{A}^2$. By our remarks at the beginning of the proof, it suffices to show that $(\mathrm{id}, \sigma_i)(L_i) = \Delta$ is preperiodic under the action of $(f_1, \tilde{f}_i)$. Furthermore, the fact that we have an infinite sequence $(z_{n,1}, z_{n,i}) \in L_i(\overline{\mathbb{Q}})$ with

$$\lim_{n\to\infty} \widehat{h}_{f_1}(z_{n,1}) = \lim_{n\to\infty} \widehat{h}_{f_i}(z_{n,i}) = 0$$

implies that we have

$$\lim_{n\to\infty} \widehat{h}_{\tilde{f}_i}(z_{n,1}) = 0,$$

because of (4.1). Fix an embedding $\theta : \overline{\mathbb{Q}} \longrightarrow \mathbb{C}$ and let f_1^θ and $\tilde{f}_i^\theta$ be the images of f_1 and $\tilde{f}_i$, respectively, in $\mathbb{C}[X]$ under this embedding. Then, by [BH05, Corollary 4.6], the Galois orbits of the points $\{z_{n,1}\}_{n\in\mathbb{N}}$ are equidistributed with respect to the equilibrium measures on the Julia sets of both f_1^θ and $\tilde{f}_i^{\theta}$. Since the support of the equilibrium measure μ_g of a polynomial $g \in \mathbb{C}[X]$ is equal to the Julia set of g ([BH05, Section 4]), we must have $J(\tilde{f}_i^\theta) = J(f_1^\theta)$.

By Lemma 3.4 (see also [BE87, AH96]), there exists a conformal Euclidean symmetry $\mu_i : z \longrightarrow a_i z + b_i$ such that $\mu_i(J(f_1^\theta)) = J(f_1^\theta)$ and $\tilde{f}_i^\theta = \mu_i \circ f_1^\theta$. Note that a_i and b_i must be in the image of $\overline{\mathbb{Q}}$ under θ since the coefficients of f_1^θ and f_i^θ are. Let τ_i be the map $\tau_i : z \longrightarrow \theta^{-1}(a_i)z + \theta^{-1}(b_i)$. Then we have $\tilde{f}_i = \tau_i \circ f_1$.

If τ_i has infinite order, then it follows from [Bea90, Lemma 4] that there exist linear polynomials γ_1, γ_i such that $\gamma_1 \circ f_1 \circ \gamma_1^{-1} = \gamma_i \circ \tilde{f}_i \circ \gamma_i^{-1} = X^d$. In this case, we reduce our problem to the usual Bogomolov conjecture for $\mathbb{G}_m^2$, proved by Zhang [Zha92]. Indeed, Zhang proves that if a curve C in $\mathbb{G}_m^2$ has an infinite family of algebraic points with height tending to zero, then it must be a torsion translate of an algebraic subgroup of $\mathbb{G}_m^2$; that is, $C = \xi A$ where ξ has finite order and A is an algebraic subgroup of $\mathbb{G}_m^2$. Since $(\xi A)^n = \xi^n A$ and ξ has finite order, it is clear that such a curve is preperiodic under the map $(X, Y) \mapsto (X^d, Y^d)$.

We may suppose then that τ_i has finite order. By Lemma 3.3, we have $f_1 \circ \tau_i = \tau_i^d \circ f_1$. Thus, we have

$$\tilde{f}_i^{\,k} = \tau_i^{(d^k-1)/(d-1)} \circ f_1^k$$

for all $k \geq 1$. Since τ_i has finite order, we conclude that the set

$$\{\tau_i^{(d^k-1)/(d-1)}\}_{k\geq 0}$$

is finite. This implies that the set of curves of the form $(f_1^k, \tilde{f}_i^{\,k})(\Delta)$ is finite, which means the diagonal subvariety Δ is preperiodic under the action of $(f_1, \tilde{f}_i)$, as desired. ☐

4.1. **Examples.** Note that Δ is periodic under Φ only if there is some n such that $f^n = g^n$. Take, for example, $f(x) = x^3$, $g(x) = -x^3$; then we have $f^2 = g^2$, so Δ has period two under the action of Φ. More generally, there exists some n such that $f^n = g^n$ if and only if $f(x) = -\beta + \gamma h(x+\beta)$ and $g(x) = -\beta + h(x+\beta)$ for some $\gamma \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and $h \in x^r\mathbb{C}[x^s]$ (with $r, s \geq 0$) such that $\gamma^s = 1$ and $\gamma^{(d^n-1)/(d-1)} = 1$ (see [GTZ08, Prop. 6.3]). On the other hand, if $f(x) = h(x^s)$ for some positive integer $s \geq 2$ and $\zeta \in \mathbb{C}$ is an s-th root of unity, and $g(x) = \zeta \cdot f(x)$, then Δ is Φ-*preperiodic*, but it is not Φ-*periodic*.

We believe that it is possible to extend the methods of the proof of Theorem 1.3 to the case of arbitrary rational maps $\varphi_1, \ldots, \varphi_m$ of the same degree, though the proof seems to be much more difficult, requiring in particular Mimar's [Mim97] results on arithmetic intersections of metrized line bundles and an analysis of Douady-Hubbard-Thurston's [DH93] classification of critically finite rational maps with parabolic orbifolds. We intend to treat this problem in a future paper.

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Dragos Ghioca, Department of Mathematics & Computer Science, University of Lethbridge, Lethbridge, AB T1K 3M4

E-mail address: `dragos.ghioca@uleth.ca`

Thomas Tucker, Department of Mathematics, Hylan Building, University of Rochester, Rochester, NY 14627

E-mail address: `ttucker@math.rochester.edu`