A VARIANT OF A THEOREM BY AILON-RUDNICK FOR ELLIPTIC CURVES

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Abstract. Given a smooth projective curve C defined over $\overline{\mathbb{Q}}$ and given two elliptic surfaces $\mathcal{E}_1 \longrightarrow C$ and $\mathcal{E}_2 \longrightarrow C$ along with sections $\sigma_{P_i}, \sigma_{Q_i}$ (corresponding to points P_i, Q_i of the generic fibers) of $\mathcal{E}_i$ (for $i = 1, 2$), we prove that if there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ such that for some integers $m_{1,t}, m_{2,t}$, we have that $[m_{i,t}](\sigma_{P_i}(t)) = \sigma_{Q_i}(t)$ on $\mathcal{E}_i$ (for $i = 1, 2$), then at least one of the following conclusions must hold: either (i) there exists an isogeny $\psi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ and also there exist nontrivial endomorphisms φ_i of $\mathcal{E}_i$ (for $i = 1, 2$) such that $\varphi_2(\sigma_{P_2}) = \psi(\varphi_1(\sigma_{P_1}))$; or (ii) Q_i is a multiple of P_i for some $i = 1, 2$. A special case of our result answers a conjecture made by Silverman.

1. Introduction

In [AR04], Ailon and Rudnick showed that for two multiplicatively independent non-constant polynomials $a, b \in \mathbb{C}[x]$, there is a nonzero polynomial $h \in \mathbb{C}[x]$, depending on a and b such that $\gcd(a^n - 1, b^n - 1) \mid h$ for all positive integer n. In this paper, we prove a similar result for elliptic curves; instead of working with the multiplicative group $\mathbb{G}_m$, we work with the group law on an elliptic curve defined over a function field. The result of Ailon-Rudnick relies crucially on the Serre-Ihara-Tate theorem (see [Lan65]), while our result relies crucially on recent Bogomolov type results for elliptic surfaces due to DeMarco and Mavraki [DM].

Throughout our article, we work with elliptic surfaces over $\overline{\mathbb{Q}}$; more precisely, given a projective, smooth curve C defined over $\overline{\mathbb{Q}}$, an *elliptic surface* $\mathcal{E}/C$ is given by a morphism $\pi : \mathcal{E} \longrightarrow C$ over $\overline{\mathbb{Q}}$ where the generic fiber of π is an elliptic curve E defined over $K = \overline{\mathbb{Q}}(C)$, while for all but finitely many $t \in C(\overline{\mathbb{Q}})$, the fiber $\mathcal{E}_t := \pi^{-1}(\{t\})$ is an elliptic curve defined over $\overline{\mathbb{Q}}$. Recall that a section σ of π (i.e. a map $\sigma : C \longrightarrow \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}|_C$) gives rise to a K-rational point of E. Conversely, a point $P \in E(K)$ corresponds to a section of π; if we need to indicate the dependence on P, we will denote it by σ_P. So, for all but finitely many $t \in C(\overline{\mathbb{Q}})$, the intersection of the image of σ_P in $\mathcal{E}$ with the fiber above t is a point $P_t := \sigma_P(t)$ on the elliptic curve $\mathcal{E}_t := \pi^{-1}(\{t\})$. For any integer k, the multiplication-by-k map $[k]$

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on E extends to a morphism on $\mathcal{E}$; if there is no risk of confusion, we still denote the extension by $[k]$.

We prove the following result.

Theorem 1.1. *Let $\pi_i : \mathcal{E}_i \longrightarrow C$ be elliptic surfaces over a curve C defined over $\overline{\mathbb{Q}}$ with generic fibers E_i, and let $\sigma_{P_i}, \sigma_{Q_i}$ be sections of π_i (for $i = 1, 2$) corresponding to points $P_i, Q_i \in E_i(\overline{\mathbb{Q}}(C))$. If there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ for which there exist some $m_{1,t}, m_{2,t} \in \mathbb{Z}$ such that $[m_{i,t}]\sigma_{P_i}(t) = \sigma_{Q_i}(t)$ for $i = 1, 2$, then at least one of the following properties must hold:*

(i) *there exist isogenies $\varphi : E_1 \longrightarrow E_2$ and $\psi : E_2 \longrightarrow E_2$ such that $\varphi(P_1) = \psi(P_2)$.*

(ii) *for some $i \in \{1, 2\}$, there exists $k_i \in \mathbb{Z}$ such that $[k_i]P_i = Q_i$ on E_i.*

We note here that, in contrast to similar results such as [AR04], the ambient algebraic group ($\mathcal{E}_1 \times \mathcal{E}_2$ in our case, as opposed to $\mathbb{G}_m$ for [AR04]) need not be defined over the field of constants in $k(C)$.

A special case of our result (when both Q_1 and Q_2 are the zero elements) answers in the affirmative [Sil04b, Conjecture 7]; this is carried out in a more general setting (over the complex numbers and also, giving a more precise connection to the original GCD problem of Ailon-Rudnick) in our Proposition 4.3 from Section 4. We also note that the special case of Theorem 1.1 when $Q_1 = Q_2 = 0$ was solved by Masser and Zannier (see [MZ14]) when both elliptic surfaces are defined over $\mathbb{C}$.

Silverman's question [Sil04b, Conjecture 7] was motivated by work of Ailon-Rudnick [AR04], who showed that the greatest common divisor of $a^n - 1$ and of $b^n - 1$ for multiplicatively independent polynomials $a, b \in \mathbb{C}[T]$ has bounded degree (see also the generalization by Corvaja-Zannier [CZ13b] along with the related results from [CZ08, CZ11, CZ13a]). In turn, the result of Ailon-Rudnick was motivated by the work of Bugeaud-Corvaja-Zannier [BCZ03] who obtained an upper bound for $\gcd(a^k - 1, b^k - 1)$ (as k varies in $\mathbb{N}$) for given $a, b \in \overline{\mathbb{Q}}$. On the other hand, Silverman [Sil04a] showed that the degree of $\gcd(a^m - 1, b^n - 1)$ could be quite large when $a, b \in \overline{\mathbb{F}_p}[T]$; see also the authors' previous paper [GHT17], where (using as technical ingredient [Ghi14] in place of [DM]) we study the $\gcd(a^m - 1, b^n - 1)$ when a and b are polynomials over arbitrary fields of positive characteristic, along with other generalizations on the same theme. Finally, we mention the work of Denis [Den11] who studied the same problem of the greatest common divisor in the context of Drinfeld modules.

As hinted in [Sil04b], this *greatest common divisor (GCD) problem* may be studied in much higher generality; for example, if one knew the result of DeMarco-Mavraki [DM] (see Theorem 2.3) in the context of abelian varieties, then our method would extend to a similar conclusion for arbitrary abelian schemes over a base curve. DeMarco-Mavraki's theorem can be interpreted as an extension of Masser-Zannier's theorem (see [MZ12]) in the same spirit as Bogomolov conjecture is an extension of the classical Manin-Mumford Conjecture. So, even though the extension to arbitrary abelian varieties of the results from [DM] is expected to be quite challenging, we mention that there is some progress in this direction due to Cinkir [Cin11], Gubler [Gub07], and Yamaki [Yam17], who proved various cases of the Bogomolov conjecture for abelian varieties defined over function fields.

Our Theorem 1.1 is related also to [BC16, Theorem 1.1] (see also the extension from [BC17]) where it is shown that given n linearly independent sections P_i on the Legendre elliptic family $y^2 = x(x-1)(x-t)$, there are at most finitely many parameters t such that the points $(P_i)_t$ satisfy two independent linear relations on the corresponding elliptic curve. Therefore, a special case of the result by Barroero and Capuano is that given sections P_1, P_2, Q_1, Q_2 on the Legendre elliptic surface, if these 4 sections are linearly independent, then there are at most finitely many t such that for some $m_t, n_t \in \mathbb{Z}$ we have that $[m_t](P_1)_t = (Q_1)_t$ and $[n_t](P_2)_t = (Q_2)_t$. However, in our Theorem 1.1 we obtain the same conclusion under the weaker hypothesis that Q_i is not a multiple of P_i for $i = 1, 2$ and also that P_1 and P_2 are linearly independent. We also note that the constant case of Barroero-Capuano's theorem is covered by the results of Habegger-Pila [HP16].

A special case of our Theorem 1.1 bears a resemblance to the classical Mordell-Lang problem proven by Faltings [Fal94] (see also [Hru96] for the case of semiabelian varieties defined over function fields). Indeed, with the notation as in Theorem 1.1, assume there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ such that for some $m_t \in \mathbb{Z}$ we have

$$[m_t](P_i)_t = (Q_i)_t \text{ for } i = 1, 2. \tag{1.2}$$

Also assume there is no $m \in \mathbb{Z}$ such that $[m]P_i = Q_i$ for $i = 1, 2$. Then the conclusion of Theorem 1.1 yields the existence of isogenies $\varphi : E_1 \longrightarrow E_2$ and $\psi : E_2 \longrightarrow E_2$ such that $\varphi(P_1) = \psi(P_2)$. Thus, using that (1.2) holds for infinitely many $t \in C(\overline{\mathbb{Q}})$ we see that

$$\varphi(Q_1) = \psi(Q_2). \tag{1.3}$$

Therefore, if we let $X \subset \mathcal{A} := \mathcal{E}_1 \times \mathcal{E}_2$ be the 1-dimensional subscheme corresponding to the section (Q_1, Q_2), and we let $\Gamma \subset \mathcal{A}$ be the cyclic subgroup spanned by (P_1, P_2), then the existence of infinitely many $\gamma \in \Gamma$ such that for some $t \in C(\overline{\mathbb{Q}})$ we have $\gamma_t \in X$ implies that X is contained in a proper algebraic subgroup of $\mathcal{A}$ (as given by the equation (1.3)). Such a statement can be viewed as a relative version of the classical Mordell-Lang problem; note that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are constant elliptic surfaces with generic fibers E_i^0 defined over $\overline{\mathbb{Q}}$, while $\Gamma \subset (E_1^0 \times E_2^0)(\overline{\mathbb{Q}})$, then this question is a special case of Faltings' theorem [Fal94] (formerly known as the Mordell-Lang conjecture). It is natural to ask whether the above relative version of the Mordell-Lang problem holds more generally when $\mathcal{A} \longrightarrow C$ is an arbitrary semiabelian scheme, $X \subset \mathcal{A}$ is a 1-dimensional scheme and $\Gamma \subset \mathcal{A}$ is an arbitrary finitely generated group. This more general question is also related to the bounded height problems studied in [AMZ17] in the context of pencils of finitely generated subgroups of $\mathbb{G}_m^n$.

In the next section of this paper, we review some preliminary material. Following that, in Section 3, we prove Theorem 1.1. The proof in the case of non-constant sections is quite similar to the proofs of the main results of [AR04] and [HT17], while the case of constant sections requires a different argument. In Section 4, we give a positive answer to Silverman's conjecture [Sil04b, Conjecture 7].

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2. Preliminaries

From now on, we fix an elliptic surface $\pi : \mathcal{E} \longrightarrow C$, where C is a projective, smooth curve defined over $\overline{\mathbb{Q}}$. We denote by E the generic fiber of $\mathcal{E}$; this is an elliptic curve defined over $\overline{\mathbb{Q}}(C)$. For all but finitely many $t \in C(\overline{\mathbb{Q}})$, we have that $\mathcal{E}_t := \pi^{-1}(\{t\})$ is an elliptic curve defined over $\overline{\mathbb{Q}}$.

2.1. **Isotriviality.** We say that $\mathcal{E}$ is *isotrivial* if the j-invariant of the generic fiber is a constant function (on C); for isotrivial elliptic surfaces $\mathcal{E}$, all smooth fibers of π are isomorphic (to the generic fiber E). If $\mathcal{E}$ is isotrivial, then there exists a finite cover $C' \longrightarrow C$ such that $\mathcal{E}' := \mathcal{E} \times_C C'$ is a *constant (elliptic) surface* over C', i.e., there exists an elliptic curve E^0 defined over $\overline{\mathbb{Q}}$ such that $\mathcal{E}' = E^0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} C'$. Furthermore, for a constant elliptic surface $E^0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} C'$, we say that σ_P is a *constant section* if $P \in E^0(\overline{\mathbb{Q}})$.

2.2. **Canonical height on an elliptic surface.** For each $t \in C(\overline{\mathbb{Q}})$ such that $\mathcal{E}_t$ is an elliptic curve, we let $\widehat{h}_{\mathcal{E}_t}$ be the Néron-Tate canonical height for the points in $\mathcal{E}_t(\overline{\mathbb{Q}})$ (for more details, see [Sil86]). There are two important properties of the canonical height which we will use:

(1) $\widehat{h}_{\mathcal{E}_t}(P_t) = 0$ if and only if P_t is a torsion point of $\mathcal{E}_t$, i.e., there exists a positive integer k such that $[k]P_t = 0$.
(2) for each $k \in \mathbb{Z}$ we have that $\widehat{h}_{\mathcal{E}_t}([k]P_t) = k^2 \cdot \widehat{h}_{\mathcal{E}_t}(P_t)$.

Also, we let $\widehat{h}_E$ be the Néron-Tate canonical height on the generic fiber E constructed with respect to the Weil height on the function field $\overline{\mathbb{Q}}(C)$ (for more details, see [Sil94a]). Property (2) above holds also on the generic fiber, i.e., $\widehat{h}_E([k]P) = k^2 \cdot \widehat{h}_E(P)$. On the other hand, property (1) above holds only if $\mathcal{E}$ is non-isotrivial. Furthermore, if $\mathcal{E} = E \times_C C$ is a constant family (where E is an elliptic curve defined over $\overline{\mathbb{Q}}$), then for any $P \in E(\overline{\mathbb{Q}}(C))$, we have that $\widehat{h}_E(P) = 0$ if and only if $P \in E(\overline{\mathbb{Q}})$.

2.3. **Variation of the canonical height.** We let h_C be a given Weil height for points in $C(\overline{\mathbb{Q}})$ corresponding to a divisor of degree 1 on C.

Let σ_P be a section of the elliptic surface $\mathcal{E} \longrightarrow C$ corresponding to a point P on the generic fiber E. Then, for all but finitely many $t \in C(\overline{\mathbb{Q}})$, we have that the intersection of the image of σ_P in $\mathcal{E}$ with the fiber above t

is a point P_t, on the elliptic curve $\mathcal{E}_t$. The following important fact will be used in our proof (see [Tat83, Sil83]):

$$\lim_{h_C(t)\to\infty} \frac{\widehat{h}_{\mathcal{E}_t}(P_t)}{h_C(t)} = \widehat{h}_E(P). \tag{2.1}$$

Furthermore, the following more precise result holds, as proven by Silverman [Sil94b],

$$\widehat{h}_{\mathcal{E}_t}(P_t) = h_{C,\eta(P)}(t) + O_P(1), \tag{2.2}$$

where $\eta(P)$ is a divisor on C of degree equal to $\widehat{h}_E(P)$ and $h_{C,\eta(P)}$ is a given Weil height for the points in $C(\overline{\mathbb{Q}})$ corresponding to the divisor $\eta(P)$, while the implicit constant from the term $O_P(1)$ is only dependent on the section σ_P (and implicitly on the divisor $\eta(P)$), but not on $t \in C(\overline{\mathbb{Q}})$.

2.4. **Points of small height on sections.** We will use the following result of DeMarco-Mavraki [DM, Theorem 1.4], which extends [DWY16] (and in turn, uses the extensive analysis from [Sil94b] regarding the variation of the canonical height in an elliptic fibration). We also note that the case of isotrivial elliptic curves from Theorem 2.3 was previously proven by Zhang [Zha98], as part of Zhang's famous proof of the classical Bogomolov conjecture.

Theorem 2.3 (DeMarco-Mavraki [DM])**.** *Let $\mathcal{E}_1, \mathcal{E}_2$ be elliptic fibrations over the same $\overline{\mathbb{Q}}$-curve C. Let P_i be a section of $\mathcal{E}_i$ (for $i = 1, 2$) with the property that there exists an infinite sequence $\{t_n\} \subset C(\overline{\mathbb{Q}})$ such that*

$$\lim_{n\to\infty} \widehat{h}_{(\mathcal{E}_i)_{t_n}}((P_i)_{t_n}) = 0 \textit{ for } i = 1, 2.$$

Then there exist group homomorphisms $\phi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ and $\psi : \mathcal{E}_2 \longrightarrow \mathcal{E}_2$, not both trivial, such that $\phi(P_1) = \psi(P_2)$.

3. Proof of our main result

Propositions 3.1 and 3.9 are key to our proof.

Proposition 3.1. *Let C be a projective, smooth curve defined over $\overline{\mathbb{Q}}$, and let $h_C(\cdot)$ be a Weil height for the algebraic points of C corresponding to a divisor of degree 1. Let P and Q be sections of an elliptic surface $\pi : \mathcal{E} \longrightarrow C$ with generic fiber E, and assume there exists no $k \in \mathbb{Z}$ such that $[k]P = Q$. In addition, assume $\widehat{h}_E(P) > 0$. If there exists an infinite sequence $\{t_i\} \subset C(\overline{\mathbb{Q}})$ such that for each $i \in \mathbb{N}$ there exists some $m_i \in \mathbb{Z}$ such that $[m_i]P_{t_i} = Q_{t_i}$, then $h_C(t_i)$ is uniformly bounded and $\lim_{i\to\infty} \widehat{h}_{\mathcal{E}_{t_i}}(P_{t_i}) = 0$.*

We note that the special case of Proposition 3.1 when $\pi : \mathcal{E} \longrightarrow C$ is a constant elliptic surface follows from [Sil83].

Proof of Proposition 3.1. Since $[m_i]P_{t_i} = Q_{t_i}$, we have

$$m_i^2 \cdot \widehat{h}_{\mathcal{E}_{t_i}}(P_{t_i}) = \widehat{h}_{\mathcal{E}_{t_i}}(Q_{t_i}). \tag{3.2}$$

Since $[k]P \neq Q$ for any $k \in \mathbb{Z}$ and the sequence $\{t_i\}$ is infinite, then

$$\lim_{i\to\infty} |m_i| = \infty. \tag{3.3}$$

We claim first that $h_C(t_i)$ is uniformly bounded. Indeed, assuming (at the expense, perhaps, of replacing $\{t_i\}$ by an infinite subsequence) that $\lim_{i\to\infty} h_C(t_i) = \infty$, equation (2.1) coupled with equations (3.2) and (3.3) yields a contradiction. To see this, we divide both sides of (3.2) by $h_C(t_i)$ and then take limits. Because $\widehat{h}_E(P) > 0$, equation (3.3) implies that the left hand side equals

$$\lim_{i\to\infty} m_i^2 \cdot \frac{\widehat{h}_{\mathcal{E}_{t_i}}(P_{t_i})}{h_C(t_i)} = \infty, \tag{3.4}$$

while the right hand side equals

$$\lim_{i\to\infty} \frac{\widehat{h}_{\mathcal{E}_{t_i}}(Q_{t_i})}{h_C(t_i)} = \widehat{h}_E(Q) < \infty, \tag{3.5}$$

which is a contradiction. So, indeed, we must have that $h_C(t_i)$ is uniformly bounded.

Next we prove that also $\widehat{h}_{\mathcal{E}_{t_i}}(Q_{t_i})$ is uniformly bounded. Using (2.2) (see [Sil94b]), we know that there exists a divisor $\eta(Q)$ of C of degree equal to $\widehat{h}_E(Q)$ such that

$$\widehat{h}_{\mathcal{E}_t}(Q_t) = h_{C,\eta(Q)}(t) + O(1), \tag{3.6}$$

where $h_{C,\eta(Q)}$ is a Weil height on $C(\overline{\mathbb{Q}})$ corresponding to the divisor $\eta(Q)$. Since h_C is a Weil height associated to a divisor D on C of degree 1, then for any positive integer $m > \deg(\eta(Q))$, the divisor $D_1 := mD - \eta(Q)$ has positive degree and therefore, is ample. Then [HS00, Proposition B.3.2] implies that any Weil height h_{C,D_1} associated to the divisor D_1 satisfies $h_{C,D_1}(t) \geq O(1)$ for all $t \in C(\overline{\mathbb{Q}})$. So,

$$mh_C(t) + O(1) \geq h_{C,\eta(Q)}(t) \text{ for } t \in C(\overline{\mathbb{Q}}). \tag{3.7}$$

Therefore $h_{C,\eta(Q)}(t_i)$ is uniformly bounded (since $h_C(t_i)$ is uniformly bounded). Then equation (3.6) provides the desired claim that

$$\widehat{h}_{\mathcal{E}_{t_i}}(Q_{t_i}) \text{ is bounded as } i \to \infty. \tag{3.8}$$

Finally, the fact that $\lim_{i\to\infty} \widehat{h}_{\mathcal{E}_i}(P_i) = 0$ follows easily from combining equations (3.2), (3.3) and (3.8). □

Proposition 3.9. *Let P and Q be sections of a constant elliptic fibration $\pi : \mathcal{E} \longrightarrow C$, and assume there exists no $k \in \mathbb{Z}$ such that $[k]P = Q$. In addition, assume P is a non-torsion, constant section. If there exists an* *infinite sequence* $\{t_i\} \subset C(\overline{\mathbb{Q}})$ *such that for each* $i \in \mathbb{N}$ *there exists some* $m_i \in \mathbb{Z}$ *such that* $[m_i]P_{t_i} = Q_{t_i}$*, then* $\lim_{i\to\infty} h_C(t_i) = \infty$*.*

Proof. We have that each fiber $\mathcal{E}_{t_i}$ is isomorphic to the generic fiber E^0, and so, because P is a constant section,

$$\widehat{h}_{\mathcal{E}_{t_i}}(P_{t_i}) = \widehat{h}_{E^0}(P^0), \tag{3.10}$$

where P^0 is the intersection of P with the generic fiber and $\widehat{h}_{E^0}(\cdot)$ is the Néron-Tate canonical height of the elliptic curve E^0 defined over $\overline{\mathbb{Q}}$ (i.e., it is not the canonical height on the generic fiber of $\mathcal{E}$ seen as an elliptic curve defined over the function field $\overline{\mathbb{Q}}(C)$).

Furthermore, since P^0 is not a torsion point of E^0, then $\widehat{h}_{E^0}(P^0) > 0$. Thus, from the equality $[m_i]P_{t_i} = Q_{t_i}$, along with equation (3.10) coupled with the fact that $|m_i| \to \infty$ (because $[k]P \neq Q$ for all integers k), we must have

$$\widehat{h}_{\mathcal{E}_{t_i}}(Q_{t_i}) = m_i^2 \widehat{h}_{E^0}(P^0) \to \infty. \tag{3.11}$$

Then, using (2.2), we have

$$\widehat{h}_{\mathcal{E}_{t_i}}(Q_{t_i}) = h_{C,\eta(Q)}(t_i) + O(1), \tag{3.12}$$

where $h_{C,\eta(Q)}$ is a Weil height on C corresponding to a certain divisor $\eta(Q)$. So, equations (3.11) and (3.12) yield $h_{C,\eta(Q)}(t_i) \to \infty$ and thus, $h_C(t_i) \to \infty$ (see [HS00, Proposition B.3.5], along with our similar argument from the proof of Proposition 3.1). □

Now we can prove our main result.

Proof of Theorem 1.1. First we note that if P_i is a torsion section (for some $i \in \{1, 2\}$), then conclusion (ii) holds trivially since then we would obtain there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ such that $(Q_i)_t = [k](P_i)_t$ for the same integer k. So, from now on, we assume that, both P_1 and P_2 are non-torsion sections on $\mathcal{E}_1, \mathcal{E}_2$ respectively. In particular, this means that if $\widehat{h}_{E_i}(P_i) = 0$, then $\mathcal{E}_i$ must be an isotrivial elliptic surface.

We assume there exists an infinite sequence $\{t_i\} \subset C(\overline{\mathbb{Q}})$ such that for each $i \in \mathbb{N}$ there exist $m_{i,1}, m_{i,2} \in \mathbb{Z}$ with the property that $[m_{i,1}](P_1)_{t_i} = (Q_1)_{t_i}$ and also $[m_{i,2}](P_2)_{t_i} = (Q_2)_{t_i}$. In addition, we assume conclusion (ii) does not hold, i.e., there is no $m \in \mathbb{Z}$ such that $[m]P_i = Q_i$ for some $i \in \{1, 2\}$. We split our analysis into two cases.

Case 1. $\widehat{h}_{E_i}(P_i) > 0$ for each $i = 1, 2$.

Applying then Proposition 3.1 to the sections P_i and Q_i, we obtain

$$\lim_{i\to\infty} \widehat{h}_{(\mathcal{E}_1)_{t_i}}((P_1)_{t_i}) = \lim_{i\to\infty} \widehat{h}_{(\mathcal{E}_2)_{t_i}}((P_2)_{t_i}) = 0. \tag{3.13}$$

Equation (3.13) along with Theorem 2.3 implies that conclusion (i) must hold in Theorem 1.1. Note that we obtain in this case that the morphisms $\varphi : E_1 \longrightarrow E_2$ and $\psi : E_2 \longrightarrow E_2$ from the conclusion of Theorem 2.3 are *both* isogenies since P_1 and P_2 are non-torsion sections.

Case 2. Either $\widehat{h}_{E_1}(P_1) = 0$ or $\widehat{h}_{E_2}(P_2) = 0$.

Without loss of generality, we assume $\widehat{h}_{E_1}(P_1) = 0$. Therefore (since P_1 is not torsion) $\mathcal{E}_1$ is an isotrivial elliptic surface, and furthermore, at the expense of replacing C by a finite cover (and also performing a base extending for $\mathcal{E}_1$ and $\mathcal{E}_2$), we may assume that $\mathcal{E}_1$ is a constant family. Thus, $\mathcal{E}_1 = E_1^0 \times_C C$ for some elliptic curve E_1^0 defined over $\overline{\mathbb{Q}}$. Then also P_1 is a constant (non-torsion) section, because $\widehat{h}_{\mathcal{E}_1}(P_1) = 0$. Finally, we let $h_C(\cdot)$ be a Weil height for the algebraic points of C with respect to a divisor of degree 1.

If $\widehat{h}_{E_2}(P_2) > 0$, then Proposition 3.1 applied to P_2 and Q_2 implies that $h_C(t_i)$ is uniformly bounded, which contradicts the conclusion of Proposition 3.9 applied to P_1 and Q_1. Therefore, we must have that $\widehat{h}_{E_2}(P_2) = 0$ and therefore, also $\mathcal{E}_2$ is an isotrivial elliptic surface. At the expense of (yet another) base extension, we may assume that also $\mathcal{E}_2 = E_2^0 \times C$ is a constant fibration. Then P_2 is a constant, non-torsion section on $\mathcal{E}_2$. We let P_i^0 be the intersection of P_i (for $i = 1, 2$) with the generic fiber of $\mathcal{E}_i$.

Now, if either Q_1 or Q_2 is also a constant section, then we get a contradiction since we assumed conclusion (ii) does not hold. Indeed, if for some $i = 1, 2$ we have that both P_i and Q_i are constant sections on the constant elliptic surface $\mathcal{E}_i$, then the existence of a point $t \in C(\overline{\mathbb{Q}})$ such that for some $k \in \mathbb{Z}$ we have $[k](P_i)_t = (Q_i)_t$ implies that actually $[k]P_i = Q_i$ on $\mathcal{E}_i$. So, we may assume that Q_1 and Q_2 are both non-constant sections on $\mathcal{E}_1$, respectively $\mathcal{E}_2$. Then, there is a (neither vertical, nor horizontal) curve $X \subset E_1^0 \times E_2^0$ containing all points $((Q_1)_t, (Q_2)_t)$ for $t \in C(\overline{\mathbb{Q}})$. Furthermore, our hypothesis means that this curve X intersects the subgroup $\Gamma \subset E_1^0 \times E_2^0$ spanned by the points $(P_1^0, 0)$ and $(0, P_2^0)$ in an infinite set. The classical Mordell-Lang conjecture (proven by Faltings [Fal94]) implies that X itself is a coset of an algebraic subgroup of $E_1^0 \times E_2^0$. Hence, because X projects dominantly onto each coordinate, there exists a nontrivial isogeny $\tau : E_1^0 \longrightarrow E_2^0$, and also there exist endomorphisms ϕ_i of E_i^0, not both trivial, such that

$$\tau(\phi_1(Q_1)) = \phi_2(Q_2). \tag{3.14}$$

Then, using (for any i such that $m_{i,1}$ and $m_{i,2}$ are nonzero) that

$$[m_{i,1}]P_1^0 = (Q_1)_{t_i} \text{ and } [m_{i,2}]P_2^0 = (Q_2)_{t_i}$$

along with the fact that $\tau\left(\phi_1\left((Q_1)_{t_i}\right)\right) = \phi_2\left((Q_2)_{t_i}\right)$, we obtain the conclusion in Theorem 1.1 with $\varphi := \tau \circ [m_{i,1}] \circ \phi_1$ and $\psi := [m_{i,2}] \circ \phi_2$. Finally, note that since P_1 and P_2 are non-torsion, then also φ and ψ are dominant morphisms. Indeed, if φ were trivial, then using that τ is an isogeny and that $m_{i,1} \neq 0$, we would obtain that ϕ_1 must be trivial. But then $\phi_2(Q_2) = 0$ (using (3.14)), which implies that $\phi_2 = 0$ because we assumed that Q_2 is a non-torsion section. So, if φ were trivial (and a completely similar argument works assuming ψ were trivial), we would get that both ϕ_1 and ϕ_2 are trivial, contradiction.

This concludes the proof of Theorem 1.1. □

4. Common divisors of elliptic sequences

In this section, we apply Theorem 1.1 to prove Silverman's conjecture [Sil04b, Conjecture 7] concerning common divisors of elliptic sequences; actually, our Proposition 4.3 provides a slightly more general statement than the original conjecture. We first recall the terminology and notation from [Sil04b] that we will use in this section.

Let k be an algebraically closed field of characteristic 0. Let C be a smooth projective curve defined over k and let $K = k(C)$ be the function field of C. For any point $\gamma \in C(k)$, we let $\mathrm{ord}_\gamma(D)$ denote the coefficient of γ in $D \in \mathrm{Div}(C)$. The *greatest common divisor* for any two effective divisors $D_1, D_2 \in \mathrm{Div}(C)$ is defined as

$$\mathrm{GCD}(D_1, D_2) = \sum_{\gamma \in C} \min\{\mathrm{ord}_\gamma(D_1), \mathrm{ord}_\gamma(D_2)\} \cdot (\gamma) \in \mathrm{Div}(C).$$

For an elliptic curve E defined over K, let $\pi : \mathcal{E} \longrightarrow C$ be an elliptic surface whose generic fiber is E and let $P \in E(K)$. Recall that the section corresponding to P is denoted by $\sigma_P : C \to \mathcal{E}$. We denote the image of σ_P by $\bar{P} := \sigma_P(C) \subset \mathcal{E}$.

Let E_1 and E_2 be elliptic curves defined over K, let $\mathcal{E}_i/C$ be elliptic surfaces with generic fibers E_i, and let $P_i \in E_i(K)$ for $i = 1, 2$. The greatest common divisor of P_1 and P_2 is given by

$$\mathrm{GCD}(P_1, P_2) = \mathrm{GCD}\left(\sigma_{P_1}^*(\overline{O}_{\mathcal{E}_1}), \sigma_{P_2}^*(\overline{O}_{\mathcal{E}_2})\right),$$

where $\overline{O}_{\mathcal{E}_i} := \sigma_{O_i}(C)$ is the zero section on $\mathcal{E}_i$ corresponding to the identity O_i of E_i and $\sigma_{P_i}^*(\overline{O}_{\mathcal{E}_i})$ is the pull-back under $\sigma_i : C \to \mathcal{E}_i$ of $\overline{O}_{\mathcal{E}_i}$ as a divisor of $\mathcal{E}_i$ for $i = 1, 2$. Thus, for any given $Q_i \in E_i(K)$, $\mathrm{GCD}(P_1 - Q_1, P_2 - Q_2)$ is the greatest common divisor of the two points $P_i - Q_i \in E_i$ for $i = 1, 2$. In the following, points P_1 and P_2 are called *(K-)dependent* if there are morphisms $\varphi : E_1 \longrightarrow E_2$ and $\psi : E_2 \longrightarrow E_2$ not both trivial such that $\varphi(P_1) = \psi(P_2)$; otherwise they are called *independent*. Note that if one of P_1 and P_2 is a torsion point, then they are automatically dependent.

Motivated by Ailon-Rudnick's result [AR04], Silverman conjectured that an elliptic analogue also exists. For the convenience of the reader, we recall his conjecture.

Conjecture 4.1 (Silverman [Sil94b, Conjecture 7])**.** *Let $K = k(C)$ be the function field of a smooth projective curve C over an algebraically closed field k of characteristic 0, let E_1/K and E_2/K be elliptic curves, and let $P_1 \in E_1(K)$ and $P_2 \in E_2(K)$ be K-independent points.*

(i) *There is a constant $c = c(K, E_1, E_2, P_1, P_2)$ so that*

$$\deg \mathrm{GCD}([n_1]P_1, [n_2]P_2) \le c \quad \textit{for all } n_1, n_2 \ge 1.$$

(ii) *Further, there is an equality*

$$\mathrm{GCD}([n]P_1, [n]P_2) = \mathrm{GCD}(P_1, P_2) \quad \text{for infinitely many } n \geq 1.$$

Remark 4.2. Silverman [Sil94b, Theorem 8] showed that Conjecture 4.1 is true provided that both E_1 and E_2 have constant j-invariant as a consequence of Raynaud's Theorem [Ray83].

As an application of Theorem 1.1, we prove that Conjecture 4.1 holds (even in a slightly stronger form); we strengthen further the conclusion from Conjecture 4.1 when $k = \overline{\mathbb{Q}}$.

Proposition 4.3. *Let k be an algebraically closed field of characteristic 0. Let C be a smooth projective curve defined over k, let $K = k(C)$ and let $E_i/K, i = 1, 2$, be elliptic curves defined over K. Let $P_i, Q_i \in E_i(K)$ for $i = 1, 2$ and furthermore, assume that P_1 and P_2 are K-independent.*

(i) *If $k = \overline{\mathbb{Q}}$, then there exists an effective divisor $D \in \mathrm{Div}(C)$ such that*
$$\mathrm{GCD}\left([n_1]P_1 - Q_1, [n_2]P_2 - Q_2\right) \leq D$$
for all integers n_i such that $[n_i]P_i \neq Q_i, i = 1, 2$.

(ii) *For an arbitrary algebraically closed field k of characteristic 0, there exists an effective divisor $D_0 \in \mathrm{Div}(C)$ such that*
$$\mathrm{GCD}\left([n_1]P_1, [n_2]P_2\right) \leq D_0$$
for all nonzero integers n_i.

(iii) *The set*
$$\{n \geq 1 : \mathrm{GCD}\left([n]P_1, [n]P_2\right) = \mathrm{GCD}\left(P_1, P_2\right)\}$$
has positive density in $\mathbb{N}$.

(iv) *For all but finitely many primes q, we have that $\mathrm{GCD}\left([q]P_1, [q]P_2\right) = \mathrm{GCD}\left(P_1, P_2\right)$.*

Remark 4.4. The conclusion of Proposition 4.3 (i) for an arbitrary algebraically closed field k of characteristic 0 would follow from our method once the validity of DeMarco-Mavraki's result [DM] (see Theorem 2.3) is extended over function fields. In turn, their result is contingent on establishing the smooth variation of the canonical height in fibers of an elliptic surface defined over a function field (over $\overline{\mathbb{Q}}$).

The proof of Proposition 4.3 relies on Theorem 1.1 and the following lemma which is a variant of [Sil04b, Lemma 4] bounding $\mathrm{ord}_\gamma(\sigma_{[n]P}^*(\overline{O}_{\mathcal{E}}))$ for $\gamma \in C$ and all integers $n \neq 0$.

Lemma 4.5. *Let k be an algebraically closed field of characteristic 0. Let E be an elliptic curve defined over $k(C)$ and let $\mathcal{E} \longrightarrow C$ be an elliptic surface whose generic fiber is E. Let $\gamma \in C(k)$ and let $P, Q \in E(k(C))$ be given. There exists a constant $m = m(\gamma, E, P, Q)$ such that $\mathrm{ord}_\gamma(\sigma_{[n]P}^*(\bar{Q})) \leq m$ for all integers n such that $[n]P \neq Q$.*

Proof. Observe that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q})) \geq 1$ if and only if $\sigma_{[n]P}(\gamma) = \sigma_Q(\gamma)$. Moreover, $\sigma_Q(\gamma)$ is a torsion point of $\mathcal{E}_\gamma$ if and only if there are more than one n such that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q})) \geq 1$.

It suffices to prove the assertion when $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q})) \geq 1$ for more than one integer n. Thus, we assume that $\sigma_Q(\gamma)$ is a torsion point of $\mathcal{E}_\gamma$. Let ℓ be the order of $\sigma_Q(\gamma)$ and assume that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q})) \geq 1$ for some integer n such that $[n]P \neq Q$. It follows that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q}))$ is finite and

$$\sigma_{[\ell n]P}(\gamma) = [\ell]\sigma_{[n]P}(\gamma) = [\ell]\sigma_Q(\gamma) = O_{\mathcal{E}_\gamma}, \tag{4.6}$$

which is the zero element for the elliptic curve $\mathcal{E}_\gamma$.

If Q is the zero element of E, then it follows from [Sil04b, Lemma 4] that the value of $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{O}_\mathcal{E}))$ is bounded independently of $n \neq 0$ and we are done in this case.

Assume that $Q \neq O$. Then (4.6) yields the inequality

$$\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q})) \leq \mathrm{ord}_\gamma(\sigma^*_{[\ell n]P}(\bar{O}_\mathcal{E})).$$

Observe that the right-hand side of the above inequality involves only $\mathrm{ord}_\gamma(\sigma^*_{[m]P}(\bar{O}_\mathcal{E}))$ which is bounded independently of the integer m in question as remarked above. Hence, we conclude that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q}))$ is bounded independently of $n \neq 0$ (and n such that $[n]P \neq Q$). As $Q \neq O$, we also have that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q}))$ is finite if $n = 0$. Thus we obtain that $\mathrm{ord}_\gamma(\sigma^*_{[n]P}(\bar{Q}))$ is bounded independently of n such that $[n]P \neq Q$, which concludes our proof. □

Proof of Proposition 4.3. We first prove part (i) in Proposition 4.3. So, for each $\gamma \in C(\overline{\mathbb{Q}})$, let $m_{i,\gamma}$ be an upper bound for $\mathrm{ord}_\gamma(\sigma^*_{[n]P_i}(\bar{Q}_i))$ as in Lemma 4.5. Set $m_\gamma = \min\{m_{1,\gamma}, m_{2,\gamma}\}$. Since P_1 and P_2 are independent, by Theorem 1.1 we may take $m_\gamma = 0$ for all but finitely many points $\gamma \in C(\overline{\mathbb{Q}})$; let S be the finite set of points $\gamma \in C(\overline{\mathbb{Q}})$ for which $m_\gamma > 0$. Let

$$D := \sum_{\gamma \in S} m_\gamma(\gamma).$$

Then, D is an effective divisor of C. Now it follows directly from Lemma 4.5 that $\mathrm{GCD}([n_1]P_1 - Q_1, [n_2]P_2 - Q_2) \leq D$ for all n_i such that $[n_i]P \neq Q_i$ for both $i = 1, 2$. Indeed,

$$\begin{aligned}
&\mathrm{GCD}([n_1]P_1 - Q_1, [n_2]P_2 - Q_2)\\
&= \mathrm{GCD}\left(\sigma^*_{[n_1]P_1 - Q_1}(\overline{O}_{\mathcal{E}_1}), \sigma^*_{[n_2]P_2 - Q_2}(\overline{O}_{\mathcal{E}_2})\right)\\
&= \mathrm{GCD}\left(\sigma^*_{[n_1]P}(\overline{Q_1}), \sigma^*_{[n_2]P_2}(\overline{Q_2})\right)\\
&= \sum_{\gamma \in C(\overline{\mathbb{Q}})} \min\left\{\mathrm{ord}_\gamma(\sigma^*_{[n_1]P_1}(\overline{Q_1})), \mathrm{ord}_\gamma(\sigma^*_{[n_2]P_2}(\overline{Q_2}))\right\}\\
&\le \sum_{\gamma \in C(\overline{\mathbb{Q}})} \min\{m_{1,\gamma}, m_{2,\gamma}\} \cdot (\gamma)\\
&\le \sum_{\gamma \in S} m_\gamma(\gamma).
\end{aligned}$$

For the proof of part (ii) in Proposition 4.3, we let $Q_i = O_i$ be the zero element of E_i for $i = 1, 2$. If $k = \overline{\mathbb{Q}}$, then the result follows immediately from part (i). Now, for the general case, we note that it suffices to prove the existence of at most finitely many $t \in C(k)$ such that both $(P_1)_t$ and $(P_2)_t$ are torsion points on the elliptic fiber $\mathcal{E}_{1,t}$ and $\mathcal{E}_{2,t}$ respectively; indeed, the fact that the multiplicity of each such t appearing in a divisor $\mathrm{GCD}([n_1]P_1, [n_2]P_2)$ is bounded follows exactly as in the proof of part (i), using Lemma 4.5. On the other hand, if there exist infinitely many $t \in C(k)$ such that both $(P_1)_t$ and $(P_2)_t$ are torsion, then (according to [MZ14, Theorem, p. 117]) P_1 and P_2 are related, which yields a contradiction.

The conclusion of part (iii) in Proposition 4.3 was proven by Silverman in [Sil04b, Theorem 8 (b)] in the case both E_1, E_2 have constant j-invariants. We generalize his argument as follows. For each of the finitely many $\gamma \in C(k)$ which does not appear in the support of $\mathrm{GCD}(P_1, P_2)$, but for which there exists some positive integer n such that γ is contained in the support of the divisor $\mathrm{GCD}([n]P_1, [n]P_2)$, or equivalently,

$$\text{the divisor } \mathrm{GCD}([n]P_1, [n]P_2) - (\gamma) \text{ is effective,} \tag{4.7}$$

we let n_γ be the smallest such positive integer n for which (4.7) holds. Then, it is easy to see that γ is contained in the support of $\mathrm{GCD}([n]P_1, [n]P_2)$ if and only if $n_\gamma \mid n$. Also, for each of these points γ which are not in the support of $\mathrm{GCD}(P_1, P_2)$, we have that $n_\gamma > 1$. This implies that for any positive integer n which is not divisible by any of the finitely many integers n_γ, we have that

$$\mathrm{GCD}([n]P_1, [n]P_2) = \mathrm{GCD}(P_1, P_2).$$

The conclusion in part (iv) in Proposition 4.3 follows from the proof of part (iii) since $\mathrm{GCD}([q]P_1, [q]P_2) = \mathrm{GCD}(P_1, P_2)$ for all primes q which do not divide any of the finitely many numbers $n_\gamma > 1$. □

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