A DYNAMICAL VARIANT OF THE ANDRÉ-OORT CONJECTURE

DRAGOS GHIOCA AND HEXI YE

Abstract. In the moduli space MP_d of degree d polynomials, special subvarieties are those cut out by critical orbit relations, and then special points are the post-critically finite polynomials. It was conjectured that in MP_d, subvarieties containing a Zariski-dense set of special points are exactly these special subvarieties. In this article, we prove the first non-trivial case for this conjecture: the case $d = 3$.

1. Introduction

Our main result is the proof of the first important case of a dynamical variant of the André-Oort Conjecture (posed by Baker and DeMarco [BD13, Conjecture 1.4]); see Theorem 1.1, which also includes a Bogomolov-type statement for our result. Since Baker-DeMarco conjecture is partly motivated by the classical André-Oort Conjecture (see [BD13, Section 1.2]), we also call the Baker-DeMarco conjecture as *the dynamical André-Oort conjecture*.

In order to state our result, we introduce a little bit of notation. We recall that a rational function f is *post-critically finite* (PCF) if each critical point of f is preperiodic. Also, we recall the classical notation and definition from algebraic dynamics that for a rational function f, its n-th iterate is denoted by f^n, and that a point c is *preperiodic* if and only if there exist integers $0 \le m < n$ such that $f^m(c) = f^n(c)$. We note that for a polynomial $f \in \overline{\mathbb{Q}}[z]$ of degree $d \ge 2$, the *canonical height* of a point $c \in \overline{\mathbb{Q}}$ (as first introduced by Call-Silverman [CS93]) is equal to

$$\hat{h}_f(c) := \lim_{n\to\infty} \frac{h(f^n(c))}{d^n}, \tag{1.1}$$

where $h(\cdot)$ is the usual Weil height; for more details on the canonical height we refer the reader to [CS93]. Then for a polynomial $f \in \overline{\mathbb{Q}}[z]$ of degree $d \ge 2$, we define the *critical height* of a polynomial (first introduced by Silverman—see also [Ing12] for a proof that the critical height is comparable to a Weil height) to be

$$\hat{h}_{\mathrm{crit}}(f) := \sum_{i=1}^{d-1} \hat{h}_f(c_i), \tag{1.2}$$

where the c_i's are the critical points of f (other than ∞). Clearly, $f \in \overline{\mathbb{Q}}[z]$ is post-critically finite if and only if $\hat{h}_{\mathrm{crit}}(f) = 0$.

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Theorem 1.1. *Let $C \subset \mathbb{C}^2$ be an irreducible complex algebraic curve and let $f_{a,b}(z) = z^3 - 3a^2z + b$; then the two critical points of $f_{a,b}(z)$ are $\pm a$. The following statements are equivalent:*

1. *there are infinitely many $(a,b) \in C$ with $f_{a,b}$ being post-critically finite.*
2. *C is the line in $\mathbb{C}^2$ given by the equation $b = 0$, or C is an irreducible component of a curve defined by one of the following equations:*
   - (i) *$f_{a,b}^n(a) = f_{a,b}^m(a)$ for some $n > m \geq 0$; or*
   - (ii) *$f_{a,b}^n(-a) = f_{a,b}^m(-a)$ for some $n > m \geq 0$; or*
   - (iii) *$f_{a,b}^n(a) = f_{a,b}^m(-a)$ for some $n, m \geq 0$.*
3. *C is defined over $\overline{\mathbb{Q}}$, and there is sequence of non-repeating points $(a_n, b_n) \in C(\overline{\mathbb{Q}})$ with $\lim_{n\to\infty} \hat{h}_{\mathrm{crit}}(f_{a_n,b_n}) = 0$.*

We describe next the background for both our result and for the Baker-DeMarco conjecture. Given a variety X, a subvariety $V \subset X$, and given a family $\mathcal{Y}$ of subvarieties $Y \subseteq X$, the principle of *unlikely intersections* predicts that

$$V \cap \bigcup_{\substack{Y\in\mathcal{Y} \\ \dim(Y)<\mathrm{codim}(V)}} Y$$

is not Zariski dense in V, unless V satisfies some rigid geometric property mirroring the varieties contained in $\mathcal{Y}$. Special cases of this principle of unlikely intersections in arithmetic geometry are the Bombieri-Masser-Zannier [BMZ99], the Pink-Zilber and the André-Oort conjectures; for a comprehensive discussion, see the book of Zannier [Zan12]. Motivated by a version of the Pink-Zilber Conjecture for semiabelian schemes, Masser and Zannier (see [MZ10, MZ12]) proved that in a non-constant elliptic family E_t parametrized by $t \in \mathbb{C}$, for any two sections $\{P_t\}_t$ and $\{Q_t\}_t$, if there exist infinitely many $t \in \mathbb{C}$ such that both P_t and Q_t are torsion points on E_t, then the two sections are linearly dependent.

The results of Masser-Zannier [MZ10, MZ12] have natural dynamical reformulations using the Lattès maps associated to the elliptic curves in the family $\{E_t\}$. More generally, one can consider the following problem: given a curve C defined over $\mathbb{C}$, given a family of rational functions $f_t \in \mathbb{C}(z)$ parametrized by the points $t \in C(\mathbb{C})$, and given $a, b : C \longrightarrow \mathbb{P}^1$, then one expects that there exist infinitely many $t \in C(\mathbb{C})$ such that both $a(t)$ and $b(t)$ are preperiodic under the action of f_t if and only if a and b are dynamically related with respect to the family $\{f_t\}$ in a precise manner. The first result in this direction, for a family f_t which was not induced by the endomorphism of an algebraic group was proven by Baker and DeMarco [BD11]. They answered a question of Zannier, thus showing that for an integer $d \geq 2$, and for two complex numbers a and b, if there exist infinitely many $t \in \mathbb{C}$ such that both a and b are preperiodic under the action of $z \mapsto z^d + t$, then $a^d = b^d$. Several new results followed in the recent years (see [FG15, GHT12, BD13, GHT15, GKN16, GKNY]), but still the question stated above remains open in its full generality; especially, it is particularly difficult to treat the case when C is an arbitrary curve, even when dealing with families of polynomials. For example, in [GHT15] (which is one of the very few articles treating the case when $C \neq \mathbb{P}^1$), the family of rational functions must have exactly one degenerate point on C and also the family $\{f_t\}$ must satisfy additional technical conditions. In [GKNY], the parameter curve is arbitrary, but the result holds only for families of unicritical polynomials, which is again quite restrictive. In this article, we release all the restrictions on the curve C, which parametrizes a family of cubic polynomials. The dynamics of cubic polynomials is already significantly more involved than the dynamics of unicritical polynomials, and therefore our analysis is significantly more involved than in the article [GKNY].

In [BD13], Baker and DeMarco posed a very general question for families of dynamical systems, which is motivated by the classical André-Oort conjecture. As a dynamical analogue to the classical André-Oort Conjecture, Baker and DeMarco's question asks that if a subvariety V of the moduli space of rational maps of given degree contains a Zariski dense set of post-critically finite points, then V itself is cut out by *critical orbit relations*, i.e., the critical points of the rational functions in the family V are related dynamically (see condition (2) in Theorem 1.1; for more details, see [BD13]). In Theorem 1.1 we prove the first case of the Dynamical André-Oort Conjecture (posed by Baker and DeMarco [BD13]), solving completely their conjecture in the moduli space MP_3 of cubic polynomials. Furthermore, we obtain a *Bogomolov-type* statement for our result, i.e., the condition that a curve $C \subset \mathrm{MP}_3$ contains infinitely many post-critically finite points can be relaxed to asking that C contains infinitely many points of small critical height.

Going back to the statement of Theorem 1.1, for the family of cubic polynomials $z^3 - 3a^2z + b$, if $b = 0$ then the two critical points $\pm a$ and also their orbits are related by $\sigma(z) = -z$, where σ is the unique symmetry of the Julia set of $f_{a,0}$ with $f_{a,0} \circ \sigma = \sigma \circ f_{a,0}$. We observe that condition (1) easily yields condition (3) in Theorem 1.1 since each post-critical point in the parameter space corresponds to a polynomial with algebraic coefficients, and furthermore (as shown in [CS93]), a point is preperiodic under the action of a polynomial if and only if its canonical height equals 0. Also, a curve of the form (2) as in the conclusion of Theorem 1.1 contains infinitely many PCF points. Therefore, the difficulty in Theorem 1.1 lies with proving that it is *only* curves cut out by orbit relations which contain an infinite sequence of points of critical height converging to 0.

We also state the following result which is an immediate consequence of our proof (see Remark 4.7).

Theorem 1.2. *Let C be an irreducible plane curve defined over a number field K, such that a is not persistently preperiodic for $f_{a,b}$ on C. The set of preperiodic parameters*

$$\mathrm{Preper}_+ := \{t \in C(\overline{K}) \colon a(t) \text{ is preperiodic for } f_{a(t),b(t)}\}$$

is equidistributed with respect to the bifurcation measure μ_+. More precisely, for any sequence of non-repeating points $t_n \in \mathrm{Preper}_+$, the discrete probability measures

$$\mu_n = \frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot t_n|} \sum_{t \in \operatorname{Gal}(\overline{K}/K) \cdot t_n} \delta_t$$

converge weakly to the normalized measure $\mu_+/\mu_+(C)$. The exact similar result holds when we replace a by $-a$ and therefore replace Preper_+ by Preper_- and also replace $\mu_+/\mu_+(C)$ by $\mu_-/\mu_-(C)$.

We note that results similar to our Theorem 1.2 appear in [BD13, GHT15, GKNY].

A key ingredient of our article (and also of all of the articles previously mentioned dealing with the problem of unlikely intersection in algebraic dynamics) is the arithmetic equidistribution of small points on an algebraic variety (in the case of $\mathbb{P}^1$, see [BR06, FRL06], in the general case of curves, see [CL06, Thu], while for arbitrary varieties, see [Yua08]). Our strategy of proof follows the general lines laid out in the article of Baker and DeMarco [BD13]. One of the significant technical difficulties of this article is to show the equidistribution of preperiodic parameters for a marked critical point of cubic polynomials on an arbitrary curve C, *not necessarily containing infinitely PCF points*. Precisely, it is delicate to prove the continuity at infinity points of potential (or escape-rate) functions $G^{\pm}$ (introduced in Section 2.3) of the bifurcation measures, especially when the bifurcation locus is not compact. Moreover, because the sets $S^{\pm}\backslash S_0^{\pm}$ appearing in (2.3) (introduced in Section 2.2) can be nonempty, *unlike* all of the previous articles on this theme, we cannot obtain the desired metrized line bundle from a uniform limit of semi-positive metrics on the same line bundle using the *exact* same approach as in [GHT15]; thus we need to use a slight modification of the setup previously used in [GHT15] in order to obtain again the desired metrized line bundles defined on the same line bundle. We note that the set $S^{\pm}$ consists of all poles for the functions a and b along the curve C (where the family of cubics is $f_{a,b}(z) = z^3 - 3a^2z + b$), while the set $S_0^{\pm}$ is the subset of $S^{\pm}$ consisting of the points $t_0 \in C$ such that $\mathrm{ord}_{t_0} f_{a,b}^n(\pm a) \to -\infty$. Proving that we can extend continuously $G^{\pm}$ by defining $G^{\pm}(t_0) = 0$ for each $t_0 \in S^{\pm} \setminus S_0^{\pm}$ is the heart of our argument, which is obtained through a careful analysis of the expansion of $f_{a,b}^n(\pm a)$ in a neighborhood of t_0 (see Lemmas 3.4 and 3.5).

Outline of the article. Using the arithmetic equidistribution theorem we show that preperiodic parameters for a marked critical point equidistribute on the curve with respect to the bifurcation measure; see Theorem 4.6. Assuming there exist infinitely many points (a, b) on the plane curve C such that $z^3 - 3a^2z + b$ is PCF, then the potential (escape-rate) functions for the bifurcation measures (with respect to the starting points $\pm a$) are proportional to each other; see Corollary 4.8. Finally, using the classification of Medvedev-Scanlon [MS14] for invariant plane curves under the coordinatewise action of polynomials, we derive a polynomial relation between the critical points of the cubic polynomial; see Theorem 5.1.

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2. Dynamics of cubic polynomials

In this section, we first introduce the moduli space MP_3 of cubic polynomials, and then study the dynamics of the two critical points of a cubic polynomial.

2.1. **Moduli space of cubic polynomials.** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. Two polynomials f and g are conjugate to each other if there is an affine map $A(z) = az + b$ with $f = A^{-1} \circ g \circ A$. The *moduli space* MP_d of polynomials consists of conjugacy classes $[f]$ of polynomials of degree $d \geq 2$, and MP_d is an affine space of dimension $d-1$. Moreover, it is well known that PCF points in MP_d are countable and Zariski-dense (see [DH93, Sil12] or [BD13, Proposition 2.6]); in particular, each PCF polynomial can be conjugate to a polynomial with coefficients in $\overline{\mathbb{Q}}$.

In this article, we focus mainly on the moduli space MP_3 of cubic polynomials. Each cubic polynomial is conjugate to a monic and *centred* polynomial, i.e.

$$f_{a,b}(z) := z^3 - 3a^2 z + b$$

with two critical points $\pm a$ (counted with multiplicity). Two polynomials f_{a_1,b_1}, f_{a_2,b_2} of such form are in the same conjugacy class if and only if $a_1 = \pm a_2$ and $b_1 = \pm b_2$. Hence the map from $\mathbb{C}^2$ to MP_3, given by $(a, b) \to [f_{a,b}]$ is a 4-to-1 map with ramification when $a = 0$ or $b = 0$. For a point $(a, b) \in \mathbb{C}^2$, the two critical points of the polynomial $f_{a,b}$ have been marked as $\pm a$, we can view $\mathbb{C}^2$ as a (ramified) double cover of the moduli space MP_3^{cm} of cubic polynomials with marked critical points. Because of the symmetry in the parametrization, we focus our analysis on the orbit of $+a$ under $f_{a,b}$.

2.2. **Order of poles.** Let C be an irreducible algebraic curve in $\mathbb{C}^2$. We denote by $\hat{C}$ the normalization of the Zariski closure of C in $\mathbb{P}^2$; hence $\hat{C}$ is a smooth projective curve. Let $c_n := f_{a,b}^n(a)$, which is a rational map $\hat{C} \longrightarrow \mathbb{P}^1$; also, c_n can be viewed as a polynomial in the two rational maps $a, b : \hat{C} \longrightarrow \mathbb{P}^1$. Finally, we note that *not both* a and b are constant on $\hat{C}$.

Lemma 2.1. *On C, the marked critical point $+a$ is persistently preperiodic under the action of $f_{a,b}$, if and only if the degree of $c_n \in \mathbb{C}(\hat{C})$ is uniformly bounded for $n \geq 1$.*

Proof. First, if $+a$ is persistently preperiodic under the action of $f_{a,b}$, then the degree of c_n is uniformly bounded (because there are finitely many distinct elements in the sequence $\{c_n\}$).

Now, assume $+a$ is not persistently preperiodic under the action of $f_{a,b}$. We note that the family of polynomials $f(z) = z^3 - 3a^2 z + b$ is not isotrivial since not both a and b are constant functions in $\mathbb{C}(\hat{C})$. Then [Bak09, Theorem 1.6] (see also [Dem16, Theorem 1.2]) yields that the canonical height $\hat{h}_f(c_n)$ is strictly positive, where the canonical height $\hat{h}_f(\cdot)$ is constructed as in (1.1) with respect to the Weil height for the function field $\mathbb{C}(\hat{C})$. So, the Weil height of c_n (as an element of the function field $\mathbb{C}(\hat{C})$) is unbounded and therefore the degree of c_n (as a function on $\hat{C}$) is unbounded. □

Let $t_0 \in \hat{C}(\mathbb{C})$ be a pole of either a or b on the smooth projective curve $\hat{C}$. Note that since not both a and b are constant on $\hat{C}$, then there must exist such a pole $t_0 \in \hat{C}$. Let $\gamma_a := -\operatorname{ord}_{t_0} a$, $\gamma_b := -\operatorname{ord}_{t_0} b$ and

$$\gamma_{\max} := \max\{3\gamma_a, \gamma_b\} > 0, \text{ and } \gamma_n := -\operatorname{ord}_{t_0} c_n \text{ for each } n \geq 1. \tag{2.1}$$

Lemma 2.2. *For a point $t_0 \in \hat{C}$ with $\gamma_{\max} > 0$, we have*

- *if* $\gamma_{\max} < 3\gamma_{n_0}$ *for some* $n_0 \geq 1$*, then for all* $n \geq n_0$*,* $\gamma_n = 3^{n-n_0} \cdot \gamma_{n_0}$*. In particular, if* $\lim_{t\to t_0} 2a^3(t)/b(t) \neq 1$*, then* $\gamma_n = 3^{n-1} \cdot \gamma_{\max}$ *for all* $n \geq 1$*.*
- *if* $\lim_{n\to\infty} \gamma_n \neq \infty$*, then* $\gamma_b = 3\gamma_a$ *and* $\gamma_n = \gamma_a$ *for all* $n \geq 1$*.*

Proof. From the fact that

$$c_{n+1} = f_{a,b}(c_n) = c_n^3 - 3a^2 c_n + b, \tag{2.2}$$

it is clear that if $3\gamma_n > \gamma_{\max}$ then $\gamma_{n+1} = 3\gamma_n$. Consequently, if $3\gamma_{n_0} > \gamma_{\max}$, then $\gamma_n = 3^{n-n_0} \cdot \gamma_{n_0}$ for all $n \geq n_0$. When $\lim_{t\to t_0} 2a^3(t)/b(t) \neq 1$, $\gamma_1 = -\operatorname{ord}_{t_0}(b - 2a^3) = \gamma_{\max}$, hence $\gamma_n = 3^{n-1} \cdot \gamma_{\max}$ for all $n \geq 1$.

Now suppose that $\lim_{n\to\infty} \gamma_n \neq \infty$. From the above analysis we get $\lim_{t\to t_0} 2a^3(t)/b(t) = 1$ and then $\gamma_b = 3\gamma_a = \gamma_{\max}$. Moreover, we have $\gamma_n \leq \gamma_a$ for all n. If there is some n_1 with $\gamma_{n_1} < \gamma_a$, then from equation (2.2), $\gamma_{n_1+1} = \gamma_{\max}$. Let $n_0 = n_1 + 1$, then $3\gamma_{n_0} > \gamma_{\max}$ and consequently $\gamma_n = 3^{n-n_0} \cdot \gamma_{\max}$ would tend to infinity as $n \to \infty$, which is a contradiction. □

From the behaviour of the order of poles in Lemma 2.2, we define the following subsets of the smooth projective curve $\hat{C}$

$$S^+ := \left\{ t_0 \in \hat{C} \colon \gamma_{max} > 0 \right\} \text{ and } S_0^+ := \left\{ t_0 \in \hat{C} \colon \lim_{n\to\infty} \gamma_n = \infty \right\} \tag{2.3}$$

It is clear that S_0^+ is subset of S^+, and similarly we can define $S_0^- \subseteq S^- \subset \hat{C}$ for the marked critical point $-a$. We note that $S^+ = S^-$ since this set consists of all points of the smooth curve $\hat{C}$ where either a or b has a pole. As we will see later in Theorem 3.1, the point $\pm a$ lies in the basin of infinity for $f_{a,b}$ for all parameters in a punctured neighborhood of each $t \in S_0^{\pm}$. As an aside, we note that each point in $S^+ = S^-$ is contained in at least one of the two sets S_0^+ and S_0^-. Indeed, as proven in Lemma 2.2, we know that if $t \in S^+ \setminus S_0^+$, then $\lim_{t\to t_0} 2a^3(t)/b(t) = 1$; similarly, we get that if $t \in S^- \setminus S_0^-$, then $\lim_{t\to t_0} 2(-a)^3(t)/b(t) = 1$. So, indeed, each point in $S^+ = S^-$ must be contained in at least one of the two sets S_0^+ and S_0^-.

Since all the poles of $c_n \in \mathbb{C}(\hat{C})$ are in S^+, then Lemma 2.1 yields the following result.

Lemma 2.3. *The marked critical point* $+a$ *(resp.,* $-a$*) is persistently preperiodic under* $f_{a,b}$ *on* C*, if and only if* S_0^+ *(resp.,* S_0^-*) is empty.*

2.3. **Escape-rate function and bifurcation.** For the marked critical points $\pm a$, the *escape-rate functions* $G^{\pm}$ are given by

$$G^{\pm}(a,b) := \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f_{a,b}^n(\pm a)|, \tag{2.4}$$

for $(a,b) \in \mathbb{C}^2$, where $\log^+|z| := \max\{\log|z|, 0\}$. In the above formula, the convergence is local and uniform on $\mathbb{C}^2$, and then the escape-rate functions $G^{\pm}(a,b)$ are continuous and plurisubharmonic on $\mathbb{C}^2$; see the proof of Lemma 3.2 or [BH88]. Hence they are subharmonic when restricted to an irreducible curve $C \subset \mathbb{C}^2$. The escape-rate function satisfies $G^{\pm}(a,b) \geq 0$ with equality if and only if the iterates $f_{a,b}^n(\pm a)$ of the critical point $\pm a$ do not tend to infinity as n tends to infinity.

The *bifurcation measures* $\mu_{\pm}$ on C corresponding to marked critical points $\pm a$ are given by

$$\mu_{\pm} := dd^c G^{\pm}. \tag{2.5}$$

The *bifurcation locus* $\mathrm{Bif}_\pm$ for marked critical points $\pm a$ is the set of parameters on $C \subset \mathbb{C}^2$, such that $\{f^n_{a(t),b(t)}(\pm a(t))\}_{n\geq 1}$ is not a normal family on any small neighbourhood of such parameters. Actually, $\mathrm{Bif}_\pm \subset C$ is the boundary of the set of parameters $t \in C$ with $G^\pm(a(t),b(t)) = 0$, and then from the continuity of the escape-rate functions it has $G^\pm = 0$ on $\mathrm{Bif}_\pm$. The supports of the bifurcation measures $\mu_\pm$ on C are exactly $\mathrm{Bif}_\pm$; see [Dem01]. The bifurcation locus Bif on C is the union of Bif_+ and Bif_-, and it is the set of parameters where the dynamical system $f_{a(t),b(t)}(z)$ is unstable when we perturb the parameter t.

Proposition 2.4. *For any irreducible curve $C \subset \mathbb{C}^2$, there are infinitely many points $(a,b) \in C$ such that $+a$ (or $-a$) is preperiodic under the iteration of $f_{a,b}$.*

Proof. If $+a$ is persistently preperiodic under $f_{a,b}$ on C, then for all $(a,b) \in C$, $+a$ is preperiodic under $f_{a,b}$. Otherwise by [DF08, Theorem 2.5], Bif_+ is not empty. Then [Dem16, Theorem 5.1] asserts that the number of points $(a,b) \in C$ such that $+a$ is preperiodic under the iteration of $f_{a,b}$ is infinite. □

We thank Xiaoguang Wang for suggesting the proof of the following proposition.

Proposition 2.5. *Let C be an irreducible curve $C \subset \mathbb{C}^2$. If $\mu^+ = r\mu^-$ on C for some $r > 0$, then we have $G^+ = rG^-$ on C.*

Proof. As $\mathrm{Bif}_\pm$ are the supports of the $\mu_\pm$, then $\mathrm{Bif} = \mathrm{Bif}_+ = \mathrm{Bif}_-$. Since $\mu^+ = r\mu^-$, $G^+ - rG^-$ is harmonic on C and zero on Bif. From [Mcm00, Theorem 1.1], in any small neighbourhood of a point in Bif, we can find a small (topological) generalized Mandelbrot set with boundary in Bif; then $G^+ - rG^-$ is zero on the boundary of this small generalized Mandelbrot set. Since this small generalized Mandelbrot set is bounded in $C \subset \mathbb{C}^2$, the harmonic function $G^+ - rG^-$ is zero in the interior of this generalized Mandelbrot set. Consequently, $G^+ - rG^-$ is zero everywhere on C as it is harmonic. □

3. A line bundle with continuous metric

In this section, we construct metrized line bundles on a smooth irreducible projective curve $\hat{C}$ associated to a complex algebraic curve $C \subset \mathbb{C}^2$; our metrics correspond to a critical point which is not persistently preperiodic on $\hat{C}$. One of the main goals of this section is to show that the metric on the line bundle is continuous, which is crucial in the next section for proving the equidistribution of small points.

3.1. **Continuity of the escape-rate function.** A priori, we have the escape-rate function G^+ defined on $\hat{C} \setminus S^+$; in the next theorem we show how to extend G^+ to the entire smooth projective curve $\hat{C}$. First, we recall that a *uniformizer* at $t_0 \in \hat{C}$ is a rational function $u \in \mathbb{C}(\hat{C})$ such that $\mathrm{ord}_{t_0} u = 1$.

Theorem 3.1. *For the escape-rate function G^+, we have*

- *G^+ can be extended to a continuous subharmonic function on $\hat{C} \backslash (S^+ \backslash S^+_0)$, and $G^+ = 0$ at any $t_0 \in S^+ \backslash S^+_0$.*
- *For any $t_0 \in S^+_0$ and u being a uniformizer at t_0, the function $G^+ + \frac{\gamma_n}{3^n} \log|u|$ can be extended to a harmonic function in a neighbourhood of t_0 for any sufficiently large n, where γ_n is defined in (2.1).*

To prove this theorem, we use Lemmas 3.2, 3.3 and 3.5. First, we mention the following convention for our forthcoming analysis. Given $t \in \hat{C}(\mathbb{C})$, for the sake of simplifying our notation, we will often drop the dependence on t and simply use $a, b, c_n := f_{a,b}^n(a)$ instead of $a(t), b(t), c_n(t)$.

Lemma 3.2. *The sequence of functions* $\frac{1}{3^n}\log^+|f_{a,b}^n(+a)|$ *converges locally uniformly to* $G^+(a,b)$ *on* $\hat{C}\backslash S^+$ *as* n *tends to infinity.*

Proof. Let W be a compact subset of $\hat{C}\backslash S^+$; it suffices to prove the functions $\frac{1}{3^n}\log^+|f_{a,b}^n(+a)|$ converge locally uniformly to $G^+(a,b)$ on W. Let M be a large positive real number such that $\max\{1, |a(t)|, |b(t)|\} < M/3$ for all $t \in W$. Recall that $c_n = f_{a,b}^n(+a)$ and $c_{n+1} = c_n^3 - 3a^2c_n + b$; also, given some point $t \in W$, we use the convention (as before) to denote $c_n(t)$ simply by c_n. We get that if $|c_n| < M$, then

$$\left|\frac{\log^+|c_{n+1}|}{3^{n+1}} - \frac{\log^+|c_n|}{3^n}\right| < \frac{\log(5M^3)}{3^{n+1}}$$

and if $|c_n| \geq M$

$$\left|\frac{\log^+|c_{n+1}|}{3^{n+1}} - \frac{\log^+|c_n|}{3^n}\right| = \left|\frac{\log|1 - 3a^2/c_n^2 + b/c_n^3|}{3^{n+1}}\right| < \frac{\log 2}{3^{n+1}}.$$

Hence $\frac{1}{3^n}\log^+|f_{a,b}^n(+a)|$ is a Cauchy sequence which converges locally uniformly. ☐

Lemma 3.3. *If* $t_0 \in S_0^+$ *and if* u *is a uniformizer for* t_0, *then*

$$\frac{1}{3^n}\log^+|f_{a,b}^n(+a)| + \frac{\gamma_n}{3^n}\log|u| = \frac{1}{3^n}\log|u^{\gamma_n} f_{a,b}^n(+a)|$$

converges locally uniformly on $\hat{C}$ *at* t_0 *as* n *tends to infinity.*

Proof. As γ_n is the order of pole of c_n at t_0, $u^{\gamma_n}c_n$ is holomorphic and non-vanishing near $t_0 \in \hat{C}$, i.e., $\frac{1}{3^n}\log|u^{\gamma_n}f_{a,b}^n(+a)|$ is harmonic near t_0 on $\hat{C}$. From Lemma 2.2, we can pick a large n_0, such that $\gamma_{n+1} = 3\gamma_n \gg \gamma_{\max}$ for any $n \geq n_0$. As $c_{n+1} = c_n^3 - 3a^2c_n + b$, inductively, $|c_n(t)| \gg 1$ grows exponentially fast as $n \to \infty$ for $t \in \hat{C}$ very close to t_0. So for t near t_0 and $n \geq n_0$, we have

$$\frac{1}{3^n}\log^+|c_n| + \frac{\gamma_n}{3^n}\log|u| = \frac{1}{3^n}\log|u^{\gamma_n}c_n|$$

and $|3a^2(t)/c_n^2(t)| \ll 1, |b(t)/c_n^3(t)| \ll 1$; hence

$$\left|\frac{\log|u^{\gamma_{n+1}}c_{n+1}|}{3^{n+1}} - \frac{\log|u^{\gamma_n}c_n|}{3^n}\right| = \left|\frac{\log|1 - 3a^2/c_n^2 + b/c_n^3|}{3^{n+1}}\right| < \frac{\log 2}{3^n},$$

as desired. ☐

It is more delicate to show the continuity of the escape-rate function G^+ on $\hat{C}$ crossing $t_0 \in S^+\backslash S_0^+ \subset \hat{C}$. We first prove a result which will be used later in Lemma 3.5.

Lemma 3.4. *Let* γ *be a positive integer, and let* $\tilde{b}(t) = 2 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \cdots$ *be a holomorphic germ at* $t = 0$. *Define*

$$g(z) := z^3 - 3z + \tilde{b}.$$

Suppose $z_n(t)$ *is a sequence of holomorphic germs at* $t = 0$, *satisfying*

- $z_{n+1}(t) = g(z_n(t))/t^{2\gamma}$ *for* $n \geq 1$.
- $\operatorname{ord}_{t=0} z_n(t) = 0$ *for all* $n \geq 1$.

We write $z_n(t) = \sum_{i=0}^{\infty} x_{n,i}t^i$ *for each* $n \geq 1$. *Then there is a finite set* T *such that* $x_{n,i} \in T$ *for all* $n \geq 1$ *and all* $0 \leq i \leq \gamma$*; moreover,* $x_{n,0}$ *is a solution of the equation* $x^3 - 3x + 2 = 0$.

The proof of Lemma 3.4 exploits the power series expansion for each $z_n(t)$ using in an essential way the fact that $\{z_n(t)\}$ satisfies a recursive formula and that $\operatorname{ord}_{t=0} z_n(t) = 0$.

Proof of Lemma 3.4. Let

$$g(z_n(t)) = z_n^3 - 3z_n + \tilde{b} =: \sum_{i=0}^{\infty} \alpha_{n,i} t^i.$$

Since $g(z_n) = z_{n+1} \cdot t^{2\gamma}$ and $\operatorname{ord}_{t=0} z_{n+1}(t) = 0$, we have

$$\alpha_{n,i} = 0, \quad \text{for } 0 \leq i < 2\gamma \text{ and } \alpha_{n,2\gamma} \neq 0.$$

Notice that since $\alpha_{n,0} = x_{n,0}^3 - 3x_{n,0} + 2 = 0$, then $x_{n,0} \in \{1, -2\}$. Now suppose $x_{n,i}$ is unique up to finitely many choices for all $0 \leq i \leq j-1 < \gamma$, which is true when $j = 1$; we prove next that also $x_{n,j}$ is unique up to finitely many choices.

Suppose $x_{n,0} \neq 1$, i.e., $x_{n,0} = -2$. An easy computation shows that

$$\alpha_{n,j} = x_{n,j}(3x_{n,0}^2 - 3) + \beta_j + F_j(x_{n,0}, \ldots, x_{n,j-1}) = 0$$

where F_j is a unique polynomial in j variables obtained from the expansion of $z_n(t)^3$. Because $x_{n,0} = -2$, then $x_{n,j}$ is uniquely determined by $x_{n,0}, \ldots, x_{n,j-1}$ and β_j.

Now suppose $x_{n,0} = 1$. Now, if $x_{n,i} = 0$ for $1 \leq i \leq j-1$, we have that

$$\alpha_{n,2j} = 3x_{n,j}^2 + \beta_{2j}.$$

If $j < \gamma$, then $\alpha_{n,2j} = 3x_{n,j}^2 + \beta_{2j} = 0$. Otherwise if $j = \gamma$, $\alpha_{n,2j} = 3x_{n,j}^2 + \beta_{2j}$ is the constant term of $z_{n+1}(t)$ which must satisfy the equation $x^3 - 3x + 2 = 0$. In any case, $x_{n,j}$ is unique up to finitely many choices.

Now, if $x_{n,i} \neq 0$ for some $1 \leq i \leq j-1$ (also under the assumption that $x_{n,0} = 1$), we let $\ell \geq 1$ be the smallest such integer with $x_{n,\ell} \neq 0$. Then

$$\alpha_{n,\ell+j} = 6x_{n,\ell}x_{n,j} + G_{j,\ell}(x_{n,\ell}, \ldots, x_{n,j-1}) + \beta_{\ell+j} = 0$$

where $G_{j,\ell}$ is a unique polynomial in $j - \ell$ variables obtained from the expansion of $z_n(t)^3$ (also taking into account that $x_{n,0} = 1$ and that $x_{n,i} = 0$ for $1 \leq i \leq \ell - 1$). Hence $x_{n,j}$ is also uniquely determined up to finitely many choices. By induction, for $0 \leq i \leq \gamma$, we get that $x_{n,i}$ is uniquely determined by $\tilde{b}$ up to finitely many choices. □

Lemma 3.5. *Fix a point* $t_0 \in S^+ \backslash S_0^+$ *and a uniformizer* u *for* t_0 *on* $\hat{C}$*. The sequence of functions* $\frac{1}{3^n} \log^+ |u^{\gamma_n} f_{a,b}^n(+a)|$ *converges uniformly on a neighborhood of* $t_0 \in \hat{C}$ *as* n *tends to infinity.*

Proof. From Lemma 2.3, $\gamma_n = \gamma_a > 0$ for all n; in particular, the convergence statement is independent on the choice of uniformizer u. For a suitable choice of the uniformizer and also for a suitable analytic parameterization of a neighborhood of $t_0 \in \hat{C}$, we can assume that

$$t_0 = 0,\ u(t) = t,\ a(t) = t^{-\gamma_a},\ b(t) = \beta_0 t^{-3\gamma_a} + \beta_1 t^{-3\gamma_a+1} + \beta_2 t^{-3\gamma_a+2} + \cdots$$

and then $c_n(t) = f^n_{a(t),b(t)}(a(t))$. From Lemma 2.3, $\mathrm{ord}_{t=0}\, c_n(t) = -\gamma_a$ for all n and

$$\lim_{t\to 0} 2a^3(t)/b(t) = 1 = 2/\beta_0 \text{ and thus } \beta_0 = 2.$$

The statement of this Lemma is equivalent to the statement that the sequence

$$\frac{1}{3^n} \log^+ |t^{\gamma_a} c_n(t)|$$

converges locally uniformly in a neighbourhood of $t = 0$ as $n \to \infty$. Also, by Lemma 3.2, $\frac{1}{3^n} \log^+ |t^{\gamma_a} c_n(t)|$ converges locally uniformly for $t \neq 0$. To prove this lemma, it suffices to show that there exists a sequence of positive real numbers r_n shrinking to zero as $n \to \infty$, such that

$$\lim_{n\to\infty} \sup_{i\geq n} \sup_{|t|<r_n} \frac{\log^+ |t^{\gamma_a} c_i(t)|}{3^i} = 0. \tag{3.1}$$

Indeed, suppose (3.1) holds. Then given any small $\epsilon > 0$ and any small $r > 0$, we can find some sufficiently large $n_0 \in \mathbb{N}$ and some positive $r_{n_0}(\epsilon) < r$ such that

$$0 \leq \frac{\log^+ |t^{\gamma_a} c_i(t)|}{3^i} < \epsilon$$

for all $i \geq n_0$ and all $|t| < r_{n_0}(\epsilon)$. Hence for all $i, j \geq n_0$ and $|t| < r_{n_0}$, we have

$$\left| \frac{\log^+ |t^{\gamma_a} c_i(t)|}{3^i} - \frac{\log^+ |t^{\gamma_a} c_j(t)|}{3^j} \right| < \epsilon \tag{3.2}$$

In addition, since $\frac{1}{3^n} \log^+ |t^{\gamma_a} c_n(t)|$ converges locally uniformly on the annuli

$$\mathcal{A} := \{t \colon r_{n_0}(\epsilon) \leq |t| \leq r\},$$

we know that there exists some positive integer $n_1 \geq n_0$ so that inequality (3.2) holds for all $i, j \geq n_1$ and all $t \in \mathcal{A}$. Putting these together yields that inequality (3.2) holds for all $i, j \geq n_1$ and all t so that $|t| \leq r$, i.e., we get the uniform convergence near $t = 0$. Therefore, the conclusion of Lemma 3.5 follows once we establish (3.1).

Since $\mathrm{ord}_{t=0}\, c_n(t) = -\gamma_a$, the Taylor series of $c_n(t)$ can be written as

$$c_n(t) = x_{n,0} t^{-\gamma_a} + x_{n,1} t^{-\gamma_a+1} + x_{n,2} t^{-\gamma_a+2} \cdots.$$

For simplicity, we let $\tilde{c}_n(t) := t^{\gamma_a} c_n(t) = x_{n,0} + x_{n,1} t + x_{n,2} t^2 + \cdots$. Because $c_{n+1} = c_n^3 - 3a^2 c_n + b$, we have that $\tilde{c}_{n+1}$ and $\tilde{c}_n$ satisfy

$$\tilde{c}_{n+1}(t) = g(\tilde{c}_n)/t^{2\gamma_a}$$

for $g(z) := z^3 - 3z + \tilde{b}$ with $\tilde{b}(t) := t^{3\gamma_a} b(t) = 2 + \beta_1 t + \beta_2 t^2 + \cdots$.

Let r_0 be a positive real number less than 1 such that if $|t| \leq r_0$ we have

$$|\tilde{b}(t)| < 3. \tag{3.3}$$

From Lemma 3.4, we know that there exists a finite set T such that $x_{n,i} \in T$ for each $n \geq 1$ and each $0 \leq i \leq \gamma_a$; moreover, one has $|x_{n,0}| < 3$. So, at the expense of replacing r_0 by a smaller positive real number, we may also assume that if $|t| \leq r_0$, then

$$\left| \sum_{i=0}^{\gamma_a} x_{n,i} t^i \right| < 4. \tag{3.4}$$

We note that for each $A > 11$, we have that

$$(A-10)^3 + 3(A-10) < A^3 - 30. \tag{3.5}$$

Let $M \geq 11$ be a real number, let n_0 be a large positive integer, and let $r_{n_0} < r_0$ be a positive real number such that for all t with $|t| \leq r_{n_0}$, we have

$$|\tilde{c}_{n_0}(t)| < M - 10. \tag{3.6}$$

Inequalities (3.5) and (3.3) yield that if $|t| \leq r_{n_0}$ then

$$|g(\tilde{c}_{n_0}(t))| = |\tilde{c}_{n_0}^3 - 3\tilde{c}_{n_0} + \tilde{b}| < (M-10)^3 + 3(M-10) + 3 < M^3 - 27 < M^3 - 10. \tag{3.7}$$

Hence for $|t| \leq r_{n_0}$, by the Maximal Value Theorem, we have that

$$|\tilde{c}_{n_0+1}(t)| = \left| \frac{g(\tilde{c}_{n_0}(t))}{t^{2\gamma_a}} \right| \leq \frac{M^3 - 10}{r_{n_0}^{2\gamma_a}},$$

and then (again using (3.5) and that $r_{n_0} < r_0 < 1$) we get

$$|\tilde{c}_{n_0+1}(t)| < \frac{M^3}{r_{n_0}^{2\gamma_a}} - 10 \text{ and so, } |g(\tilde{c}_{n_0+1})| < \left(\frac{M^3}{r_{n_0}^{2\gamma_a}} \right)^3 - 10. \tag{3.8}$$

From (3.6, 3.7, 3.8), inductively, we know that for $|t| \leq r_{n_0}$ and all $i \geq 0$, one has

$$|\tilde{c}_{n_0+i}(t)| < \frac{M^{3^i}}{r_{n_0}^{2\gamma_a(3^{i-1}+3^{i-2}+\cdots+3^0)}} - 10 < \frac{M^{3^i}}{r_{n_0}^{2\gamma_a(3^{i-1}+3^{i-2}+\cdots+3^0)}}. \tag{3.9}$$

So we have

$$\sup_{i \geq n_0} \sup_{|t| \leq r_{n_0}} \frac{\log^+ |\tilde{c}_i(t)|}{3^i} \leq \frac{\log M}{3^{n_0}} - \frac{\log r_{n_0}^{2\gamma_a}}{3^{n_0}}.$$

Next, we are going to shrink r_{n_0} properly to r_{n_0+i} such that r_{n_0+i} will not tend to zero too fast as $i \to \infty$. Let $M_1 = M^3$. Since $|\tilde{c}_{n_0}(t)| < M - 10$ if $|t| \leq r_{n_0}$, inequalities (3.5), (3.3) and (3.4) yield that

$$\left| g(\tilde{c}_{n_0}(t)) - \sum_{i=0}^{\gamma_a} x_{n+1,i} t^{2\gamma_a+i} \right| < M^3 - 30 + 3 + 4 < M_1 - 20.$$

Notice that the order of $g(\tilde{c}_{n_0}) - \sum_{i=0}^{\gamma_a} x_{n+1,i} t^{2\gamma_a+i}$ at $t = 0$ is at least $3\gamma_a + 1$; so, again by the Maximal Value Theorem, for all $|t| \leq r_{n_0}$ we have that

$$\frac{\left| g(\tilde{c}_{n_0}(t)) - \sum_{i=0}^{\gamma_a} x_{n+1,i} t^{2\gamma_a+i} \right|}{|t|^{3\gamma_a+1}} \leq \frac{M_1 - 20}{r_{n_0}^{3\gamma_a+1}},$$

i.e.,

$$\left| \tilde{c}_{n_0+1} - \sum_{i=0}^{\gamma_a} x_{n+1,i} t^i \right| \leq \frac{M_1 - 20}{r_{n_0}^{3\gamma_a+1}} \cdot |t|^{\gamma_a+1}.$$

Let $r_{n_0+1} := r_{n_0}^{(3\gamma_a+1)/(\gamma_a+1)}$. We have that for all t with $|t| \le r_{n_0+1}$,

$$\left|\tilde{c}_{n_0+1}(t) - \sum_{i=0}^{\gamma_a} x_{n+1,i}t^i\right| < M_1 - 20.$$

Since $r_{n_0+1} < r_{n_0} < r_0$, inequality (3.4) yields that if $|t| \le r_{n_0+1}$, then

$$|\tilde{c}_{n_0+1}| < M_1 - 10.$$

Replacing n_0 and M in (3.9) by n_0+1 and M_1, we get that for all $|t| < r_{n_0+1}$ and all $i \ge 0$

$$|\tilde{c}_{n_0+1+i}| < \frac{M_1^{3^i}}{r_{n_0+1}^{2\gamma_a(3^{i-1}+3^{i-2}+\cdots+3^0)}} - 10 < \frac{M_1^{3^i}}{r_{n_0+1}^{2\gamma_a(3^{i-1}+3^{i-2}+\cdots+3^0)}}.$$

Consequently,

$$\sup_{i\ge n_0+1}\sup_{|t|<r_{n_0+1}} \frac{\log^+|\tilde{c}_i(t)|}{3^i} \le \frac{\log M_1}{3^{n_0+1}} - \frac{\log r_{n_0+1}^{2\gamma_a}}{3^{n_0+1}}.$$

Because $r_{n_0+1} = r_{n_0}^{(3\gamma_a+1)/(\gamma_a+1)}$ and $M_1 = M^3$, the above inequality becomes

$$\sup_{i\ge n_0+1}\sup_{|t|<r_{n_0+1}} \frac{\log^+|\tilde{c}_i(t)|}{3^i} \le \frac{\log M}{3^{n_0}} - \left(\frac{3\gamma_a+1}{3\gamma_a+3}\right)\frac{\log r_{n_0}^{2\gamma_a}}{3^{n_0}}.$$

Inductively, for all $j \ge 0$ and $r_{n_0+j} := r_{n_0}^{(3\gamma_a+1)^j/(\gamma_a+1)^j}$

$$\sup_{i\ge n_0+j}\sup_{|t|<r_{n_0+j}} \frac{\log^+|\tilde{c}_i(t)|}{3^i} \le \frac{\log M}{3^{n_0}} - \left(\frac{3\gamma_a+1}{3\gamma_a+3}\right)^j\frac{\log r_{n_0}^{2\gamma_a}}{3^{n_0}}.$$

Since M is fixed and we can pick arbitrarily large n_0 and j, the above inequality yields (3.1). This concludes the proof of Lemma 3.5. □

Proof of Theorem 3.1. Lemma 3.2 implies that G^+ is a continuous subharmonic function on $\hat{C}\backslash(S^+\backslash S_0^+)$. For $t_0 \in S^+\backslash S_0^+$, from Lemma 2.2, γ_n is a constant independent on n. Hence Lemma 3.5 indicates that G^+ can be continued to $S^+\backslash S_0^+$, and from limit (3.1) we have $G^+(t_0) = 0$ for $t_0 \in S^+\backslash S_0^+$.

The only thing left to prove is that $G^+ + \frac{\gamma_n}{3^n}\log|u|$ can be extended to a harmonic function in some neighborhood of each $t_0 \in S_0^+$. For $t_0 \in S_0^+$, from Lemma 2.2, when n is large, $\gamma_n/3^n$ is a constant independent on n. So for large n, Lemma 3.3 implies that $G^+ + \frac{\gamma_n}{3^n}\log|u|$, which is the limit of

$$\frac{1}{3^i}\log^+|f_{a,b}^i(+a)| + \frac{\gamma_i}{3^i}\log|u|$$

as i tends to infinity, can be extended to a harmonic function in some neighborhood of t_0. □

3.2. **Metrized line bundle.** Let L be a line bundle of the projective, smooth curve $\hat{C}$. A *metric* $\|\cdot\|$ on L is a collection of norms on $L(t)$, one for each $t \in \hat{C}$, such that for a section s, we have

$$\|\alpha \cdot s(t)\| = |\alpha| \cdot \|s(t)\|$$

for all constants α. The metric $\|\cdot\|$ is said to be *continuous*, if $\|s(t)\|$ varies continuously as we vary t locally. All metrics considered in this article are continuous. A continuous metric

$\|\cdot\|$ on L is said to be *semi-positive* if, locally $\log\|s(t)\|^{-1}$ is a subharmonic function for any non vanishing holomorphic section s, and the *curvature* of this metric is given by

$$c_1(L) = dd^c \log \|s(t)\|^{-1}, \tag{3.10}$$

which can be viewed as a measure on $\hat{C}$. It is clear the curvature does not depend on the choice of the section s.

We work with curves where the marked critical point $+a$ (respectively $-a$) is not persistently preperiodic. Then by Lemma 2.3, S_0^+ (respectively S_0^-) is non empty. For each $n \geq 1$, let D_n^+ be the Weil divisor on the smooth projective curve $\hat{C}$ defined as

$$D_n^+ := \sum_{t_0 \in S_0^+} \gamma_n \cdot t_0$$

where γ_n is the order of pole of c_n at t_0 (see (2.1)). Similarly, we can define the divisor D_n^- corresponding to the marked critical point $-a$ on $\hat{C}$. By Lemma 2.2, for any sufficiently large n and all $i \geq 1$, we have

$$D_{n+i}^{\pm} = 3^i \cdot D_n^{\pm}. \tag{3.11}$$

Let $L_{D_n^{\pm}}$ be the line bundles of $\hat{C}$ associated to $D_n^{\pm}$. From Theorem 3.1, when n is big enough, we can construct a metric $\|\cdot\|$ on $L_{D_n^{\pm}}$, which is continuous and semi-positive. Indeed, first we choose an isomorphism

$$\psi : L_{D_n^{\pm}}|_{\hat{C}\setminus S^{\pm}} \simeq (\hat{C}\setminus S^{\pm}) \times \mathbb{C}, \tag{3.12}$$

and then let $\varphi := \mathrm{pr}_2 \circ \psi$, where pr_2 is the projection of $(\hat{C}\setminus S^{\pm}) \times \mathbb{C}$ onto its second factor. Then for any section $s(t)$ of $L_{D_n^{\pm}}$ on $\hat{C}\setminus S^{\pm}$, the metric $\|\cdot\|$ on s is given by

$$\|s(t)\| := e^{-3^n \cdot G^{\pm}} \cdot |\varphi(s(t))|. \tag{3.13}$$

The definition of $\|s(t)\|$ depends on the choice of the trivialization ψ; however, a different choice for ψ leads to a change of $\|s(t)\|$ by a nonzero multiple which would not affect the definition of the height from (4.3) due to the product formula.

Theorem 3.1 asserts that when n is sufficiently large, this metric can be extended to a continuous and semi-positive metric on $L_{D_n^{\pm}}$. If we consider the metric on $L_{D_n^{\pm}}^{\otimes 3^i}$ inherited from the metric on $L_{D_n^{\pm}}$, then for large n (using (3.11) and (3.13)), we have that

$$L_{D_n^{\pm}}^{\otimes 3^i} \simeq L_{D_{n+i}^{\pm}}$$

is an isometry.

4. Equidistribution of small points

In this section, we show that the parameters at which the marked critical point is preperiodic are equidistributed on $\hat{C}$ with respect to the induced measure on $\hat{C}$ by the bifurcation measure on C of the marked critical point, thus proving Theorem 1.2 (see Theorem 4.6 and also Remark 4.7).

4.1. **Height functions.** Let K be a number field and let $\overline{K}$ be its algebraic closure. It is well known that K is naturally equipped with a set Ω_K of pairwise inequivalent nontrivial absolute values, together with positive integers N_v for each $v \in \Omega_K$ such that

- for each $\alpha \in K^*$, we have $|\alpha|_v = 1$ for all but finitely many places $v \in \Omega_K$.
- for every $\alpha \in K^*$, we have the *product formula*:

$$\prod_{v \in \Omega_K} |\alpha|_v^{N_v} = 1. \tag{4.1}$$

For each $v \in \Omega_K$, let K_v be the completion of K at v, let $\overline{K}_v$ be the algebraic closure of K_v and let $\mathbb{C}_v$ denote the completion of $\overline{K}_v$. The field $\mathbb{C}_v$ is algebraic closed; when v is Archimedean, $\mathbb{C}_v \simeq \mathbb{C}$. We fix an embedding of $\overline{K}$ into $\mathbb{C}_v$ for each $v \in \Omega_K$; hence we have a fixed extension of $|\cdot|_v$ on K to $\overline{K}$. Let $f \in \overline{\mathbb{Q}}[z]$ be a polynomial of degree $d \geq 2$. There is a *canonical height* $\hat{h}_f$ on $\overline{K} = \overline{\mathbb{Q}}$ (see also (1.1)); for $x \in \overline{K}$ such that also $f \in K[z]$, we have

$$\hat{h}_f(x) := \frac{1}{[K(x):\mathbb{Q}]} \lim_{n\to\infty} \sum_{y \in \mathrm{Gal}(\overline{K}/K)\cdot x} \sum_{v \in \Omega_K} N_v \cdot \frac{\log^+ |f^n(y)|_v}{d^n}. \tag{4.2}$$

As proven in [CS93], for any $x \in \overline{K}$, we have that $\hat{h}_f(x) \geq 0$ with equality if and only if x is preperiodic under the iteration of f.

Counted with multiplicity, a degree $d \geq 2$ polynomial f has exactly $d-1$ critical points (other than infinity). As introduced by Silverman, the *critical height* $\hat{h}_{\mathrm{crit}}$ of a polynomial is given by the sum of the canonical heights of its $(d-1)$ critical points (see (1.2)). Clearly, $\hat{h}_{\mathrm{crit}}(f) \geq 0$ with equality if and only if f is PCF.

4.2. **Arithmetic equidistribution theorem.** Let X be an irreducible projective curve defined over a number field K, and L be an ample line bundle of X. Replacing the absolute value $|\cdot|$ in Subsection 3.2 by $|\cdot|_v$, we can define metrics $\|\cdot\|_v$ on L corresponding to each $v \in \Omega$. Let $X^{an}_{\mathbb{C}_v}$ be the analytic space associated to $X(\mathbb{C}_v)$. When v is Archimedean, $X^{an}_{\mathbb{C}_v} \simeq X(\mathbb{C})$. For Archimedean v, a continuous metric $\|\cdot\|_v$ on a line bundle is *semi-positive* if its curvature $dd^c \log \|\cdot\|_v$ is non negative. For non-Archimedean places v, $X^{an}_{\mathbb{C}_v}$ is the Berkovich space associated to $X(\mathbb{C}_v)$, and Chambert-Loir [CL06, CL11] constructed an analog of curvature on $X^{an}_{\mathbb{C}_v}$ using methods from Berkovich spaces.

An *adelic metrized line bundle* $\overline{L} := \{L, \{\|\cdot\|_v\}_{v\in\Omega_K}\}$ over L is a collection of metrics on L, one for each place $v \in \Omega_K$, satisfying certain continuity and coherence conditions; see [Zha95a, Zha95b]. Precisely, the metric $\|\cdot\|_v$ on L should be continuous for each $v \in \Omega_K$. Moreover, we require that: there exists a model $(\mathfrak{X}, \mathscr{L}, e)$ over the ring of integers of K inducing the given metrics for all but finitely many places $v \in \Omega_K$. An adelic metrized line bundle is said to be *semi-positive* if the metrics are semi-positive at all places; see [Thu] for more details.

For example, we can construct an adelic metrized line bundle from a non constant morphism $g : X \to \mathbb{P}^1$ for an irreducible smooth projective curve X defined over a number field K. Let

$$D := \sum_{x \in X:\ \mathrm{ord}_x g < 0} (-\mathrm{ord}_x g) \cdot x$$

be the Weil divisor corresponding to the poles of g. For each $v \in \Omega_K$, the metric $\|\cdot\|_v$ on $L_D|_{X\backslash D}$ is defined as

$$\|s(x)\|_v := e^{-\log^+|g|_v} \cdot |\varphi(s(x))|_v$$

for s being a section of $L_D|_{X\backslash D}$ and $\varphi = \mathrm{pr}_2 \circ \psi$ corresponding to a trivialization ψ of $L_D|_{X\backslash D} \simeq (X\backslash D) \times \mathbb{C}_v$. This metric can be extended to the whole L_D, and the extended metric is continuous and semi-positive. The metrized line bundle $\overline{L}_D := \{L_D, \{\|\cdot\|_v\}_{v\in\Omega_K}\}$ constructed this way is adelic. The simplest one is when $X = \mathbb{P}^1$ and g is the identity map, i.e., we get an adelic metrized line bundle $\overline{\mathcal{O}}_{\mathbb{P}^1}(1)$. For general non constant morphisms $g : X \to \mathbb{P}^1$, it is obvious that $\overline{L}_D \simeq g^*\overline{\mathcal{O}}_{\mathbb{P}^1}(1)$.

For a semi-positive line bundle $\overline{L}$ of X, there is a height $\hat{h}_{\overline{L}}(Y)$ associated to it for each subvariety Y of X; see [Zha95b]. In the case of points $x \in X(\overline{K})$, the height is given by

$$\hat{h}_{\overline{L}}(x) := \frac{1}{[K:\mathbb{Q}]} \sum_{y\in \mathrm{Gal}(\overline{K}/K)\cdot x} \sum_{v\in\Omega_K} \frac{-N_v \cdot \log\|s(y)\|_v}{|\operatorname{Gal}(\overline{K}/K)\cdot x|} \tag{4.3}$$

where $|\operatorname{Gal}(\overline{K}/K)\cdot x|$ is the number of points in the Galois orbit $\operatorname{Gal}(\overline{K}/K)\cdot x$, and s is any meromorphic section of L with support disjoint from the Galois orbit of x. A sequence of points $x_n \in X(\overline{K})$ is said to be *small* if $\lim_{n\to\infty} \hat{h}_{\overline{L}}(x_n) = \hat{h}_{\overline{L}}(X)$. For the cases considered in this article, we will see that $\hat{h}_{\overline{L}}(X)$ is always zero.

Theorem 4.1. [Thu, Yua08] *Suppose X is a projective curve over a number field K and $\overline{L}$ is a semi-positive adelic metrized line bundle on X with L being ample. Let $\{x_n\}$ be a non-repeating sequence of points in $X(\overline{K})$ which is small. Then for any $v \in \Omega_K$, the Galois orbits of the sequence $\{x_n\}$ are equidistributed in the analytic space $X^{an}_{\mathbb{C}_v}$ with respect to the probability measure $\mu_v = c_1(\overline{L})_v/\deg_L(X)$.*

Here $c_1(\overline{L})_v$ is the curvature of the metric on L for each place $v \in \Omega_K$.

4.3. **Equidistribution of parameters with small heights.** Analogous to (2.4), for each $v \in \Omega_K$, we define

$$G_v^{\pm}(a,b) := \lim_{n\to\infty} \frac{\log^+|f^n_{a,b}(\pm a)|_v}{3^n}.$$

There is a metric $\|\cdot\|_v$ on each $L_{D_n^{\pm}}$ (hence metrics $\|\cdot\|_v$ on $L^{\otimes 3^i}_{D_n^{\pm}}$ for $i \geq 1$) for n sufficiently large given by (see (3.13))

$$\|s(t)\|_v := e^{-3^n\cdot G_v^{\pm}} \cdot |\varphi(s(t))|_v. \tag{4.4}$$

More precisely, n needs to be sufficiently large so that for each $t_0 \in S_0^{\pm}$, we have that $\gamma^{\pm}_{m+1} = 3\cdot\gamma^{\pm}_m$ for $m \geq n$, where $\gamma^{\pm}_m$ is the order of the pole at t_0 of $f^m_{a,b}(\pm a)$ (see Lemma 2.2). For such a positive integer n, we have

$$L_{D^{\pm}_{n+i}} = L^{\otimes 3^i}_{D^{\pm}_n} \text{ for each } i \geq 1. \tag{4.5}$$

Line by line examination of Section 3 indicates that $\|\cdot\|_v$ on $L_{D_n^{\pm}}$ is continuous and semi-positive (see also Remark 4.3) for each $v \in \Omega$. Actually, we use only Taylor series, triangle inequality, Maximum Value Theorem and elementary algebra for the proofs of lemmas in Section 3, and all these work when dealing with a non-Archimedean norm $|\cdot|_v$.

Now, let n_0 be a sufficiently large positive integer so that (4.5) holds, and let $\overline{L}_{D^{\pm}_{n_0}} := \{L_{D^{\pm}_{n_0}}, \{\|\cdot\|_v\}_{v\in\Omega_K}\}$ be the metrized line bundle constructed above.

Lemma 4.2. *If $+a$ is not persistently preperiodic under $f_{a,b}$ on $\hat{C}$, then there is a non constant rational function g on $\hat{C}$ such that*

$$3^{n_0}\cdot G_v^+ = \log^+|g|_v$$

on $\hat{C}$ for all but finitely many $v\in\Omega_K$.

Proof. Replacing the divisor $d^N(d-1)D$ in [Ing13, Lemma 6] by $D^+_{n_0}$, we obtain the desired result from [Ing13, Lemma 13]. □

For g being the rational map as in the conclusion of Lemma 4.2, we get that the isomorphism $L_{D^+_{n_0}} \simeq g^*\mathcal{O}_{\mathbb{P}^1}(1)$ is an isometry for all but finitely many places $v\in\Omega_K$. Similarly, Lemma 4.2 holds for the marked critical point $-a$ and its escape-rate function G_v^-.

Remark 4.3. We note that Thuillier [Thu] defined the notion of being W^1-regular for a metric on a line bundle at an arbitrary place v (both Archimedean and non-Archimedean); then Thuillier [Thu, Theorem 4.3.7] proves the equidistribution theorem for points of small height with respect to adelic, W^1-regular metrics on ample line bundles on a curve. Also, as proven by Thuillier (we thank Laura DeMarco and Myrto Mavraki for sharing with us Thuillier's note), every continuous subharmonic metric is W^1-regular, and therefore Theorem 4.1 holds for such metrics, as it is the case for the metrics we constructed on $L_{D^{\pm}_{n_0}}$ (by Theorem 3.1).

Remark 4.4. We also thank Xinyi Yuan for pointing out that Theorem 4.1 holds for the metrics constructed on $L_{D^{\pm}_{n_0}}$ by employing [YZ, Proposition A.8]; a remarkable feature of the article by Yuan and Zhang is that they can prove that the limit of semi-positive metrics is still semi-positive, even when the underlying line bundles in the limit process are not the same. Finally, we thank the referee for pointing out that in [FG, Lemma 3.11], it is shown that a continuous, subharmonic metric is also semi-positive in the sense of Yuan and Zhang.

Combining Lemma 4.2 with the setup from Subsections 4.2 and 4.3, one obtains the following:

Corollary 4.5. *If $+a$ (resp. $-a$) is not persistently preperiodic under $f_{a,b}$ on $\hat{C}$, then $\overline{L}_{D^+_{n_0}}$ (resp. $\overline{L}_{D^-_{n_0}}$) is a semi-positive adelic metrized line bundle on $\hat{C}$.*

For $t\in\hat{C}(\overline{K})$, we let $\hat{h}_{\pm}(t) := \hat{h}_{f_{a(t),b(t)}}(\pm a(t))$ be the height of the two critical points of $f_{a,b}$. From the definition (4.3) of the height corresponding to the metrized line bundle $\overline{L}_{D^{\pm}_{n_0}}$, for any $t\in\hat{C}(\overline{K})\backslash S^{\pm}$, one has

$$\begin{aligned}
\hat{h}_{\overline{L}_{D^{\pm}_{n_0}}}(t) &= \frac{1}{[K:\mathbb{Q}]}\sum_{y\in\mathrm{Gal}(\overline{K}/K)\cdot t}\sum_{v\in\Omega_K}\frac{-N_v\cdot\log\|s(y)\|_v}{|\mathrm{Gal}(\overline{K}/K)\cdot t|}\text{ , by (4.4)}\\
&= \frac{1}{[K:\mathbb{Q}]}\sum_{y\in\mathrm{Gal}(\overline{K}/K)\cdot t}\sum_{v\in\Omega_K}\frac{N_v\cdot(3^{n_0}G_v^{\pm}-\log|\varphi(s(y))|_v)}{|\mathrm{Gal}(\overline{K}/K)\cdot t|}\text{ , by formula (4.1)}\\
&= \frac{3^{n_0}}{[K:\mathbb{Q}]}\sum_{y\in\mathrm{Gal}(\overline{K}/K)\cdot t}\sum_{v\in\Omega_K}\frac{N_v\cdot G_v^{\pm}(a(y),b(y))}{|\mathrm{Gal}(\overline{K}/K)\cdot t|}\\
&= \frac{3^{n_0}}{[K:\mathbb{Q}]}\sum_{y\in\mathrm{Gal}(\overline{K}/K)\cdot t}\sum_{v\in\Omega_K}\frac{\lim_{i\to\infty}\frac{N_v\cdot\log^+\left|f^i_{a(y),b(y)}(\pm a(y))\right|_v}{3^i}}{|\mathrm{Gal}(\overline{K}/K)\cdot t|}\\
&= 3^{n_0}\cdot\lim_{i\to\infty}\frac{\sum_{y\in\mathrm{Gal}(\overline{K}/K)\cdot t}\sum_{v\in\Omega_K}\frac{N_v\cdot\log^+\left|f^i_{a(y),b(y)}(\pm a(y))\right|_v}{3^i}}{[K:\mathbb{Q}]\cdot|\mathrm{Gal}(\overline{K}/K)\cdot t|}\\
&= 3^{n_0}\lim_{i\to\infty}h\left(f^i_{a(t),b(t)}(\pm a(t))\right)/3^i = 3^{n_0}\cdot\hat{h}_{f_{a(t),b(t)}}(\pm a(t)) = 3^{n_0}\cdot\hat{h}_{\pm}(t).
\end{aligned}$$

Here we can take the limit outside of the summation because $|f^i_{a(y),b(y)}(\pm a(y))|_v\le 1$ for all but finitely many places $v\in\Omega_K$ and all $i\ge 0$ (for any given $y\in\mathrm{Gal}(\overline{K}/K)\cdot t$).

At the expense of replacing K by a finite extension, we may assume that $S_0^+\subset\hat{C}(K)$. Let $t_0\in S_0^+$, and let γ_{n_0} be the multiplicity of the point t_0 in the divisor $D^{\pm}_{n_0}$. Let s be a non-vanishing section of $\overline{L}_{D^{\pm}_{n_0}}$ locally defined in some neighborhood of $t_0\in\hat{C}$, and let $u\in K(\hat{C})$ be a uniformizer at t_0. For a choice of a trivialization φ of $\overline{L}_{D^{\pm}_{n_0}}$ in a neighborhood of t_0, we have that the function $\varphi(s(t))u^{\gamma_{n_0}}(t)$ is a non vanishing morphism from a neighborhood of $t_0\in\hat{C}$ to the affine line $\mathbb{A}^1$. From the definition of metric (3.13) for the line bundle, we have

$$\|s(t)\|_v = e^{-3^{n_0}G_v^{\pm}(t)}\cdot|\varphi(s(t))|_v = e^{-3^{n_0}\left(G_v^{\pm}(t)+\frac{\gamma_{n_0}}{3^{n_0}}\log|u|_v\right)}\cdot|\varphi(s(t))\cdot u^{\gamma_{n_0}}(t)|_v \tag{4.6}$$

We notice that

$$u^{\gamma_{n_0+i}}f^{n_0+i}_{a,b}(\pm a)(t_0) = [u^{\gamma_0}f^{n_0}_{a,b}(\pm a)(t_0)]^{3^i}. \tag{4.7}$$

Indeed, letting $c_n^{\pm} := f^n_{a,b}(\pm a)$ as before, since

$$c^{\pm}_{n+1} = (c^{\pm}_n)^3 - 3a^2c^{\pm}_n + b$$

and $c^{\pm}_n$ has a pole at t_0 of order γ_n, which is larger than $\max\{-\mathrm{ord}_{t_0}(a), -\mathrm{ord}_{t_0}(b)/3\}$ (see Lemma 2.2), then we get (4.7).

From Lemma 3.3 and (4.7), we obtain that

$$\begin{aligned}\lim_{t\to t_0}\left(G_v^{\pm}(t)+\frac{\gamma_{n_0}}{3^{n_0}}\log|u(t)|_v\right) &= \lim_{t\to t_0}\left(G_v^{\pm}(t)+\frac{\gamma_{n_0+i}}{3^{n_0+i}}\log|u(t)|_v\right)\\ &= \lim_{t\to t_0}\lim_{i\to\infty}\frac{\log^+|f_{a,b}^{n_0+i}(\pm a)(t)|_v}{3^{n_0+i}}+\frac{\gamma_{n_0+i}\cdot\log|u(t)|_v}{3^{n_0+i}}\\ &= \lim_{i\to\infty}\frac{1}{3^{n_0+i}}\log|u^{\gamma_{n_0+i}}f_{a,b}^{n_0+i}(\pm a)(t_0)|_v\\ &= \frac{1}{3^{n_0}}\log|u^{\gamma_{n_0}}f_{a,b}^{n_0}(\pm a)(t_0)|_v\end{aligned}$$

Hence from (4.6), one has

$$\|s(t_0)\|_v=\frac{|\varphi(s)\cdot u^{\gamma_{n_0}}(t_0)|_v}{|u^{\gamma_{n_0}}f_{a,b}^{n_0}(\pm a)(t_0)|_v}.$$

We recall that both $\varphi(s)\cdot u^{\gamma_{n_0}}(t_0)$ and $u^{\gamma_{n_0}}f_{a,b}^{n_0}(\pm a)(t_0)$ are nonzero. Then, from the definition of the height and by the product formula, we have

$$\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(t_0)=\frac{1}{[K:\mathbb{Q}]}\sum_{v\in\Omega_K}-N_v\cdot\log\|s(t_0)\|_v=\frac{1}{[K:\mathbb{Q}]}\sum_{v\in\Omega_K}-N_v\cdot\log\left|\frac{\varphi(s)u^{\gamma_{n_0}}}{f_{a,b}^{n_0}(\pm a)u^{\gamma_{n_0}}}(t_0)\right|_v=0.$$

Also, for $t_0\in S^{\pm}\backslash S_0^{\pm}$, from (3.1) we get that $G_v^{\pm}(a(t_0),b(t_0))=0$ (see Theorem 3.1 (i)) for each $v\in\Omega_K$; therefore $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(t_0)=0$.

Theorem 1.2 follows from Theorem 4.6 (see also Remark 4.7).

Theorem 4.6. *Let $\hat C$ be an irreducible smooth curve defined over some number field K such that $\pm a$ is not persistently preperiodic for $f_{a,b}$ on $\hat C$, and let n_0 be a sufficiently large integer (as in the conclusion of Lemma 2.2). Then for any non-repeating sequence of points $t_n\in\hat C(\overline{K})$ with $\lim_{n\to\infty}\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(t_n)=0$, the Galois orbits of $\{t_n\}_{n\geq 1}$ equidistribute with respect to the probability measure $\mu_v^{\pm}=c_1(\overline{L}_{D_{n_0}^{\pm}})_v/\deg(D_{n_0}^{\pm})$ on $\hat C_{\mathbb{C}_v}^{an}$ for each $v\in\Omega_K$.*

Proof. To prove this theorem, by Theorem 4.1 and Corollary 4.5, it suffices to show that $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(\hat C)=0$. We know that $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(t)=3^{n_0}\cdot\hat h_{\pm}(t)\geq 0$ for $t\in\hat C\setminus S^{\pm}$ with equality if and only if $\pm a(t)$ is preperiodic under $f_{a(t),b(t)}$. Hence $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}$ is non-negative on $\hat C$; note also that $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(t)=0$ if $t\in S^{\pm}$. Also, there are infinitely many $t\in\hat C(\overline{K})$ with $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(t)=0$ (see Proposition 2.4). Hence $\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(\hat C)=0$ by [Zha95b, Theorem 1.10] (note that $\overline{L}_{D_{n_0}^{\pm}}$ is integrable since it is semi-positive and thus we can apply [Zha95b, Theorem 1.10]). Indeed, using the notation from [Zha95b, Section (1.9)], both essential minima $e_1\left(\overline{L}_{D_{n_0}^{\pm}}\right)$ and $e_2\left(\overline{L}_{D_{n_0}^{\pm}}\right)$ equal 0; since there exist infinitely many points on the curve of height equal to 0, we have that $e_2\left(\overline{L}_{D_{n_0}^{\pm}}\right)=\inf_{x\in C(\overline{\mathbb{Q}})}\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(x)=0$ and also that $e_1\left(\overline{L}_{D_{n_0}^{\pm}}\right)=\sup_{Y\subsetneq\hat C}\inf_{x\in(\hat C\setminus Y)(\overline{\mathbb{Q}})}\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(x)=0$; thus [Zha95b, Theorem 1.10] yields that

$$\left(e_1(\overline{L}_{D_{n_0}^{\pm}})+e_2(\overline{L}_{D_{n_0}^{\pm}})\right)/2=0\leq\hat h_{\overline{L}_{D_{n_0}^{\pm}}}(\hat C)\leq 0=e_1(\overline{L}_{D_{n_0}^{\pm}}),$$

as desired. □

Remark 4.7. If v is Archimedean, then $\hat{C}^{an}_{\mathbb{C}_v} \simeq \hat{C}(\mathbb{C})$ and on $\hat{C}(\mathbb{C})\backslash S^{\pm}$ the measure $\mu_v^{\pm}$ is the normalization (total mass 1 on $\hat{C}$) of the bifurcation measure $\mu_{\pm}$ introduced in (2.5). If $\hat{C}$ is defined over $\overline{\mathbb{Q}}$, then a point $t \in \hat{C}(\overline{\mathbb{Q}}) \setminus S^{\pm}$ has height zero for the adelic metrized line bundle $\overline{L}_{D^{\pm}_{n_0}}$ if and only if the critical point $\pm a(t)$ is preperiodic under $f_{a(t),b(t)}$. And since $\mu_v^{\pm}$ admits no atom on $\hat{C}(\mathbb{C})$ (the potential function is continuous), the set of parameters on $\hat{C}$ for which the marked critical point $\pm a$ preperiodic under $f_{a,b}$ is equidistributed with respect to the normalized bifurcation measure $\mu_{\pm}$.

Corollary 4.8. *Let $C \subset \mathbb{C}^2$ be an irreducible curve defined over a number field K, satisfying*

- *neither $+a$ nor $-a$ is persistently preperiodic under $f_{a,b}$ on C.*
- *there is a sequence of non-repeating points $t_n \in C(\overline{K})$ with*

$$\lim_{n\to\infty} \hat{h}_{\mathrm{crit}}(f_{a(t_n),b(t_n)}) = 0.$$

Then for any sufficiently large n_0, $\deg(D^-_{n_0}) \cdot G_v^+ = \deg(D^+_{n_0}) \cdot G_v^-$ on $\hat{C}$ for all $v \in \Omega_K$.

Proof. As $\hat{h}_{\mathrm{crit}}(f_{a(t),b(t)}) = \hat{h}_+(t) + \hat{h}_-(t) = \left(\hat{h}_{\overline{L}_{D^+_{n_0}}}(t) + \hat{h}_{\overline{L}_{D^-_{n_0}}}(t)\right)/3^{n_0}$, we have

$$\lim_{n\to\infty} \hat{h}_{\overline{L}_{D^-_{n_0}}}(t_n) = \lim_{n\to\infty} \hat{h}_{\overline{L}_{D^+_{n_0}}}(t_n) = 0.$$

Consequently, the Galois orbits of $\{t_n\}$ equidistribute on $\hat{C}^{an}_{\mathbb{C}_v}$ with respect to the measures $\mu_v^{\pm}$ appearing in Theorem 4.6; hence we have

$$\mu_v^+ = \mu_v^-.$$

When v is Archimedean, $\hat{C}^{an}_{\mathbb{C}_v} \simeq \hat{C}(\mathbb{C})$ and from (2.5, 3.10, 3.13), one has

$$3^{n_0} \cdot \mu_{\pm} = \deg(D^{\pm}_{n_0}) \cdot \mu_v^{\pm}.$$

Combining the above two formulas with Proposition 2.5, then for each Archimedean $v \in \Omega_K$ we get the following equality on $\hat{C}$:

$$\deg(D^-_{n_0}) \cdot G_v^+ = \deg(D^+_{n_0}) \cdot G_v^-.$$

By looking at the growth of $G_v^{\pm}(a(t), b(t))$ for $t \to t_0 \in S_0^{\pm}$ (see Theorem 3.1), the above equation indicates $\deg(D^-_{n_0}) \cdot D^+_{n_0} = \deg(D^+_{n_0}) \cdot D^-_{n_0}$. So we have $L_{D^+_{n_0}}^{\otimes \deg(D^-_{n_0})} \simeq L_{D^-_{n_0}}^{\otimes \deg(D^+_{n_0})}$, and the two canonical metrics on this line bundle induce the same curvature on $\hat{C}$ for each $v \in \Omega_K$ because $\mu_v^+ = \mu_v^-$. From [Yua12, Theorem 3.3], we know that if two continuous semi-positive metrics on a line bundle have the same curvature, then these two metrics are proportional to each other. So for each $v \in \Omega_K$, we have

$$\deg(D^-_{n_0}) \cdot G_v^+ = \deg(D^+_{n_0}) \cdot G_v^- + M(v)$$

on C with $M(v)$ being a constant depending only on v. Because for all $t \in C$, we have

$$\hat{h}_{\mathrm{crit}}(f_{a(t),b(t)}) \geq \hat{h}_{\pm}(t) \geq G_v^{\pm}(a(t), b(t)) \geq 0,$$

we have $\lim_{n\to\infty} G_v^+(a(t_n), b(t_n)) = \lim_{n\to\infty} G_v^-(a(t_n), b(t_n)) = 0$, i.e., $M(v) = 0$ for all $v \in \Omega$. □

5. An algebraic relation between critical points

In this section, under certain conditions, we prove an algebraic relation of the iterated critical points on a curve C; see Theorem 5.1. Our strategy is similar to the one employed by Baker and DeMarco in [BD13, Section 5].

Let K be a number field and $C \subset \mathbb{C}^2$ be an irreducible curve defined over K. As before, we let $\hat{C}$ be a smooth projective curve birational to C, and we let

$$c_n^{\pm} := f_{a,b}^n(\pm a) \in K(\hat{C}).$$

Theorem 5.1. *Suppose the following two conditions are satisfied:*

- *neither $+a$ nor $-a$ is persistently preperiodic under $f_{a,b}$ on C.*
- *there is a sequence of non-repeating points $t_n \in \hat{C}(\overline{K})$ with*

$$\lim_{n\to\infty} \hat{h}_{\mathrm{crit}}(f_{a(t_n),b(t_n)}) = 0.$$

Then there exist non constant polynomials $P_{\pm}(z) \in K(\hat{C})[z]$ with

$$P_+(c_n^+) = P_-(c_n^-) \in K(\hat{C})$$

for all sufficiently large n.

5.1. **Böttcher coordinate.** For each $f_{a,b}(z) = z^3 - 3a^2z + b$ there is a unique uniformized Böttcher coordinate $\Phi_{a,b}(z)$, i.e., a univalent, analytic function defined on a neighbourhood of infinity, which is uniquely determined by the conditions

$$\Phi_{a,b}(f_{a,b}(z)) = \Phi_{a,b}(z)^3, \text{ and } \Phi_{a,b}(z) = z + o(1).$$

Moreover, for the escape-rate function of the polynomial $f_{a,b}$

$$G_{a,b}(z) := \lim_{n\to\infty} \frac{\log^+ |f_{a,b}^n(z)|}{3^n},$$

we have

$$\log |\Phi_{a,b}(z)| = G_{a,b}(z)$$

for any $z \in \mathbb{C}$ with $G_{a,b}(z) > \max\{G_{a,b}(+a), G_{a,b}(-a)\}$. Let

$$\Phi_{a,b}(z) =: z + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \frac{\alpha_3}{z^3} + \cdots$$

We note that there is no constant term in the above formula since $f_{a,b}$ is monic and centred; one can compare the Laurent series for both sides of the equation $\Phi_{a,b}(f_{a,b}(z)) = \Phi_{a,b}(z)^3$ and see that the constant term of $\Phi_{a,b}(z)$ must be zero. More generally, for any polynomial $g(z)$ of degree $d \geq 2$ in *normal form* (i.e., monic and with no nonzero term of degree $d-1$), the corresponding Böttcher coordinate is of the form $z + \sum_{i\geq 1} \alpha_i/z^i$.

Similarly by comparing the Laurent series for both sides of $\Phi_{a,b}(f_{a,b}(z)) = \Phi_{a,b}(z)^3$, recursively one can show that α_i is a polynomial of a and b for $i = 1, 2, \cdots$. Moreover, let the k-th power of $\Phi_{a,b}(z)$ be written as:

$$\Phi_{a,b}(z)^k = P_k(z) + \frac{\alpha_{k,1}}{z} + \frac{\alpha_{k,2}}{z^2} + \frac{\alpha_{k,3}}{z^3} + \cdots \tag{5.1}$$

where $P_k(z) \in \mathbb{C}[a,b][z]$ and $\alpha_{k,i}$ are polynomial functions in a and b.

The following result is implicitly used in the proof of [BD13, Lemma 5.5] and the ingredients for its proof are all contained in [BD13], as it was kindly pointed out to us by Laura DeMarco, whom we thank warmly for her help.

Lemma 5.2. *We work under the notation as before for the plane curve C, its projective smooth model $\hat{C}$ and the corresponding $\Phi_{a(t),b(t)}$ for the specialization of the Böttcher's coordinate along $\hat{C}$; we also let $f_t(z) := z^3 - 3a(t)^2 z + b(t)$ for $t \in \hat{C}(\mathbb{C})$. Let $t_0 \in S^{\pm}$ and let $k \in \mathbb{N}$. Then for each $c \in \mathbb{C}(\hat{C})$ with the property that $-\operatorname{ord}_{t_0} f_t^n(c(t)) \to \infty$, we have that*

$$\operatorname{ord}_{t_0}\left(\Phi_{a(t),b(t)}(f_t^n(c(t)))^k - P_k(f_t^n(c(t)))\right) \to \infty,$$

as $n \to \infty$.

Proof. First, we know that each $\alpha_{k,i}$ is a polynomial in a and b and so, specializing a and b along the curve $\hat{C}$ we obtain that each $\alpha_{k,i}$ becomes a function $\alpha_{k,i}(t) \in \mathbb{C}(\hat{C})$. We note that the order of the pole of $\alpha_{k,i}(t)$ at t_0 grows at most linearly fast as $i \to \infty$ (for k fixed) (see [BD13, Lemma 5.4]).

Secondly, arguing as in the proof of Lemma 3.5, we choose a suitable analytic parametrization of a neighborhood of t_0 and a suitable uniformizer u at t_0 such that we may assume $t_0 = 0$ and $u = t$. Thus (from the definition of the Böttcher's coordinate) we obtain that the power series

$$F(t,z) := \sum_{i \geq 1} \frac{\alpha_{k,i}(t)}{z^i},$$

converges on open sets of $\mathbb{C} \times \mathbb{C}$ of the form

$$\{(t,z) \colon |t| < s \text{ and } |z| > R_t^1\}, \tag{5.2}$$

for any $s < s_0$, where s_0 is a given small positive real number, and R_t^1 is sufficiently large so that the Böttcher's coordinate of the polynomial $f_t(z)$ converges for $|z| > R_t^1$; for example, one may take (see the proof of [BD13, Lemma 5.1])

$$R_t^1 = O\left(|t|^{2\min\{\operatorname{ord}_0(a), \operatorname{ord}_0(b)/3\}}\right). \tag{5.3}$$

Since $t_0 = 0 \in S^{\pm}$, we know that $\min\{\operatorname{ord}_0(a), \operatorname{ord}_0(b)/3\} < 0$ and therefore R_t^1 is (potentially) larger when $|t|$ gets smaller. Using the estimate from (5.3), we can construct (perhaps after replacing s_0 by a smaller positive real number) a decreasing real-valued function $s \mapsto R_s$ on the interval $(0, s_0)$ such that

$$R_s \geq \max\{R_t^1 : |t| = s\} \text{ for each } s < s_0.$$

Next we consider the function

$$U^{\ell}(t) := \sum_{i \geq 1} \frac{\alpha_{k,i}(t)}{f_t^{\ell}(c(t))^i}, \tag{5.4}$$

for $\ell \in \mathbb{N}$, where $c \in \mathbb{C}(\hat{C})$ is a function such that $\lim_{n\to\infty} -\operatorname{ord}_0 f_t^n(c(t)) = \infty$. Our goal is to conclude that

$$\lim_{\ell \to \infty} \operatorname{ord}_0 U^{\ell}(t) = +\infty. \tag{5.5}$$

First we note that, perhaps at the expense of replacing $c(t)$ by an iterate of it under f_t, we have that $|f_t^\ell(c(t))| > R_t$ for each ℓ whenever $|t| < s_0$ (see (5.3)); therefore $U^\ell(t)$ is analytic if $|t| < s_0$.

Now, since $F(t,z)$ is analytic in both t and z, then the series defining $F(t,z)$ converges uniformly on sets of the form (see also (5.2))

$$\{t \in \hat{C}(\mathbb{C}) : s_1 < |t| < s_2\} \times \{z : |z| > R_{s_1}\},$$

where $0 < s_1 < s_2 < s_0$.

Fix now some $s_1 < s_2$ in the interval $(0, s_0)$. Then let $\ell \in \mathbb{N}$ with the property that if $|t| < s_2$, we have that $|f_t^\ell(c(t))| > R_{s_1}$. Then the series $U^\ell(t)$ converges uniformly on

$$\{t \in C(\mathbb{C}) \colon s_1 < |t| < s_2\}.$$

In particular, this means that the difference between $U^\ell(t)$ and the partial sum $U_n^\ell(t)$ which is the sum of the first n terms in the series (5.4) is uniformly bounded on the above annuli.

However, because both $U^\ell(t)$ and also $U_n^\ell(t)$ are analytic functions when $|t| < s_2$, then their difference is bounded by their difference on the closed set $|t| = s_2$ (according to the Maximum Value Theorem), which is in turn uniformly bounded (as we vary n) because of the above uniform convergence on the annuli. So, $\{U_n^\ell\}$ converges uniformly to U^ℓ when $|t| < s_2$. Since each U_n^ℓ vanishes at 0 of high order (note that each U_n^ℓ is simply a finite sum and each term vanishes at 0 of order depending on the order of the pole of $f_t^\ell(c(t))$, while the order of the pole at 0 for $\alpha_{k,i}(t)$ grows only linearly in i), then also $U^\ell(t)$ vanishes at 0 of high order, thus proving (5.5). □

5.2. **Proof of Theorem 5.1.** From the hypotheses of Theorem 5.1, by Corollary 4.8, it is clear that for each large n_0,

$$\deg(D_{n_0}^-) \cdot D_{n_0}^+ = \deg(D_{n_0}^+) \cdot D_{n_0}^-.$$

Let $t_0 \in S_0^\pm$; then for $t \in \hat{C}$ near t_0, we know that for large n, both $c_n^\pm(t)$ are in the univalent domain of $\Phi_{a(t),b(t)}(z)$ (this is proven in [BD13, Lemma 5.1]). Since

$$|\Phi_{a(t),b(t)}(c_n^\pm(t))| = 3^n \cdot G_{a(t),b(t)}(\pm a(t)) = 3^n \cdot G^\pm(a(t), b(t))$$

by Corollary 4.8, for all t close to t_0, one has

$$\deg(D_{n_0}^-) \cdot |\Phi_{a(t),b(t)}(c_n^+(t))| = \deg(D_{n_0}^+) \cdot |\Phi_{a(t),b(t)}(c_n^-(t))|,$$

i.e., there is $\zeta_{t_0,n} \in \mathbb{C}$ of absolute value 1 such that

$$\Phi_{a(t),b(t)}^{\deg(D_{n_0}^-)}(c_n^+(t)) = \zeta_{t_0,n} \cdot \Phi_{a(t),b(t)}^{\deg(D_{n_0}^+)}(c_n^-(t)).$$

Lemma 5.3. *The number $\zeta_{t_0,n}$ is a root of unity.*

Proof. Since $\Phi_{a(t),b(t)}(c_n^\pm(t)) = c_n^\pm(t) + o(1)$, then by Lemma 5.2 we see that

$$\zeta_{t_0,n} = \lim_{t \to t_0} \frac{(c_n^+(t))^{\deg(D_{n_0}^-)}}{(c_n^-(t))^{\deg(D_{n_0}^+)}}. \tag{5.6}$$

In particular, this yields that $\zeta_{t_0,n} \in K$ (because a and b are rational functions on the curve $\hat{C}$ which is defined over K). Now we want to show that for any non-Archimedean place

$v \in \Omega_K$, we also have $|\zeta_{t_0,n}|_v = 1$. For large n, we know that for t close to t_0 in the topology determined by the non-Archimedean place v,

$$|f_{a,b}(c_n^{\pm}(t))|_v = |c_n^{\pm}(t)|_v^3 \gg |2a^2(t)|_v, |b(t)|_v$$

and inductively

$$|f_{a,b}^i(c_n^{\pm}(t))|_v = |c_n^{\pm}(t)|_v^{3^i}.$$

Hence from the definition of $G_v^{\pm}$, we have $G_v^{\pm}(a(t), b(t)) = \frac{\log|c_n^{\pm}(t)|_v}{3^n}$. Again, by Corollary 4.8, for $t \in \hat{C}$ close to t_0, we have

$$\deg(D_{n_0}^-) \cdot \log|c_n^+(t)|_v = \deg(D_{n_0}^+) \cdot \log|c_n^-(t)|_v. \tag{5.7}$$

Then equalities (5.6) and (5.7) yield $|\zeta_{t_0,n}|_v = 1$. As $|\zeta_{t_0,n}|_v = 1$ for all $v \in \Omega_K$, we conclude that $\zeta_{t_0,n}$ a root of unity. ☐

Since $\Phi_{a,b}(c_{n+i}^{\pm}(t)) = \Phi_{a,b}(f_{a,b}^i(c_n^{\pm}(t))) = \Phi_{a,b}^{3^i}(c_n(t))$, if we increase n to $n+i$, then we get that $\zeta_{t_0,n+i} = \zeta_{t_0,n}^{3^i}$.

We pick a large k, such that $\zeta_{t_0,n}^k = 1$ for all $t_0 \in S_0^{\pm}$ (and all large n). Also we let

$$P_+(z) := P_{k\cdot\deg(D_{n_0}^-)}(z) \text{ and } P_-(z) := P_{k\cdot\deg(D_{n_0}^+)}(z),$$

where the polynomials $P_m(z)$ are defined as in (5.1). Then Lemma 5.2 coupled with the fact

$$\Phi_{a(t),b(t)}^{k\cdot\deg(D_{n_0}^-)}(c_n^+(t)) = \zeta_{t_0,n}^k \cdot \Phi_{a(t),b(t)}^{k\cdot\deg(D_{n_0}^+)}(c_n^-(t)) = \Phi_{a(t),b(t)}^{k\cdot\deg(D_{n_0}^+)}(c_n^-(t)),$$

yield that for each $t_0 \in S_0^+ = S_0^-$, we have

$$\lim_{n\to\infty} \operatorname{ord}_{t_0}\left(P_+(c_n^+(t)) - P_-(c_n^-(t))\right) = \infty. \tag{5.8}$$

Moreover, as $P_{\pm}(z) \in \mathbb{C}[a,b][z]$ and also $c_n^{\pm} \in \mathbb{C}[a,b]$, then $P_{\pm}(c_n^{\pm}(t))$ may have poles only at the points contained in $S^{\pm}$. However, (5.8) coupled with the fact that for $t_0 \in S^{\pm} \setminus S_0^{\pm}$, the functions $c_n^{\pm}(t)$ have poles of bounded order at t_0 yields that $P_{\pm}(c_n^{\pm}(t))$ may only have poles of bounded order (as we vary n) at the finitely many points contained in $S^{\pm} \setminus S_0^{\pm}$. But then for large n, (5.8) yields that

$$P_+(c_n^+) - P_-(c_n^-) = 0 \in K(\hat{C}).$$

6. Proof of the main theorem

In this section we finish the proof of the main theorem stated in the introduction.

Proof of Theorem 1.1. As before, we let $\hat{C}$ be a smooth, projective curve birational to C. Notice that PCF points $(a,b) \in \mathbb{C}^2$ (in the moduli space of cubic polynomials) are the intersections of the zero loci of $f_{a,b}^{n_1}(+a) - f_{a,b}^{m_1}(+a) = 0$ and $f_{a,b}^{n_2}(-a) - f_{a,b}^{m_2}(-a) = 0$ for some $n_i > m_i \geq 0$ with $i = 1, 2$. From Thurston's rigidity [DH93] or a recent short argument [BD13, Proposition 2.6] which in turn uses the compactness of the connectedness locus proven in [BH88], there are only countable many PCF points in MP_3, hence all the parameters (a,b) with $f_{a,b}$ being PCF are in $\overline{\mathbb{Q}}^2 \subset \mathbb{C}^2$. If an irreducible curve $C \subset \mathbb{C}^2$ contains a Zariski-dense set of points in $\overline{\mathbb{Q}}^2$, then C must be a curve defined over $\overline{\mathbb{Q}}$. Since $\hat{h}_{\mathrm{crit}}(f_{a,b}) \geq 0$ with equality if and only if $f_{a,b}$ is PCF, we conclude that statement (1) implies statement (3) in the conclusion of Theorem 1.1.

Suppose now that statement (2) holds. If one of the marked critical points is persistently preperiodic on C, then by Proposition 2.4, there are infinitely many $(a,b)\in C$ with $f_{a,b}$ being PCF. If $f_{a,b}^n(+a)=f_{a,b}^m(-a)$ on C for some $n,m\ge 0$, then for any $(a,b)\in C$, $+a$ is preperiodic if and only if $-a$ is preperiodic under $f_{a,b}$. Moreover, if $b=0$ on C, then $f_{a,0}^n(-a)=-f_{a,0}^n(+a)$ for all n on C, i.e., $+a$ has finite forward orbit under $f_{a,0}$ if and only if so does $-a$. Hence by Proposition 2.4, in all these cases there are infinitely many $(a,b)\in C$ with $f_{a,b}$ being PCF. Therefore, statement (2) implies statement (1) in the conclusion of Theorem 1.1.

The only implication left to prove is that statement (3) implies statement (2) in the conclusion of Theorem 1.1. So, suppose now that statement (3) holds. We observe that we may assume $ab\ne 0$ since otherwise either $b=0$ and thus statement (2) holds, or $a=0$ and then the critical points of $f_{0,b}$ are both equal to 0, and thus statement (2) holds because $f_{0,b}^0(+0)=f_{0,b}^0(-0)$.

Let K be a number field such that C and $\hat{C}$ are defined over K. If one of the marked critical points $\pm a$ is persistently preperiodic, then the second statement holds. We assume neither of the critical points $\pm a$ is persistently preperiodic under $f_{a,b}$ on C. From Theorem 5.1, there exist polynomials $P_\pm(z)\in K(\hat{C})[z]$ such that for all $n\ge n_0$ (for some large positive integer n_0), we have that $P_+(c_n^+)=P_-(c_n^-)$ as functions in $K(\hat{C})$. Consider the plane curve given by the equation

$$\{(x,y)\colon P_+(x)-P_-(y)=0\}.$$

Since $\{(c_n^+,c_n^-)\}_{n\ge n_0}\subset K(\hat{C})\times K(\hat{C})$ is an infinite set lying on $P_+(x)-P_-(y)=0$, which is also invariant by the coordinatewise action of $f_{a,b}$ on $(\mathbb{P}^1)^2$, we can find an irreducible component of $P_+(x)-P_-(y)=0$ containing infinitely many points in $\{(c_n^+,c_n^-)\}_{n\ge n_0}$ and periodic under $(f_{a,b},f_{a,b})$. By [MS14, Theorem 6.24], such an irreducible component of $P_+(x)-P_-(y)=0$ is a graph given by

$$x=g(y)\text{ or }y=g(x)$$

for some $g(z)\in K(\hat{C})[z]$ which commutes with $f_{a,b}^\ell(z)\in K(\hat{C})[z]$ for some $\ell>0$ (note that $f_{a,b}(z)$ is not conjugate to z^3 or to the third Chebyshev polynomial z^3-3z since not both a and b are constant functions). Without loss of generality, we assume the curve is given by $x=g(y)$ and then $c_{n_1}^+=g(c_{n_1}^-)$ for some large positive integer n_1. We will show next that g must be equal with an iterate of $f_{a,b}$.

In order to prove this we use the description of all polynomials commuting with an iterate of a given polynomial, as provided by [Ngu15, Proposition 2.3] (which builds on the works of Medvedev-Scanlon [MS14], Ritt [Rit23] and Schmidt-Steinmetz [SS95]). First we claim that since $ab\ne 0$, then there are no linear polynomials μ commuting with some iterate $f_{a,b}^\ell$. Indeed, an easy induction on ℓ yields that

$$f_{a,b}^\ell(z)=z^{3^\ell}-3^\ell a^2 z^{3^\ell-2}+3^{\ell-1}bz^{3^\ell-3}+\text{ lower order terms.}$$

So, if μ commutes with $f_{a,b}^\ell$, then we immediately get that $\mu(z)=\zeta\cdot z$ for some root of unity ζ satisfying $\zeta^{3^\ell-3}=\zeta^{3^\ell-4}=1$; hence $\zeta=1$, as claimed.

Second, clearly $f_{a,b}$ is not a compositional power of another polynomial (since it has prime degree). Therefore, [Ngu15, Proposition 2.3 part (d)] yields that $g = f_{a,b}^r$ for some nonnegative integer r. This concludes the proof of Theorem 1.1. □

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Dragos Ghioca, Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: dghioca@math.ubc.ca

Hexi Ye, Department of Mathematics, Zhejiang University, Hangzhou, 310027, P.R.China

E-mail address: yehexi@gmail.com