

# DENSITY OF ORBITS OF ENDOMORPHISMS OF ABELIAN VARIETIES

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ABSTRACT. Let  $A$  be an abelian variety defined over  $\bar{\mathbb{Q}}$ , and let  $\varphi$  be a dominant endomorphism of  $A$  as an algebraic variety. We prove that either there exists a non-constant rational fibration preserved by  $\varphi$ , or there exists a point  $x \in A(\bar{\mathbb{Q}})$  whose  $\varphi$ -orbit is Zariski dense in  $A$ . This provides a positive answer for abelian varieties of a question raised by Medvedev and the second author in [MS14]. We prove also a stronger statement of this result in which  $\varphi$  is replaced by any commutative finitely generated monoid of dominant endomorphisms of  $A$ .

## 1. INTRODUCTION

The following conjecture was raised in [MS14, Conjecture 7.14] (motivated by a conjecture of Zhang [Zha10] for polarizable endomorphisms of projective varieties).

**Conjecture 1.1.** *Let  $K_0$  be an algebraically closed field of characteristic 0, let  $X$  be an irreducible algebraic variety defined over  $K_0$ , and let  $\varphi : X \rightarrow X$  be a dominant rational self-map. We suppose there exists no positive dimensional algebraic variety  $Y$  and dominant rational map  $f : X \rightarrow Y$  such that  $f \circ \varphi = f$ . Then there exists  $x \in X(K_0)$  whose forward  $\varphi$ -orbit is Zariski dense in  $X$ .*

We denote by  $\mathcal{O}_\varphi(x)$  the forward  $\varphi$ -orbit, i.e. the set of all  $\varphi^n(x)$  for  $n \geq 0$ , where by  $\varphi^n$  we denote the  $n$ -th compositional power of  $\varphi$ . Conjecture 1.1 was proven in [MS14, Theorem 7.16] in the special case  $X = \mathbb{A}^m$ , and  $\varphi := (f_1, \dots, f_m)$  is given by the coordinatewise action of  $m$  one-variable polynomials  $f_i$ . In this paper we prove Conjecture 1.1 when  $X$  is an abelian variety. As a convention, for us, *endomorphisms* of an abelian variety  $A$  are self-morphisms of  $A$  in the category of algebraic varieties, while the *group endomorphisms* of  $A$  are self-morphisms of  $A$  in the category of abelian varieties. Our result is the fourth known case of Conjecture 1.1 (besides the case proven by Medvedev and the second author in [MS14], Amerik, Bogomolov and Rovinsky [ABR11] exploited the local dynamical behaviour

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of the map  $\varphi$  to prove a special case of Conjecture 1.1 assuming there is a *good*  $p$ -adic analytic parametrization for the orbit  $\mathcal{O}_\varphi(x)$ , and recently, the case when  $X$  is a surface was proven in [BGT] using also  $p$ -adic methods).

**Theorem 1.2.** *Let  $K_0$  be an algebraically closed field of characteristic 0. Let  $A$  be an abelian variety defined over  $K_0$ , and let  $\sigma : A \rightarrow A$  be a dominant map of algebraic varieties. Then the following statements are equivalent:*

- (1) *there exists  $x \in A(K_0)$  such that  $\mathcal{O}_\sigma(x)$  is Zariski dense in  $A$ .*
- (2) *there exists no non-constant rational map  $f : A \rightarrow \mathbb{P}^1$  such that  $f \circ \sigma = f$ .*

The motivation for Conjecture 1.1 comes from two different directions. First, Zhang [Zha10, Conjecture 4.1.6] proposed a variant of Conjecture 1.1 for polarizable endomorphisms  $\varphi$  of projective varieties  $X$  defined over  $\bar{\mathbb{Q}}$  (we say that  $\varphi$  is *polarizable* if there exists an ample line bundle  $\mathcal{L}$  on  $X$  so that  $\varphi^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{L}^{\otimes d}$  for some integer  $d > 1$ ). The polarizability condition imposed by Zhang is stronger than the hypothesis from Conjecture 1.1 that  $\varphi$  preserves no non-constant fibration of  $X$ . The motivation for the stronger hypothesis appearing in [Zha10, Conjecture 4.1.6] lies in the fact that in his seminal paper [Zha10], Zhang was interested in the arithmetic properties exhibited by the dynamics of endomorphisms of projective varieties. In particular, Zhang was interested in formulating good dynamical analogues of the classical Manin-Mumford and Bogomolov Conjectures, and thus he wanted to use the canonical heights associated to polarizable endomorphisms (previously introduced by Call and Silverman [CS93]). The second motivation for Conjecture 1.1 comes from the fact that its conclusion is known assuming  $K_0$  is an uncountable field of characteristic 0 (see [AC08]). More precisely, in [AC08], Amerik and Campana proved that if  $\varphi$  preserves no non-constant rational fibration, then there exist countably many proper subvarieties  $Y_i$  of  $X$  so that for each  $x \in X(K_0) \setminus \cup_i Y_i(K_0)$ , the orbit  $\mathcal{O}_\varphi(x)$  is Zariski dense in  $X$ . However, if  $K_0$  is countable, then the result of Amerik and Campana leaves open the possibility that each  $K_0$ -valued point of  $X$  is also a  $K_0$ -valued point of some subvariety  $Y_i$  for some positive integer  $i$ . Hence, Conjecture 1.1 raises a deeper arithmetical question.

We are able to extend Theorem 1.2 to the action of any commutative finitely generated monoid of dominant endomorphisms of an abelian variety. For a monoid  $S$  of endomorphisms of an abelian variety  $A$ , and for any point  $x \in A$ , we let  $\mathcal{O}_S(x)$  be the  $S$ -orbit of  $x$ , i.e. the set of all  $\psi(x)$ , where  $\psi \in S$ .

**Theorem 1.3.** *Let  $K_0$  be an algebraically closed field of characteristic 0, and let  $S$  be a finitely generated, commutative monoid of dominant endomorphisms of an abelian variety  $A$  defined over  $K_0$ . Then either there exists  $x \in A(K_0)$  such that  $\mathcal{O}_S(x)$  is Zariski dense in  $A$  or there exists a non-constant rational map  $f : A \rightarrow \mathbb{P}^1$  such that  $f \circ \sigma = f$  for each  $\sigma \in S$ .*

It is reasonable to formulate an extension of Conjecture 1.1 to the setting of a monoid action of rational self-maps on an algebraic variety  $X$ . However,

there are several additional complications arising from such a generalization even in the case of the dynamics of endomorphisms of an abelian variety  $A$ , such as:

- (i) Should we impose any restriction on the monoid  $S$ ? Theorem 1.3 is valid only for finitely generated, commutative monoids, and our method of proof does not seem to extend beyond this case (at least not in the case of arbitrary endomorphisms of an abelian variety  $A$ ; if  $S$  is an arbitrary commuting monoid of dominant group endomorphisms of  $A$ , then the conclusion of Theorem 1.3 holds easily). As an aside, note that there are many examples of infinitely generated commutative monoids of endomorphisms of  $A$ ; simply take infinitely many points of  $A$  (linearly independent over  $\mathbb{Z}$ ) and then consider the monoid spanned by translations of  $A$  by these points. Once again, the difficulty in extending Theorem 1.3 lies not necessarily with this last example, but with the *mixed* case, i.e., when the endomorphisms in the monoid are compositions of translations with algebraic group endomorphisms of  $A$ .
- (ii) Assuming there is no non-constant fibration preserved by the entire monoid  $S$ , is it true that there exists some  $\sigma \in S$  and there exists  $x \in A(K_0)$  such that  $\mathcal{O}_\sigma(x)$  is Zariski dense in  $A$ ? We have examples of non-commuting monoids  $S$  generated by two group homomorphisms of  $A$  such that there is no non-constant fibration preserved by  $S$ , even though for *each*  $\sigma \in S$  there exists a non-constant fibration preserved by  $\sigma$ . On the other hand, if  $S$  is a commutative monoid of group homomorphisms of  $A$ , then it is easy to see that the above question has a positive answer.

Finally, we note that Amerik-Campana's result [AC08] was extended in [BGZ] for arbitrary monoids  $S$  acting on an algebraic variety  $X$  through dominant rational endomorphisms, i.e. if there is no non-constant rational fibration preserved by  $S$ , then there exist countably many proper subvarieties  $Y_i \subset X$  such that for each  $x \in X(K_0) \setminus \cup_i Y_i(K_0)$ , the orbit  $\mathcal{O}_S(x)$  is Zariski dense in  $X$ . Again, similar to [AC08], the result of [BGZ] leaves open the possibility that if  $K_0$  is countable, then  $X(K_0)$  may be covered by  $\cup_i Y_i(K_0)$ .

Here is the strategy for our proof. By the classical theory of abelian varieties, we know that each endomorphism  $\varphi$  of an abelian variety  $A$  is of the form  $T_y \circ \tau$  (for a translation map  $T_y$ , with  $y \in A$ ) and some (algebraic) group homomorphism  $\tau$ . Since the endomorphisms  $\varphi$  from the given monoid  $S$  commute with each other, we obtain that also the corresponding group homomorphisms  $\tau$  commute with each other. This gives us a lot of control on the action of the corresponding group homomorphisms  $\tau$ ; in particular, if all endomorphisms from  $S$  would also be group homomorphisms, then Theorem 1.3 would follow easily. Essentially, in that special case, the problem would reduce to the following dichotomy: either there exists a positive dimensional algebraic subgroup of  $A$  which is fixed by a finite index

submonoid of  $S$ , or there exists a *single* element  $\sigma$  of  $S$ , and there exists an algebraic point  $x$  of  $A$  whose  $\sigma$ -orbit is Zariski dense in  $A$  (essentially, such a point  $x$  has the property that the cyclic subgroup generated by  $x$  is Zariski dense in  $A$ ). So, if  $S$  consists only of group homomorphisms, the conclusion of Theorem 1.3 holds even in a stronger form. However, if the endomorphisms from  $S$  are not all group endomorphisms of  $A$ , then the proof is much more complicated. One can still find a necessary and sufficient condition under which there exists a non-constant rational fibration preserved by all elements in  $S$ , but that condition is very technical. Also, it is quite difficult to obtain a quantitative statement about the number of algebraic points  $x \in A$  whose orbit under the semigroup  $S$  of endomorphisms of  $A$  is Zariski dense. Essentially, our method yields that any point  $x \in A(K_0)$  which is linearly independent from the points of  $A$  defined over a certain subfield  $K_1$  of  $K_0$  have a Zariski dense orbit under  $S$  (see Remark 7.2). The difficulty in obtaining a quantitative statement in terms of the heights of the points of  $A$  lies with the fact that  $K_1$  is itself an infinitely generated field; however, we are able to show that  $K_0/K_1$  is also an infinite extension (see Claim 5.3) which guarantees the existence of *many* points  $x$  with a Zariski dense  $S$ -orbit.

We note that in our proof we use Faltings' Theorem (originally known as Mordell-Lang Conjecture; see [Fal94]) as follows. For any point  $x \in A$ , the orbit of  $x$  under  $S$  is contained in a finitely generated subgroup of  $A$  (see Fact 3.6). This yields that the orbit of  $x$  under  $S$  is not Zariski dense in  $A$  if and only if there exist finitely many translates of some proper algebraic subgroups of  $A$  which contain the orbit of  $x$ . Finally, we note that the exact same proof works to prove a variant of Theorem 1.3 with the abelian variety  $A$  replaced by a power of the torus. On the other hand, our proof does not seem to generalize to the case of semiabelian varieties due to the failure of the Poincaré Reducibility Theorem (see Fact 3.2) for semiabelian varieties which are not isogenous to split semiabelian varieties.

The plan of the paper is as follows. In Section 2 we note several easy statements regarding monoids. We continue by stating some basic facts about abelian varieties in Section 3. Then, in Section 4 and then in Section 5 we prove various reductions of the Theorem 1.3, respectively some auxiliary results needed later. In Section 6 we prove Theorem 1.2 as a way to introduce the reader to the more elaborate argument needed for the proof of Theorem 1.3 (which is completed in Section 7). While Theorem 1.2 is a special case of Theorem 1.3, we have chosen to prove them separately because we believe it is easier for the reader to read first the argument done for a cyclic monoid (Theorem 1.2), which avoids some of the technicalities appearing in the proof of Theorem 1.3.

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2. GENERAL RESULTS REGARDING MONOIDS

We need some basic facts about finitely generated, commutative monoids. First we need a definition.

**Definition 2.1.** *Let  $S$  be any finitely generated, commutative monoid. For each submonoid  $T \subseteq S$ , we denote by  $\bar{T}$  the set of all  $x \in S$  with the property that  $xT \cap T \neq \emptyset$ .*

So,  $\bar{T}$  is the set of all  $x \in S$  such that there exist  $y, z \in T$  such that  $xy = z$ . Because  $T$  is a submonoid of  $S$ , then also  $\bar{T}$  is a submonoid of  $S$ .

**Definition 2.2.** *A monoid  $S$  is called left cancellative if whenever  $xy = xz$  for  $x, y, z \in S$ , then  $y = z$ .*

We note that a monoid of dominant endomorphisms of a given algebraic variety is a left cancellative monoid. Since we will only be working with commutative monoids, we will simply call them cancellative if they satisfy Definition 2.2.

**Lemma 2.3.** *Let  $S$  be a cancellative, commutative monoid generated by the elements  $\gamma_1, \dots, \gamma_s$ , and let  $T$  be a submonoid of  $S$  such that  $\bar{T} = S$ . Then there exists a finitely generated submonoid  $T_0 \subset T$  and there exists a positive integer  $n$  such that  $\gamma_i^n \in \bar{T}_0$  for each  $i = 1, \dots, s$ .*

*Proof.* Let  $f : \mathbb{N}^s \rightarrow S$  be the homomorphism of monoids given by  $f(e_i) = \gamma_i$ , where  $e_i \in \mathbb{N}^s$  is the  $s$ -tuple consisting only of zeros with the exception of the  $i$ -th entry which equals 1. Let  $U$  be the set of all  $a \in \mathbb{N}^s$  such that  $f(a) \in T$ , and let  $H$  be the subgroup of  $\mathbb{Z}^s$  generated by  $U$ . Since  $\bar{T} = S$ , then  $H = \mathbb{Z}^s$ . Therefore there exist  $s$  linearly independent tuples in  $U$ ; call them  $u_1, \dots, u_s$ . We claim that the monoid  $T_0$  spanned by  $f(u_1), \dots, f(u_s)$  satisfies the conclusion of our Lemma.

Indeed, we first show that  $\bar{T}_0 = f(H_0 \cap \mathbb{N}^s)$  where  $H_0$  is the subgroup of  $\mathbb{Z}^s$  generated by  $u_1, \dots, u_s$ . To see this, on one hand, it is clear that  $\bar{T}_0 \subseteq f(H_0 \cap \mathbb{N}^s)$ . Now, to see the reverse inclusion, note that  $\bar{T}_0$  satisfies  $(\bar{T}_0) = \bar{T}_0$ . Indeed, if  $x_1, x_2 \in \bar{T}_0$  and  $x \in S$  such that  $xx_1 = x_2$ , we show that  $x \in \bar{T}_0$ . We have that there exist  $y_i, z_i \in T_0$  such that  $x_i y_i = z_i$  for  $i = 1, 2$ . Then we claim that  $x(y_2 z_1) = y_1 z_2$  which would indeed show that  $x \in \bar{T}_0$  because  $y_2 z_1, y_1 z_2 \in T_0$  (note that  $T_0$  is a submonoid). To see the above equality in the cancellative monoid  $S$ , it suffices to prove that  $x_1 x y_2 z_1 = x_1 y_1 z_2$ . Using that  $x_1 x = x_2$ ,  $x_2 y_2 = z_2$  and  $x_1 y_1 = z_1$ , and that  $S$  is commutative, we obtain the desired equality; hence  $(\bar{T}_0) = \bar{T}_0$  and thus  $\bar{T}_0 = f(H_0 \cap \mathbb{N}^s)$ .

Now, since  $u_1, \dots, u_s$  are linearly independent over  $\mathbb{Z}$  (as elements of  $\mathbb{Z}^s$ ), then  $H_0$  has finite index in  $\mathbb{Z}^s$ . So, there exists a positive integer  $n$  such that  $ne_i \in H_0$  for each  $i = 1, \dots, s$ , and therefore  $f(ne_i) = \gamma_i^n \in \bar{T}_0$ .  $\square$

We also need some simple results from linear algebra. The first is a consequence of the Lie-Kolchin triangularization theorem [Kol48].

**Fact 2.4.** *Let  $S_0$  be a finitely generated, commuting monoid of matrices with entries in  $\bar{\mathbb{Q}}$ . Then there exists an invertible matrix  $C$  (with entries in  $\bar{\mathbb{Q}}$ ) such that for each  $A \in S_0$ , the matrix  $C^{-1}AC$  is upper triangular.*

Fact 2.4 will be used repeatedly throughout our proof. An important consequence of it is that the eigenvalues of each matrix in a commuting monoid  $S_0$  are simply the entries on the diagonal (after a suitable change of coordinates). In particular, this has the following easy corollaries.

**Lemma 2.5.** *Let  $S_0$  be a commuting monoid of matrices with entries in  $\bar{\mathbb{Q}}$ , generated by matrices  $A_1, \dots, A_s$ . Then there exists a positive integer  $n$  such that for each matrix  $A$  contained in the submonoid of  $S_0$  generated by  $A_1^n, \dots, A_s^n$ , if  $\lambda$  is an eigenvalue of  $A$  which is also a root of unity, then  $\lambda = 1$ .*

*Proof.* The conclusion holds with  $n$  being the cardinality of the group of roots of unity contained in the number field  $L$  which is generated by all the eigenvalues of the matrices  $A_i$ .  $\square$

**Lemma 2.6.** *Let  $S_0$  be a finitely generated, commuting monoid of matrices with the property that for each matrix  $A$  in  $S_0$ , if  $\lambda$  is an eigenvalue of  $A$  which is a root of unity, then  $\lambda = 1$ . Let  $\mathcal{U}_0$  be the set of matrices in  $S_0$  with the property that the eigenspace corresponding to the eigenvalue 1 has the smallest dimension among all the matrices in  $S_0$ . Let  $U_0$  be the submonoid generated by  $\mathcal{U}_0$ . Then  $\bar{U}_0 = S_0$ .*

*Proof.* Using Fact 2.4, we can choose a basis so that each matrix in  $S$  is represented by an upper triangular matrix. Furthermore, we may assume each matrix in  $\mathcal{U}$  has the first  $r$  entries on the diagonal equal to 1, and none of the other entries on the diagonal are equal to 1 (or to a root of unity). Indeed, we know each matrix in  $\mathcal{U}$  has  $r$  entries on the diagonal equal to 1; if these entries equal to 1 would not be in the same places of the diagonal for two distinct matrices  $A$  and  $B$  in  $\mathcal{U}$ , then for some positive integers  $m$  and  $n$  we would have that  $A^m B^n$  has fewer than  $r$  entries equal to 1 on the diagonal. So, indeed the  $r$  entries equal to 1 appear in the same position on the diagonal for each matrix in  $\mathcal{U}$ ; so we may assume they are the first  $r$  entries, while the remaining  $\ell - r$  entries on the diagonal of each matrix in  $\mathcal{U}$  is not a root of unity.

Let  $A \in \mathcal{U}$ . Now, for each matrix  $B \in S$ , even if there exist entries in the positions  $i = r + 1, \dots, \ell$  on the diagonal which are equal to 1, there exists a positive integer  $n$  such that the entries on the diagonal of  $A^n B$  in the positions  $i = r + 1, \dots, \ell$  are not equal to 1. This completes our proof.  $\square$

### 3. ABELIAN VARIETIES

First we recall several results regarding abelian varieties (see [Mil] or [Mum70] for more details). The setup will be as follows:  $A$  is an abelian variety defined over a field  $K$  of characteristic 0; since one needs only finitely

many parameters in order to define  $A$ , then we may assume  $K$  is a finitely generated extension of  $\mathbb{Q}$ . We let  $\overline{K}$  be a fixed algebraic closure of  $K$ . At the expense of replacing  $K$  by a finite extension we may assume that all algebraic group endomorphisms of  $A$  are defined over  $K$ ; we denote by  $\text{End}(A)$  the ring of all these endomorphisms. Since the torsion subgroup  $C_{\text{tor}}$  of any algebraic subgroup  $C \subseteq A$  is Zariski dense in  $C$ , we conclude that any algebraic subgroup of  $A$  is defined over  $K(A_{\text{tor}})$ . Frequently we will use the following facts.

First, as a matter of notation, always the connected component of an algebraic subgroup  $B$  of  $A$  is denoted by  $B^0$  (we recall that  $B^0$  is the connected algebraic subgroup of  $B$  of maximal dimension).

**Fact 3.1.** *Let  $B$  and  $C$  be algebraic subgroups of the abelian variety  $A$ . Then  $(B + C)^0 = (B^0 + C^0)$ .*

*Proof.* The algebraic group  $B^0 + C^0$  is the image of the connected group  $B^0 \times C^0$  under the sum map and is therefore connected. As  $B^0 \times C^0$  has finite index in  $B \times C$ , its image under the sum map has finite index in  $B + C$ . Hence,  $B^0 + C^0 = (B + C)^0$ .  $\square$

The following result is proven in [Mil, Proposition 10.1].

**Fact 3.2** (Poincaré's Reducibility Theorem). *If  $B \subseteq A$  is an abelian subvariety of  $A$ , then there exists an abelian subvariety  $C \subseteq A$  such that  $A = B + C$  and  $B \cap C$  is finite; in particular  $A/B$  and  $C$  are isogenous.*

Poincaré's Reducibility Theorem yields that any abelian variety is isogenous with a direct product of finitely many simple abelian varieties, i.e.  $A \xrightarrow{\sim} A_0 := \prod_{i=1}^r C_i^{k_i}$ , where each  $C_i$  is simple. Then  $\text{End}(A) \xrightarrow{\sim} \text{End}(A_0)$  (see also [Mil, Section 1.10]), and moreover  $\text{End}(A_0) \xrightarrow{\sim} \prod_{i=1}^r M_{k_i}(R_i)$ , where  $M_{k_i}(R_i)$  is the ring of all  $k_i$ -by- $k_i$  matrices with entries in the ring  $R_i := \text{End}(C_i)$ . For any simple abelian variety  $C$ , the ring  $R := \text{End}(C)$  is a finite integral extension of  $\mathbb{Z}$ . Therefore we have the following fact.

**Fact 3.3.** *Let  $A$  be an abelian variety defined over a field of characteristic 0. For each algebraic group endomorphism  $\phi : A \rightarrow A$  there exists a minimal monic polynomial  $f \in \mathbb{Z}[t]$  of degree at most  $2 \dim(A)$  such that  $f(\phi) = 0$ .*

The following result is proven in [Mil, Corollary 1.2].

**Fact 3.4** (Rigidity Theorem). *Each endomorphism  $\psi : A \rightarrow A$  is of the form  $T_y \circ \phi$  for some  $y \in A$ , where  $T_y : A \rightarrow A$  is the translation map  $x \mapsto x + y$ , and  $\phi \in \text{End}(A)$  is an algebraic group endomorphism. In particular, if  $\psi$  is dominant, then  $\phi : A \rightarrow A$  is an isogeny. Furthermore, the pair  $(T_y, \phi)$  is uniquely determined by  $\psi$ .*

As a simple consequence of Fact 3.4, we obtain.

**Lemma 3.5.** *Let  $\psi_1, \psi_2 : A \rightarrow A$  be endomorphisms of the form  $\psi_i := T_{y_i} \circ \varphi_i$  (for  $i = 1, 2$ ) where  $\varphi_i : A \rightarrow A$  are group endomorphisms. If  $\psi_1 \circ \psi_2 = \psi_2 \circ \psi_1$ , then  $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$ .*

The following result is an immediate application of the structure theorem for the ring of group endomorphisms of an abelian variety.

**Fact 3.6.** *Let  $S$  be a finitely generated commutative monoid of endomorphisms of an abelian variety  $A$  as an algebraic variety. Then for each point  $x \in A$ , there exists a finitely generated subgroup  $\Gamma \subset A$  containing  $\mathcal{O}_S(x)$ .*

*Proof.* Let  $\{\gamma_1, \dots, \gamma_s\}$  be a set of generators for  $S$ . For each  $i = 1, \dots, s$ , we let  $\gamma_i := T_{y_i} \circ \tau_i$  for some translations  $T_{y_i}$  (where  $y_i \in A$ ) and some group endomorphisms  $\tau_i$ . Let  $d := \dim(A)$ . Then, by Fact 3.3, for each  $i = 1, \dots, s$ , there exist integers  $c_{i,j}$  such that

$$\tau_i^{2d} + c_{i,2d-1}\tau_i^{2d-1} + \dots + c_{i,1}\tau_i + c_{i,0} \cdot \text{id} = 0,$$

where  $\text{id}$  always represents the identity map. Then  $\mathcal{O}_S(x)$  is contained in the subgroup  $\Gamma \subset A$  generated by  $\gamma(x), \gamma(y_1), \dots, \gamma(y_s)$ , where  $\gamma$  varies among the finitely many elements of  $S$  of the form  $\gamma := \gamma_1^{m_1} \circ \dots \circ \gamma_s^{m_s}$ , with  $0 \leq m_i < 2d$ , for each  $i = 1, \dots, s$ .  $\square$

The next result is a relatively simple application of Fact 3.2.

**Lemma 3.7.** *Let  $B \subseteq A$  be an algebraic subgroup of the abelian variety  $A$ . Then  $B \neq A$  if and only there exists a nonzero algebraic group endomorphism  $\psi : A \rightarrow A$  such that  $\psi(B) = \{0\}$ .*

*Proof.* Clearly, if  $B = A$ , then there exists no nonzero endomorphism  $\psi$  of  $A$  such that  $\psi(B) = \{0\}$ . Now, assume  $B \neq A$ . We note that it suffices to prove the existence of  $\psi \in \text{End}(A)$  such that  $B^0 \subseteq \ker(\psi)$ , where  $B^0$  is the connected component of  $B$  containing 0 for if  $B^0 \subseteq \ker(\psi)$  and  $N := [B : B^0]$  is the index of  $B^0$  in  $B$ , then  $B \subseteq \ker(\phi)$  where  $\phi = [N] \cdot \psi$ . So, from now on assume  $B$  is an abelian subvariety of  $A$ . We let  $\pi : A \rightarrow A/B$  be the canonical quotient map. By Fact 3.2, we obtain that there exists an abelian subvariety  $C \subseteq A$  and an isogeny  $\tau : A/B \rightarrow C$ . So, letting  $\iota : C \rightarrow A$  be the canonical injection map, we get that  $\psi := \iota \circ \tau \circ \pi : A \rightarrow A$  is an endomorphism with the property that  $\psi(B) = \{0\}$ . We claim that  $\psi \neq 0$ . Indeed, by construction, the image of  $\psi$  is  $C$  which is a positive dimensional variety (since  $B \neq A$ ).  $\square$

The following result is the famous consequence of Mordell-Lang Conjecture proven by Faltings [Fal94].

**Fact 3.8** (Faltings' theorem; Mordell-Lang Conjecture). *Let  $V \subset A$  be an irreducible subvariety with the property that there exists a finitely generated subgroup  $\Gamma \subseteq A(\overline{K})$  such that  $V(\overline{K}) \cap \Gamma$  is Zariski dense in  $V$ . Then  $V$  is a coset of an abelian subvariety of  $A$ .*

We will also employ the following easy result.

**Lemma 3.9.** *Let  $A$  be an abelian variety. If  $x \in A$  is a point generating a cyclic group which is Zariski dense in  $A$ , then for each positive integer  $\ell$ , the cyclic group generated by  $\ell x$  is Zariski dense in  $A$ .*



*Proof.* Let  $H$  be the Zariski closure of the cyclic group generated by  $\ell x$ ; then  $H$  is an algebraic subgroup of  $A$ . Furthermore, because the cyclic group generated by  $x$  is Zariski dense in  $A$ , then

$$A = \bigcup_{i=0}^{\ell-1} (ix + H).$$

Since  $A$  is connected, we conclude that  $H = A$ , as desired.  $\square$

Finally, for any simple abelian variety  $A$  defined over a field  $K$  of characteristic 0, the action of  $\text{Gal}(\overline{K}/K)$  on  $A_{\text{tor}}$  yields the following result.

**Fact 3.10.** *The group  $\text{Gal}(K(A_{\text{tor}})/K)$  embeds into  $\text{GL}_{2d}(\hat{\mathbb{Z}})$ , where  $d = \dim(A)$  and  $\hat{\mathbb{Z}}$  is the ring of finite adèles.*

#### 4. REDUCTIONS

Next we proceed with several preliminary results used later in the proof of Theorem 1.2. The following result was proven in the case of a cyclic group  $S$  of automorphisms in [BRS10]; we thank Jason Bell for pointing out how to extend the result from [BRS10] to our setting.

**Lemma 4.1.** *It suffices to prove Theorem 1.3 for a submonoid of  $S$  spanned by iterates of each of the generators of  $S$ .*

*Proof.* We consider a finite generating set  $\mathcal{U} := \{\gamma_1, \dots, \gamma_s\}$  for the monoid  $S$ . We assume  $S$  does not fix a non-constant fibration of  $A$  (otherwise Theorem 1.3 holds). We let  $S'$  be the submonoid of  $S$  spanned by the endomorphisms in  $\mathcal{U}' := \{\gamma_1^{m_1}, \dots, \gamma_s^{m_s}\}$  (for some positive integers  $m_i$ ). We assume Theorem 1.3 holds for  $S'$ . If also  $S'$  does not fix a non-constant fibration, then there exists  $x \in A(K_0)$  such that the  $S'$ -orbit of  $x$  is Zariski dense in  $A$ ; hence also  $\mathcal{O}_S(x)$  is Zariski dense in  $A$ . So, it remains to prove that  $S'$  cannot fix a non-constant fibration if  $S$  does not fix a non-constant fibration.

We assume  $f \circ \gamma_i^{m_i} = f$  for some non-constant map  $f : A \rightarrow \mathbb{P}^1$  (for each  $i$ ). Let  $S_{\text{rep}}$  be a finite set of representatives for the cosets of  $S'$  in  $S$  (note that  $S/S'$  is a finite group since it is a finite monoid in which each element is invertible); without loss of generality we assume the identity is part of  $S_{\text{rep}}$ . Let  $m := |S_{\text{rep}}|$ , and let  $S_{\text{rep}} := \{\sigma_1, \dots, \sigma_m\}$ . Let  $s_1, \dots, s_m$  be the elementary symmetric functions  $s_i : (\mathbb{P}^1)^m \rightarrow \mathbb{P}^1$  and let  $g_i := s_i(f \circ \sigma_1, \dots, f \circ \sigma_m)$  (for  $i = 1, \dots, m$ ). Clearly,  $\gamma_i$  preserves each fibration  $g_j$ ; hence if one  $g_j$  is non-constant, then we are done. If each  $g_j$  is a constant, then we obtain a contradiction because  $f = f \circ \text{id}$  would be a root of the polynomial (with constant coefficients)

$$X^m - g_1 X^{m-1} + g_2 X^{m-2} + \dots + (-1)^m g_m = 0.$$

This completes the proof of Lemma 4.1.  $\square$

**Lemma 4.2.** *With the notation as in Theorem 1.3, let  $T$  be a submonoid of  $S$  such that  $\bar{T} = S$ . If the conclusion of Theorem 1.3 holds for  $T$ , then it holds for  $S$ .*

*Proof.* We assume that there exists no non-constant fibration preserved by all elements of  $S$ , and it suffices to prove that there is also no non-constant fibration preserved by the elements of  $T$ . Assume, by contradiction that there exists  $f : A \rightarrow \mathbb{P}^1$  such that  $f \circ \psi = f$  for each  $\psi \in T$ . Now, let  $\sigma \in S$ ; then there exist  $\psi_1, \psi_2 \in T$  such that  $\gamma\psi_1 = \psi_2$ . So, using also that  $S$  is commutative, we get

$$f \circ \gamma = f \circ \psi_1 \circ \gamma = f \circ \psi_2 = f.$$

Hence  $f$  must be constant, as desired.  $\square$

Combining Lemmas 4.1 and 4.2 we obtain the following reduction of Theorem 1.3.

**Lemma 4.3.** *With the notation from Theorem 1.3, assume the monoid  $S$  is generated by the maps  $\gamma_1, \dots, \gamma_s$ . Then it suffices to prove the conclusion of Theorem 1.3 for a finitely generated submonoid  $T$  of  $S$  with the property that  $\gamma_i^n \in T$  for each  $i = 1, \dots, s$ , for some positive integer  $n$ .*

## 5. AUXILIARY RESULTS

In this Section we present several technical results useful for our proof of Theorems 1.2 and 1.3.

**Lemma 5.1.** *Let  $K_0$  be an algebraically closed field of characteristic 0. Let  $\psi_1, \dots, \psi_s : B \rightarrow C$  be algebraic group morphisms of abelian varieties, and let  $y_1, \dots, y_s \in C(K_0)$ . Then there exists  $x \in B(K_0)$  such that for each  $i = 1, \dots, s$ , the Zariski closure of the subgroup generated by  $\psi_i(x) + y_i$  is the algebraic group generated by  $\psi_i(B)$  and  $y_i$ .*

*Proof.* Let  $K$  be a finitely generated subfield of  $K_0$  such that  $B, C, \psi_1, \dots, \psi_s$  are defined over  $K$ , and moreover, each  $y_i \in C(K)$ . Without loss of generality, we may assume  $K_0$  is a fixed algebraic closure  $\bar{K}$  of  $K$  (a priori,  $K_0$  may be a proper extension of  $\bar{K}$ , and thus showing the conclusion with  $\bar{K}$  in place of  $K_0$  suffices).

We let  $B = A_1 + \dots + A_m$  written as a sum of simple abelian varieties.

Let  $i = 1, \dots, s$ ; then  $\psi_i(B)$  equals the sum  $\psi_i(A_1) + \dots + \psi_i(A_m)$  (each algebraic group being either simple or trivial). We find an algebraic point  $x_i \in \psi_i(B)$  such that the Zariski closure of the cyclic group generated by  $x_i + y_i$  is the algebraic group generated by  $\psi_i(B)$  and  $y_i$ ; moreover we ensure that

$$\bigcap_{i=1}^s \psi_i^{-1}(\{x_i\})$$

is nonempty (in  $B$ ). We find  $x_i$  as a sum  $x_{i,1} + \dots + x_{i,m}$ , where each  $x_{i,j} \in \psi_i(A_j)$ . If for some  $j$  we have  $\psi_i(A_j) = \{0\}$ , we simply pick  $x_{i,j} = 0$ . Then

our goal is to construct the sequence  $\{x_{i,j}\}$  such that for each  $j = 1, \dots, m$ , the set

$$(5.1.1) \quad \bigcap_{i=1}^s (\psi_i)|_{A_j}^{-1}(\{x_{i,j}\})$$

is nonempty (in  $A_j$ ). Obviously when  $\psi_i(A_j) = \{0\}$ , we might as well disregard the set

$$(\psi_i)|_{A_j}^{-1}(\{x_{i,j}\}) = (\psi_i)|_{A_j}^{-1}(\{0\}) = A_j$$

from the above intersection. Let now  $j = 1, \dots, m$  such that  $\psi_i(A_j)$  is nontrivial. We will show that there exists  $x_{i,j} \in \psi_i(A_j)$  such that for any positive integer  $n$  we have

$$(5.1.2) \quad nx_{i,j} \notin (\psi_i(A_j))(K(C_{\text{tor}}, x_{i,1}, \dots, x_{i,j-1})).$$

*Claim 5.2.* If the above condition (5.1.2) holds for each  $j = 1, \dots, m$  such that  $\psi_i(A_j) \neq \{0\}$ , then the Zariski closure of the cyclic group generated by  $x_i + y_i$  is the algebraic subgroup  $B_i$  generated by  $\psi_i(B)$  and  $y_i$ .

*Proof of Claim 5.2.* Indeed, assume there exists some algebraic subgroup  $D \subseteq C$  (not necessarily connected) such that  $x_i + y_i \in D(\bar{K})$ . Let  $j \leq m$  be the largest integer such that  $x_{i,j} \neq 0$ ; then we have

$$x_{i,j} \in ((-y_i - x_{i,1} - \dots - x_{i,j-1}) + D) \cap \psi_i(A_j).$$

Assume first that  $\psi_i(A_j) \cap D$  is a proper algebraic subgroup of  $\psi_i(A_j)$ . Since  $\psi_i(A_j)$  is a simple abelian variety, then  $D \cap \psi_i(A_j)$  is a 0-dimensional algebraic subgroup of  $C$ ; hence there exists a nonzero integer  $n$  such that  $n \cdot (D \cap \psi_i(A_j)) = \{0\}$ . Then  $nx_{i,j}$  is the only (geometric) point of the subvariety  $n \cdot ((-y_i - x_{i,1} - \dots - x_{i,j-1}) + D) \cap \psi_i(A_j)$  which is thus rational over  $K(C_{\text{tor}}, x_{i,1}, \dots, x_{i,j-1})$ . But by our construction,

$$nx_{i,j} \notin \psi_i(A_j)(K(C_{\text{tor}}, x_{i,1}, \dots, x_{i,j-1}))$$

which is a contradiction. Therefore  $\psi_i(A_j) \subseteq D$  if  $j$  is the largest index  $\leq m$  such that  $x_{i,j} \neq 0$  (or equivalently, such that  $\psi_i(A_j) \neq \{0\}$ ). So,  $x_i + y_i \in D$  yields now  $x'_i + y_i \in D$ , where  $x'_i := x_{i,1} + \dots + x_{i,j-1}$ . Repeating the exact same argument as above for the next positive integer  $j_1 < j$  for which  $\psi_i(A_{j_1}) \neq \{0\}$ , and then arguing inductively we obtain that each  $\psi_i(A_j)$  is contained in  $D$ , and therefore  $\psi_i(B) \subseteq D$ . But then  $x_i \in \psi_i(B) \subseteq D$  and so,  $y_i \in D$  as well, which yields that the Zariski closure of the group generated by  $x_i + y_i$  is the algebraic subgroup  $B_i$  of  $C$  generated by  $\psi_i(B)$  and  $y_i$ .  $\square$

We just have to show that we can choose  $x_{i,j}$  both satisfying (5.1.2) and also such that the above intersection (5.1.1) is nonempty. So, the problem reduces to the following:  $L$  is a finitely generated field of characteristic 0,  $\varphi_1, \dots, \varphi_\ell$  are algebraic group homomorphisms (of finite kernel) between a simple abelian variety  $A$  and another abelian variety  $C$  all defined over  $L$ ,

and we want to find  $z \in A(\overline{K})$  such that for each positive integer  $n$ , and for each  $i = 1, \dots, \ell$ , we have

$$(5.2.1) \quad n\varphi_i(z) \notin \varphi_i(A)(L(C_{\text{tor}})).$$

Indeed, with the above notation,  $A := A_j$ ,  $L$  is the extension of  $K$  generated by  $x_{i,k}$  (for  $i = 1, \dots, s$ , and  $k = 1, \dots, j-1$ ), and the  $\varphi_i$ 's are the homomorphisms  $\psi_i$ 's (restricted on  $A = A_j$ ) for which  $\psi_i(A_j)$  is nontrivial.

Let  $d$  be the maximum of the degree of the isogenies  $\varphi'_i : A \rightarrow \varphi_i(A) \subset C$ . In particular, this means that for each  $w \in C(\overline{K})$ , and for each  $z \in A(\overline{K})$  such that  $\phi_i(z) = w$  we have

$$(5.2.2) \quad [L(z) : L] \leq d \cdot [L(w) : L].$$

For any subfield  $M \subseteq \overline{K}$ , we let  $M^{(d)}$  be the compositum of all extensions of  $M$  of degree at most equal to  $d$ .

*Claim 5.3.* Let  $L$  be a finitely generated field of characteristic 0, let  $C$  be an abelian variety defined over  $L$ , let  $L_{\text{tor}} := L(C_{\text{tor}})$ , and let  $d$  be a positive integer. Then there exists a normal extension of  $L_{\text{tor}}^{(d)}$  whose Galois group is not abelian.

*Proof of Claim 5.3.* As proven in [Tho13], the field  $L_{\text{tor}}$  is Hilbertian (note that  $L$  itself is Hilbertian since it is a finitely generated field of characteristic 0). For each positive integer  $n$ , according to [FJ08, Corollary 16.2.7 (a)], there exists a Galois extension  $L_n$  of  $L_{\text{tor}}$  such that  $\text{Gal}(L_n/L_{\text{tor}}) \xrightarrow{\sim} S_n$  (the symmetric group on  $n$  letters). Assume there exists an abelian extension  $L_0$  of  $L_{\text{tor}}^{(d)}$  containing  $L_n$ . If  $n > \max\{5, d!\}$ , we will derive a contradiction from our assumption.

We let  $G_1 := \text{Gal}(L_0/L_{\text{tor}}^{(d)})$  and  $G_0 := \text{Gal}(L_0/L_{\text{tor}})$ . Then there exists a surjective group homomorphism  $f : G_0 \rightarrow S_n$ . Because  $G_1$  is a normal subgroup of  $G_0$  (and  $f$  is a surjective group homomorphism), we get that  $f(G_1)$  is a normal subgroup of  $S_n$ , and moreover, it is abelian since  $G_1$  is abelian. Because  $n \geq 5$ , the only proper normal subgroup of  $S_n$  is  $A_n$ , which is not abelian. Hence,  $G_1 \subseteq \ker(f)$ , and therefore,  $f$  induces a surjective group homomorphism (also denoted by  $f$ ) from  $G_0/G_1$  to  $S_n$ ; more precisely, we have a surjective group homomorphism  $f : G^{(d)} \rightarrow S_n$ , where  $G^{(d)} := \text{Gal}(L_{\text{tor}}^{(d)}/L_{\text{tor}})$ . But  $G^{(d)}$  is a group of exponent  $d!$ , and so,  $S_n = f(G^{(d)})$  is also a group of exponent  $d!$ , which is a contradiction with the fact that  $n > d!$ .  $\square$

Claim 5.3 yields that there exists a point  $z \in A(\overline{K})$  which is not defined over an abelian extension of  $L(C_{\text{tor}})^{(d)}$ ; i.e.,  $nz \notin A(L(C_{\text{tor}})^{(d)})$  for all positive integers  $n$ . Hence,  $n\phi_i(z) \notin \phi_i(A)(L(C_{\text{tor}}))$  (see (5.2.2)), which concludes the proof of Lemma 5.1.  $\square$

The next result will be used (only) in the proof of Theorem 1.2.

**Lemma 5.4.** *It suffices to prove Theorem 1.2 for a conjugate  $\gamma^{-1} \circ \sigma \circ \gamma$  of the automorphism  $\sigma$  under some automorphism  $\gamma$ .*

*Proof.* Since  $\mathcal{O}_{\gamma^{-1}\sigma\gamma}(\gamma^{-1}(x)) = \gamma^{-1}(\mathcal{O}_\sigma(x))$ , we obtain that there exists a Zariski dense orbit of a point under the action of  $\sigma$  if and only if there exists a Zariski dense orbit of a point under the action of  $\gamma^{-1} \circ \sigma \circ \gamma$ . Also,  $\sigma$  preserves a non-constant fibration  $f : A \rightarrow \mathbb{P}^1$  if and only if  $\gamma^{-1}\sigma\gamma$  preserves the non-constant fibration  $f \circ \gamma$ .  $\square$

The conclusion of the next result shares the same philosophy as the conclusion of Lemma 5.1: one can find an algebraic point in an abelian variety so that it is *sufficiently* generic with respect to any given set of finitely many points.

**Lemma 5.5.** *Let  $K_0$  be an algebraically closed field of characteristic 0, let  $\Gamma \subseteq A(K_0)$  be a subgroup such that  $\text{End}(A) \otimes_{\mathbb{Z}} \Gamma$  is a finitely generated  $\text{End}(A)$ -module, and let  $B \subseteq A$  be a nontrivial abelian subvariety. Then there exists  $x \in B(K_0)$  such that for each  $\psi \in \text{End}(A)$  satisfying  $\psi(x) \in \Gamma$ , we must have that  $B \subseteq \ker(\psi)$ .*

*Proof.* Each abelian variety is isogenous to a product of simple abelian varieties; so let  $\pi : A \rightarrow A_0 := \prod_{i=1}^r C_i^{k_i}$  be such an isogeny, where each  $C_i$  is a simple abelian variety defined over  $K_0$ . Then it suffices to find an algebraic point  $y \in C := \pi(B)$  such that for each  $\phi \in \text{End}(A_0)$ , if  $\phi(y) \in \pi(\Gamma)$ , then  $C \subseteq \ker(\phi)$ .

At the expense of replacing  $C$  with an isogenous abelian variety, we may assume that  $C := \prod_{i=1}^r C_i^{m_i}$  with  $0 \leq m_i \leq k_i$ . Each endomorphism  $\phi \in \text{End}(A_0)$  is of the form  $(J_1, \dots, J_r)$  where each  $J_i \in M_{k_i}(R_i)$ , where  $M_{k_i}(R_i)$  is the  $k_i$ -by- $k_i$  matrices with entries in the ring  $R_i$  of endomorphisms of  $C_i$  (note that  $R_i$  is a finite integral extension of  $\mathbb{Z}$ ). We let  $\Gamma_i$  be the finitely generated  $R_i$ -module generated by the projections of  $\pi(\Gamma)$  on each of the  $k_i$  copies of  $C_i$  contained in the presentation of  $A_0 = \prod_{i=1}^r C_i^{k_i}$ . We let  $y_{i,1}, \dots, y_{i,\ell_i}$  be generators of the free part of  $\Gamma_i$  as an  $R_i$ -module. Without loss of generality, we may assume the points  $y_{i,1}, \dots, y_{i,\ell_i}$  are linearly independent over  $R_i$ .

Then it suffices to pick  $x \in C$  of the form

$$(x_{1,1}, \dots, x_{1,m_1}, x_{2,1}, \dots, x_{2,m_2}, \dots, x_{r,1}, \dots, x_{r,m_r}),$$

where each  $x_{i,j} \in C_i$  such that for each  $i$ , the points  $x_{i,1}, \dots, x_{i,m_i}, y_{i,1}, \dots, y_{i,\ell_i}$  are linearly independent over  $R_i$ . The existence of such points  $x_{i,j}$  follows from the fact that each  $C_i(\overline{K}) \otimes_{R_i} \text{Frac}(R_i)$  has the structure of a  $\text{Frac}(R_i)$ -vector space of infinite dimension.  $\square$

The next result is an application of Fact 3.8.

**Lemma 5.6.** *Let  $K_0$  be an algebraically closed field of characteristic 0, let  $y_1, \dots, y_r \in A(K_0)$ , and let  $P_1, \dots, P_r \in \mathbb{Q}[z]$  such that  $P_i(n) \in \mathbb{Z}$  for each  $n \geq 1$  and for each  $i = 1, \dots, r$ , while  $\deg(P_r) > \dots > \deg(P_1) > 0$ . For*

an infinite subset  $S \subseteq \mathbb{N}$ , let  $V := V(S; P_1, \dots, P_r; y_1, \dots, y_r)$  be the Zariski closure of the set

$$\{P_1(n)y_1 + \dots + P_r(n)y_r : n \in S\}.$$

Then there exist nonzero integers  $\ell_1, \dots, \ell_r$  such that  $V$  contains a coset of the subgroup  $\Gamma$  generated by  $\ell_1 y_1, \dots, \ell_r y_r$ .

*Proof.* Let  $\Gamma_0$  be the subgroup of  $A$  generated by  $y_1, \dots, y_r$ . Because  $V(K_0) \cap \Gamma_0$  is Zariski dense in  $V$ , then by Fact 3.8 we obtain that  $V$  is a finite union of cosets of algebraic subgroups of  $A$ . So, at the expense of replacing  $S$  by an infinite subset, we may assume  $V = z + C$ , for some  $z \in A(K_0)$  and some irreducible algebraic subgroup  $C$  of  $A$ . This is equivalent with the existence of an endomorphism  $\psi : A \rightarrow A$  such that  $\ker(\psi)^0 = C$  (the construction of  $\psi$  is identical with the one given in the proof of Lemma 3.7); hence  $\psi$  is constant on the set  $\{P_1(n)y_1 + \dots + P_r(n)y_r\}_{n \in S}$ . We will show there exist nonzero integers  $\ell_i$  such that  $\ell_i y_i \in \ker(\psi)$  for each  $i = 1, \dots, r$ ; since  $\ker(\psi)^0 = C$ , then we obtain the desired conclusion.

We proceed by induction on  $r$ . The case  $r = 1$  is obvious since then  $\{P_1(n)\}_{n \in S}$  takes infinitely many distinct integer values (note that  $\deg(P_1) \geq 1$ ), and so, if  $\psi$  is constant on the set  $\{P_1(n)y_1\}_{n \in S}$ , then  $\psi(\ell y_1) = 0$  for some nonzero  $\ell := P_1(n) - P_1(n_0)$  with distinct  $n_0, n \in S$ . Next we assume the statement holds for all  $r < s$  (where  $s \geq 2$ ), and we prove it for  $r = s$ .

Let  $n_0 \in S$ . At the expense of replacing each  $P_i(n)$  by  $P_i(n) - P_i(n_0)$ , we may assume from now on that the set  $\{P_1(n)y_1 + \dots + P_s(n)y_s\}_{n \in S}$  lies in the kernel of  $\psi$ . Let  $n_1 \in S$  such that  $P_1(n_1) \neq 0$  (note that  $\deg(P_1) \geq 1$ ), and for each  $i = 2, \dots, s$  we let  $Q_i(z) := P_1(n_1) \cdot P_i(n) - P_1(n) \cdot P_i(n_1)$ . Then the set  $\{\sum_{i=2}^s Q_i(n)y_i\}_{n \in S}$  is in the kernel of  $\psi$ . Because  $\deg(Q_i) = \deg(P_i)$  for each  $i = 2, \dots, s$ , we can use the induction hypothesis and conclude that there exist nonzero integers  $\ell_2, \dots, \ell_s$  such that  $\ell_i y_i \in \ker(\psi)$  for each  $i$ . Since  $\psi(P_1(n_1)y_1 + \dots + P_s(n_1)y_s) = 0$  and  $P_1(n_1) \neq 0$ , then also  $(P_1(n_1) \cdot \prod_{i=2}^s \ell_i) y_1 \in \ker(\psi)$ . This concludes our proof.  $\square$

Lemma 5.6 has the following important consequence for us.

**Lemma 5.7.** *Let  $K_0$  be an algebraically closed field of characteristic 0, let  $A$  be an abelian variety defined over  $K_0$ , let  $\tau \in \text{End}(A)$  with the property that there exists a positive integer  $r$  such that  $(\tau - \text{id})^r = 0$ , let  $y \in A(K_0)$ , let  $\sigma : A \rightarrow A$  be an endomorphism as algebraic varieties such that  $\sigma = T_y \circ \tau$ , and let  $x \in A(K_0)$ . Let  $\gamma \in \text{End}(A)$  with the property that there exists an infinite set  $S$  of positive integers such that  $\gamma$  is constant on the set  $\{\sigma^n(x) : n \in S\}$ . Then there exists a positive integer  $\ell$  such that  $\ell \cdot (\beta(x) + y) \in \ker(\gamma)$ , where  $\beta := \tau - \text{id}$ .*

*Proof.* We compute  $\sigma^n(x)$  for any  $n \in \mathbb{N}$ ; first of all, we have

$$(5.7.1) \quad \sigma^n(x) = \tau^n(x) + \sum_{i=0}^{n-1} \tau^i(y).$$

Then (since  $\beta = \tau - \text{id}$  and also) noting that  $\beta^r = 0$  we have

$$(5.7.2) \quad \sigma^n(x)$$

$$(5.7.3) \quad = \sum_{i=0}^n \binom{n}{i} \beta^i(x) + \sum_{i=0}^{n-1} \tau^i(y)$$

$$(5.7.4) \quad = \sum_{i=0}^{r-1} \binom{n}{i} \beta^i(x) + \sum_{i=0}^{n-1} \sum_{j=0}^i \binom{i}{j} \beta^j(y)$$

$$(5.7.5) \quad = \sum_{j=0}^{r-1} \binom{n}{j} \beta^j(x) + \sum_{j=0}^{r-1} \left( \sum_{i=j}^{n-1} \binom{i}{j} \right) \beta^j(y)$$

$$(5.7.6) \quad = \sum_{j=0}^{r-1} \binom{n}{j} \beta^j(x) + \sum_{j=0}^{r-1} \binom{n}{j+1} \beta^j(y)$$

$$(5.7.7) \quad = x + \sum_{j=1}^r \binom{n}{j} \beta^j(x) + \sum_{j=1}^r \binom{n}{j} \beta^{j-1}(y)$$

$$(5.7.8) \quad = x + \sum_{j=1}^r \binom{n}{j} \beta^{j-1}(\beta(x) + y).$$

Since  $\gamma$  is constant on the set  $\{\sigma^n(x) : n \in S\}$ , then letting  $n_1 \in S$  we have that for each  $n \in S$ ,

$$(5.7.9) \quad \sum_{j=1}^r \left( \binom{n}{j} - \binom{n_1}{j} \right) \beta^{j-1}(\beta(x) + y) \in \ker(\gamma).$$

Using Lemma 5.6 and (5.7.9), we obtain the desired conclusion.  $\square$

Then the following result is an immediate consequence of Lemma 5.7 and of Lemma 3.9.

**Corollary 5.8.** *With the notation as in Lemma 5.7, if the cyclic group generated by  $\beta(x) + y$  is Zariski dense in  $A$ , then  $\gamma = 0$ . Moreover, the set  $\{\sigma^n(x) : n \in S\}$  is Zariski dense in  $A$ .*

*Proof.* Indeed, Lemmas 3.9 and 5.7 yield that any group homomorphism  $\gamma$  which is constant on the set  $U := \{\sigma^n(x) : n \in S\}$  must be trivial.

Now, for the ‘moreover’ part of Corollary 5.8, Fact 3.6 yields that  $U$  (along with  $\mathcal{O}_\sigma(x)$ ) is contained in a finitely generated subgroup of  $A$ , and so, Fact 3.8 yields that the Zariski closure of  $U$  is a finite union of cosets of algebraic subgroups of  $A$ . Pick such a coset  $w + H$  which contains infinitely many  $\sigma^n(x)$ . Then another application of Lemma 5.7 (coupled with Lemmas 3.7 and 3.9) yields that  $H = A$ , thus completing our proof that  $U$  is Zariski dense in  $A$ .  $\square$

## 6. THE CYCLIC CASE

Now we are ready to prove Theorem 1.3 for cyclic monoids.

*Proof of Theorem 1.2.* Let  $K$  be a finitely generated subfield of  $K_0$  such that both  $A$  and  $\sigma$  are defined over  $K$ . Let  $\overline{K}$  be the algebraic closure of  $K$  inside  $K_0$ ; clearly, it suffices to prove Theorem 1.2 with  $K_0$  replaced by  $\overline{K}$ .

By Fact 3.4, there exists an isogeny  $\tau : A \rightarrow A$ , and there exists  $y \in A(K)$ , such that  $\sigma(x) = \tau(x) + y$  for all  $x \in A$ . At the expense of replacing  $\sigma$  by an iterate  $\sigma^n$  (and in particular, replacing  $\tau$  by  $\tau^n$ ; see also (5.7.1)), we may assume  $\dim \ker(\tau^m - \text{id}) = \dim(\ker(\tau - \text{id}))$  for all  $m \in \mathbb{N}$  (see Lemma 4.1 which shows that it is sufficient to prove Theorem 1.2 for an iterate of  $\sigma$ ). In other words, we may assume that the only root of unity, if any, which is a root of the minimal polynomial  $f$  (with coefficients in  $\mathbb{Z}$ ) of  $\tau \in \text{End}(A)$  is equal to 1.

Let  $r$  be the order of vanishing at 1 of  $f$ , and let  $f_1 \in \mathbb{Z}[t]$  such that  $f(t) = f_1(t) \cdot (t-1)^r$ . Then  $f_1$  is also a monic polynomial, and if  $r = 0$ , then  $f_1 = f$ . Let  $A_1 := (\tau - \text{id})^r(A)$  and let  $A_2 := f_1(\tau)(A)$ , where  $f_1(\tau) \in \text{End}(A)$  and  $\text{id}$  is the identity map on  $A$ . If  $r = 0$ , then  $A_2 = 0$  and therefore  $A_1 = A$ . By definition, both  $A_1$  and  $A_2$  are connected algebraic subgroups of  $A$ , hence they are both abelian subvarieties of  $A$ . Furthermore, by definition, the restriction of  $\tau|_{A_1} \in \text{End}(A_1)$  has minimal polynomial equal to  $f_1$  whose roots are not roots of unity. On the other hand,  $(\tau - \text{id})^r|_{A_2} = 0$ .

**Lemma 6.1.** *With the above notation,  $A = A_1 + A_2$  and  $A_1 \cap A_2$  is finite.*

*Proof of Lemma 6.1.* By the definition of  $r$  and of  $f_1$ , we know that the polynomials  $f_1(t)$  and  $(t-1)^r$  are coprime; so there exist polynomials  $g_1, g_2 \in \mathbb{Z}[t]$  and there exists a nonzero integer  $k$  (the resultant of  $f_1(t)$  and of  $(t-1)^r$ ) such that

$$f_1(t) \cdot g_1(t) + (t-1)^r \cdot g_2(t) = k.$$

Let  $x \in A(\overline{K})$  and let  $x_0 \in A(\overline{K})$  such that  $kx_0 = x$ . Then clearly

$$x_1 := (\tau - \text{id})^r(g_2(\tau)x_0) \in A_1 \text{ and } x_2 := f_1(\tau)(g_1(\tau)x_0) \in A_2,$$

and moreover,  $x_1 + x_2 = kx_0 = x$ , as desired.

Arguing similarly, one can show that  $A_1 \cap A_2 \subseteq A[k]$  since if  $x \in A_1 \cap A_2$  then  $f_1(\tau)x = 0 = (\tau - \text{id})^r x$  and thus

$$kx = (g_1(\tau)f_1(\tau) + g_2(\tau)(\tau - \text{id})^r)x = 0,$$

as desired.  $\square$

Let  $y_1 \in A_1$  and  $y_2 \in A_2$  such that  $y = y_1 + y_2$ ; furthermore, we may assume that if  $y_1 \in A_2$  then  $y_1 = 0$ . We note that  $\tau$  restricts to an endomorphism to each  $A_1$  and  $A_2$ ; we denote by  $\tau_i$  the action of  $\tau$  on each  $A_i$ . Let  $y_0 \in A_1(\overline{K})$  such that  $(\text{id} - \tau_1)(y_0) = y_1$  (note that  $(\text{id} - \tau_1) : A_1 \rightarrow A_1$  is an isogeny because the minimal polynomial  $f_1$  of  $\tau_1 \in \text{End}(A_1)$  does not have the root 1). Using Lemma 5.4, it suffices to prove Theorem 1.2 for  $T_{-y_0} \circ \sigma \circ T_{y_0}$ ; so, we may and do assume that  $y_1 = 0$ .



Let  $\sigma_i : A_i \rightarrow A_i$  be given by  $\sigma_1(x) = \tau_1(x)$  and  $\sigma_2(x) = \tau_2(x) + y_2$ . Then for each  $x \in A$ , we let  $x_1 \in A_1$  and  $x_2 \in A_2$  such that  $x = x_1 + x_2$ ; we have:

$$\sigma(x) = \sigma(x_1 + x_2) = \tau(x_1 + x_2) + y_2 = \tau(x_1) + \tau(x_2) + y_2 = \sigma_1(x_1) + \sigma_2(x_2).$$

Moreover,  $\sigma^n(x_1 + x_2) = \sigma_1^n(x_1) + \sigma_2^n(x_2)$  for all  $n \in \mathbb{N}$ .

We let  $\beta := (\tau_2 - \text{id})|_{A_2} \in \text{End}(A_2)$ ; then  $\beta^r = 0$ . Let  $B$  be the Zariski closure of the subgroup of  $A_2$  generated by  $\beta(A_2)$  and  $y_2$ . Then  $B$  is an algebraic subgroup of  $A_2$ .

**Lemma 6.2.** *If  $B \neq A_2$ , then  $\sigma$  preserves a nonconstant fibration.*

*Proof of Lemma 6.2.* If  $B \neq A_2$ , then  $\dim(B) < \dim(A_2)$  (note that  $A_2$  is connected) and since  $A_2 \cap A_1$  is finite, we conclude that the algebraic subgroup  $C := A_1 + B$  is a proper abelian subvariety of  $A$ . We let  $f : A \rightarrow A/C$  be the quotient map; we claim that  $f \circ \sigma = f$ . Indeed, for each  $x \in A$ , we let  $x_1 \in A_1$  and  $x_2 \in A_2$  such that  $x = x_1 + x_2$  and then

$$f(\sigma(x)) = f(\sigma(x_1 + x_2)) = f(\sigma_1(x_1) + \sigma_2(x_2)) = f(\sigma_2(x_2)) = f(x_2) = f(x).$$

Since  $A/C$  is a positive dimensional algebraic group and  $f : A \rightarrow A/C$  is the quotient map, then we conclude that  $\sigma$  preserves a nonconstant fibration.  $\square$

From now on, assume  $B = A_2$ . We will prove that there exists  $x \in A(\overline{K})$  such that  $\mathcal{O}_\sigma(x)$  is Zariski dense in  $A$ . First we prove there exists  $x_2 \in A_2(\overline{K})$  such that  $\mathcal{O}_{\sigma_2}(x_2)$  is Zariski dense in  $A_2$ .

Because we assumed that the group generated by  $\beta(A_2)$  and  $y_2$  is Zariski dense in  $A$ , then Lemma 5.1 yields the existence of  $x_2 \in A_2(\overline{K})$  such that the group generated by  $\beta(x_2) + y_2$  is Zariski dense in  $A_2$ . Then Corollary 5.8 yields that any infinite subset of  $\mathcal{O}_{\sigma_2}(x_2)$  is Zariski dense in  $A_2$ . If  $A_1$  is trivial, then  $A_2 = A$  and  $\sigma_2 = \sigma$  and Theorem 1.2 is proven. So, from now on, assume that  $A_1$  is positive dimensional.

Let  $\Gamma$  be the subgroup of  $A(\overline{K})$  generated by all  $\phi(x_2)$  and  $\phi(y_2)$  as we vary  $\phi \in \text{End}(A)$ . Then  $\Gamma$  is a finitely generated  $\text{End}(A)$ -module. Using Lemma 5.5, we may find  $x_1 \in A_1(\overline{K})$  with the property that if  $\psi \in \text{End}(A)$  has the property that  $\psi(x_1) \in \Gamma$ , then  $A_1 \subseteq \ker(\psi)$ . Let  $x := x_1 + x_2$ ; we will prove that  $\mathcal{O}_\sigma(x)$  is Zariski dense in  $A$ .

Let  $V$  be the Zariski closure of  $\mathcal{O}_\sigma(x)$ . The orbit  $\mathcal{O}_\sigma(x)$  is contained in a finitely generated group (see Fact 3.6). Then Fact 3.8 yields that  $V$  is a finite union of cosets of algebraic subgroups of  $A$ . So, if  $V \neq A$ , then there exists a coset  $c + C$  of a proper algebraic subgroup  $C \subset A$  which contains  $\{\sigma^n(x)\}_{n \in S}$  for some infinite subset  $S \subseteq \mathbb{N}$ . By Lemma 3.7, there exists a nonzero  $\psi \in \text{End}(A)$  such that  $\psi(\sigma^n(x)) = \psi(c)$  for each  $n \in S$ , i.e.  $\psi$  is constant on the set  $\{\sigma^n(x) : n \in S\}$ .

Let  $n > m$  be two elements of  $S$ . Then  $\psi(\sigma^n(x) - \sigma^m(x)) = 0$ , and so,

$$\psi(\tau_1^n - \tau_1^m)(x_1) = \psi(\sigma_2^m - \sigma_2^n)(x_2) \in \Gamma.$$

Using the fact that  $x_1 \in A_1$  was chosen to satisfy the conclusion of Lemma 5.5 with respect to  $\Gamma$  and the fact that  $\tau_1^n - \tau_1^m = \tau_1^m(\tau_1^{n-m} - \text{id})$  is an isogeny on  $A_1$ , we obtain that  $\psi(A_1) = 0$ . Thus  $\psi$  is constant on  $\{\sigma_2^n(x_2)\}_{n \in S}$ . Then Corollary 5.8 yields that  $A_2 \subseteq \ker(\psi)$ . Hence  $A_1 + A_2 = A \subseteq \ker(\psi)$  which contradicts the fact that  $\psi \neq 0$ . This concludes our proof.  $\square$

## 7. THE GENERAL CASE

The proof of Theorem 1.3 follows the same strategy as the proof of Theorem 1.2.

*Proof of Theorem 1.3.* We let  $\gamma_1, \dots, \gamma_s$  be a set of generators for  $S$ . As before, we let  $K$  be a finitely generated subfield of  $K_0$  such that  $A$  and each  $\gamma_i$  are defined over  $K$ . Also, we may (and do) assume that  $K_0$  is a given algebraic closure  $\overline{K}$  of  $K$ .

We let  $S_0$  be the monoid of group endomorphisms of  $A$  consisting of all  $\tau : A \rightarrow A$  such that there exists some  $y \in A$  such that  $T_y \circ \tau \in S$ . We let  $\mathcal{U} := \{\gamma_1, \dots, \gamma_s\}$ , and also let  $\mathcal{U}_0$  be a finite set of generators for  $S_0$  corresponding to the elements in  $\mathcal{U}$  (i.e., for each  $\varphi \in \mathcal{U}_0$ , there exists  $y \in A$  such that  $T_y \circ \varphi \in \mathcal{U}$ ).

By Fact 3.2,  $A$  is isogenous with a product of simple abelian varieties  $\prod_i A_i^{r_i}$  and so,  $\text{End}(A)$  (the ring of group endomorphisms of  $A$ ) is isomorphic to  $\prod_i M_{r_i}(\text{End}(A_i))$ . We let  $R_i := \text{End}(A_i)$  and  $F_i := \text{Frac}(R_i)$ . Then each element in  $S_0$  is represented by a tuple of matrices in  $\prod_i M_{r_i}(R_i)$ ; from now on, we use freely this identification of the group endomorphisms from  $S_0$  with tuples of matrices in  $\prod_i M_{r_i}(R_i)$ . Using Lemma 2.5 and also Lemma 4.1, it suffices to assume that for each  $\tau \in S_0$ , and for each positive integer  $n$ , we have

$$(7.0.1) \quad \dim \ker(\tau - \text{id}) = \dim \ker(\tau^n - \text{id}).$$

Let  $U_0$  be the submonoid of  $S_0$  generated by all  $\tau \in S_0$  such that

$$(7.0.2) \quad \max_{n \geq 1} \dim \ker(\tau - \text{id})^n$$

is minimal as we vary  $\tau$  in  $S_0$ . Then, by Lemma 2.6,  $\overline{U}_0 = S_0$ . Let  $U$  be the submonoid of  $S$  corresponding to  $U_0$ , i.e. the set of all  $\sigma \in S$  such that there exists some  $\tau \in U_0$  and there exists a translation  $T_y$  on  $A$  for which  $\sigma = T_y \circ \tau$ . Because  $\overline{U}_0 = S_0$ , then also  $\overline{U} = S$ . Using Lemma 2.3, there exists a finitely generated submonoid  $U'$  of  $U$  (and therefore of  $S$ ) and there exists a positive integer  $n$  such that for each  $i = 1, \dots, s$ , we have  $\gamma_i^n \in \overline{U}'$ . By Lemma 4.3, it suffices to prove Theorem 1.3 for  $U'$ . So, from now on, we assume  $U' = S$ . In particular, this means that  $S_0$  is generated (as a monoid) by finitely many endomorphisms  $\tau$  satisfying (7.0.2); we denote this set by  $\mathcal{U}_0$  (as before). Finally, we recall our notation that  $\mathcal{U} = \{\gamma_1, \dots, \gamma_s\}$  is a finite set of generators of  $S$ , and that for each generator  $\tau \in \mathcal{U}_0$  of  $S_0$  there exists some translation  $T_y$  and some  $i = 1, \dots, s$  such that  $T_y \circ \tau = \gamma_i$ .

Let  $\tau_1, \tau_2$  in  $\mathcal{U}_0$ . Assume  $r_1$  is the order of the root 1 of the minimal polynomial for  $\tau_1$ , and let  $B_2 := \ker(\tau_1 - \text{id})^{r_1}$ . Since  $\tau_2$  commutes with  $\tau_1$ , we obtain that  $\tau_2$  acts on  $B_2$ . Furthermore, because both  $\tau_1$  and  $\tau_2$  are in  $\mathcal{U}_0$ , it must be that the restriction of the action of  $\tau_2$  on  $B_2$  is also unipotent (see also the proof of Lemma 2.6); otherwise for some positive integer  $m$ , the element  $\tau := \tau_2^m \tau_1 \in S_0$  would have the property that

$$\max_{n \geq 1} \dim \ker(\tau - \text{id})^n$$

is smaller than  $\dim B_2$  (which is minimal among all elements of  $S_0$ ).

We let  $B_1$  be a complementary connected algebraic subgroup of  $A$  such that  $A = B_1 + B_2$ , and moreover, each element of  $S$  induces an endomorphism of  $B_1$ . So, we reduced to the case that each element of  $S$  is of the form  $T_y \circ \tau$ , where  $\tau$  acts on  $A = B_1 + B_2$  as follows:

- (i)  $\tau$  restricted to  $B_2$  acts unipotently, i.e. there exists some positive integer  $r_\tau$  such that  $(\tau - \text{id})^{r_\tau}|_{B_2} = 0$ ;
- (ii) for each  $\tau \in \mathcal{U}_0$ , the action of  $\tau$  on  $B_1$  (which by abuse of notation, we also denote by  $\tau$ ) has the property that  $\tau^n - \text{id}$  is a dominant map for each positive integer  $n$  (see (7.0.1)).

We proceed similarly to the case  $S$  is cyclic. Then for each  $\sigma_i \in \mathcal{U}$  (for  $i = 1, \dots, s$ ), we let  $\tau_i \in \mathcal{U}_0$ ,  $z_i \in B_1$  and  $y_i \in B_2$  such that  $\sigma_i = T_{y_i+z_i} \circ \tau_i$ . Note that it may be that  $\tau_i = \tau_j$  for some  $i \neq j$ , but this is not relevant for the proof. We let  $C_i$  be the algebraic subgroup of  $B_2$  spanned by  $y_i$  and  $(\tau_i - \text{id})(B_2)$  (for each  $i = 1, \dots, s$ ). We recall that  $\beta_i := (\tau_i - \text{id})|_{B_2}$  is a nilpotent endomorphism of  $B_2$ ; we let  $\mathcal{U}_1$  be the finite set of all  $\beta_i$ . Finally, we let  $C_S$  be the algebraic subgroup of  $B_2$  generated by all  $C_i$ .

If the algebraic subgroup  $C_S + B_1$  does not equal  $A$ , then the exact same argument as in Lemma 6.2 yields the existence of a non-constant rational map fixed by each  $\sigma \in S$ . Essentially, the projection map  $\pi : A \rightarrow A/(B_1 + C_S)$  is a non-constant morphism with the property that  $\pi \circ \sigma = \pi$  for each  $\sigma \in S$ .

Next assume  $C_S + B_1 = A$ ; we will show there exists  $x \in A(\overline{K})$  whose orbit under  $S$  is Zariski dense. The strategy is the same as in the case  $S$  is cyclic. We can find algebraic points  $x_1 \in B_1$  and  $x_2 \in B_2$  such that the  $S$ -orbit of  $x = x_1 + x_2$  is Zariski dense in  $A$ . First we choose  $x_2 \in B_2(\overline{K})$  as in Lemma 5.1 with respect to the algebraic group endomorphisms  $\beta_i$  and the points  $y_i$ , for  $i = 1, \dots, s$ ; hence the Zariski closure of the group generated by  $\beta_i(x_2) + y_i$  is  $C_i$  for each  $i$ .

Let  $\Gamma$  be the  $\text{End}(A)$ -module spanned by  $x_2, y_1, \dots, y_s, z_1, \dots, z_s$ , which is a finitely generated subgroup of  $A(\overline{K})$ . Then (using Lemma 5.5) we choose  $x_1 \in B_1(\overline{K})$  such that if  $\psi \in \text{End}(A)$  has the property that  $\psi(x_1) \in \Gamma$ , then  $B_1 \subseteq \ker(\psi)$ . Let  $x := x_1 + x_2$ ; we will prove that  $\mathcal{O}_S(x)$  is Zariski dense in  $A$ .

Using Facts 3.6 and 3.8, the Zariski closure of  $\mathcal{O}_S(x)$  is a union of finitely many cosets  $w_j + H_j$  of algebraic subgroups of  $A$ .

**Lemma 7.1.** *There exists a coset  $w + H$  of an algebraic subgroup appearing as a component of the Zariski closure of  $\mathcal{O}_S(x)$ , and there exists a positive integer  $N$  such that  $w + H$  is invariant under  $\gamma_i^N$  for each  $i = 1, \dots, s$ .*

*Proof.* So, we know that the Zariski closure of  $\mathcal{O}_S(x)$  is the union of cosets of (irreducible) algebraic subgroups  $\cup_{i=1}^{\ell} (w_i + H_i)$ . Let  $\gamma \in S$ . Then, using that  $\gamma(\mathcal{O}_S(x)) \subseteq \mathcal{O}_S(x)$ , we obtain

$$\cup_{i=1}^{\ell} (\gamma(w_i) + \gamma(H_i)) \subseteq \cup_{i=1}^{\ell} (w_i + H_i).$$

On the other hand, each  $\gamma \in S$  is a dominant endomorphism of  $A$ , and therefore, for each  $i = 1, \dots, \ell$ , we have  $\dim(\gamma(H_i)) = \dim(H_i)$ . So, that means  $\gamma$  permutes the subgroups  $H_i$  of maximal dimension appearing above. In particular, there exists a positive integer  $N_0$  such that for each  $i = 1, \dots, s$ , the endomorphism  $\gamma_i^{N_0}$  fixes each algebraic group  $H_i$  of maximal dimension.

Let  $S^{(N_0)}$  be the submonoid of  $S$  consisting of all  $\gamma^{N_0}$  for  $\gamma \in S$ . Now, let  $H$  be one such algebraic group of maximal dimension among the algebraic groups  $H_i$  (for  $i = 1, \dots, \ell$ ). Let  $w_i + H$  with  $i = 1, \dots, k$  be all the cosets of  $H$  appearing as irreducible components of the Zariski closure of  $\mathcal{O}_S(x)$ . Then each element  $\gamma \in S^{(N_0)}$  induces a map  $f_\gamma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$  given by  $f_\gamma(w_i + H) = w_{f_\gamma(i)} + H$ ; the map is not necessarily bijective. Moreover, we get a homomorphism of monoids  $f : S^{(N_0)} \rightarrow F_k$  given by  $f(\gamma) := f_\gamma$ , where  $F_k$  is the monoid of all functions from the set  $\{1, \dots, k\}$  into itself. Clearly, there exists  $j \in \{1, \dots, k\}$ , and there exists a positive integer  $N_1$  such that  $f_{\gamma^{N_1}}(j) = j$  for each generator  $\gamma \in \{\gamma_1^{N_0}, \dots, \gamma_s^{N_0}\}$  of  $S^{(N_0)}$ . Then Lemma 7.1 holds with  $N := N_0 \cdot N_1$ .  $\square$

Let  $w + H$  be one coset as in the conclusion of Lemma 7.1, and let  $N$  be the positive integer from the conclusion of Lemma 7.1 with respect to the coset  $w + H$ . We let  $S'$  be the submonoid of  $S$  generated by  $\gamma_i^N$  for  $i = 1, \dots, s$ . Then  $w + H$  contains a set of the form  $\mathcal{O}_{S'}(x')$ , for some  $x' \in \mathcal{O}_S(x)$ ; in other words,  $w + H$  contains a set of the form

$$\{\gamma_1^{m_1+Nn_1} \dots \gamma_s^{m_s+Nn_s}(x) : n_1, \dots, n_s \geq 0\},$$

for some positive integers  $m_1, \dots, m_s$ .

Let then  $\pi : A \rightarrow A/H$  be the canonical projection. Then

$$\pi \left( \gamma_1^{m_1+Nn_1} \dots \gamma_s^{m_s+Nn_s}(x) \right) = w$$

for all  $n_1, \dots, n_s \geq 0$ . Restricted on  $B_1$ , for each group endomorphism  $\tau_i$  (for  $i = 1, \dots, s$ ), the action on the tangent space of  $B_1$  has no eigenvalue which is a root of unity (see (ii) above); hence

$$\psi_1 := \left( \tau_1^{m_1+N} \tau_2^{m_2} \dots \tau_s^{m_s} - \tau_1^{m_1} \tau_2^{m_2} \dots \tau_s^{m_s} \right) |_{B_1} \text{ is an isogeny.}$$

So, we get that  $(\pi \circ \psi_1)(x_1) \in \Gamma$ . Because of our choice for  $x_1$  and the fact that  $\psi_1$  is an isogeny on  $B_1$ , we conclude that  $B_1 \subseteq \ker(\pi)$  (note also that  $B_1$  is connected by our assumption). Thus  $B_1 \subseteq H$ . So, we can view  $\pi$  as

a group homomorphism  $\pi : B_2 \rightarrow A/H$  with the property that for each  $n_1, \dots, n_s \geq 0$  we have

$$\pi \left( \gamma_1^{m_1+n_1N} \dots \gamma_s^{m_s+n_sN} - \gamma_1^{m_1} \dots \gamma_s^{m_s} \right) (x_2) = 0.$$

Letting  $\gamma' := \gamma_1^{m_1} \dots \gamma_s^{m_s}|_{B_2}$ , we have that  $\pi \circ \gamma'$  is constant (equal to  $w$ ) on each orbit  $\mathcal{O}_{\gamma_i^N}(x_2)$ . Then Corollary 5.8 yields that the connected component of the Zariski closure  $C_i$  of the cyclic group generated by  $(\tau_i - \text{id})(x_2) + y_i$  is contained in the kernel of  $\pi \circ \gamma'$ . Since the  $C_i$ 's generate the algebraic group  $C_S$  (and therefore the connected components of the  $C_i$ 's generate the connected component of  $C_S$ ; see also Fact 3.1), and furthermore, the connected component of  $C_S$  contains the connected component of  $B_2$ , we conclude that  $\pi \circ \gamma'$  is identically 0 on  $B_2$ . Because  $\gamma'$  is an isogeny, we conclude that  $B_2 \subseteq \ker(\pi)$ , and therefore  $H = A$  since  $H$  contains both  $B_1$  and  $B_2$ . This concludes our proof.  $\square$

*Remark 7.2.* Our proof of Theorem 1.3 yields that any point  $x \in A(K_0)$  which satisfies the following two properties:

- (a) the cyclic group generated by  $x$  is Zariski dense in  $A$ ; and
- (b)  $x$  is linearly independent (over  $\text{End}(A)$ ) from the points of  $A$  defined over  $K_1 := L(A_{\text{tor}})^{(d)}$ , where  $L$  is a finitely generated subfield of  $K_0$  (depending only on the endomorphisms generating  $S$ ) and  $d$  is a positive integer (also depending only on  $S$ )

has a Zariski dense  $S$ -orbit. Condition (a) is employed in Lemma 5.5, and it is satisfied by *most* points  $x \in A(K_0)$  (this statement can be made precise either by saying that  $x$  is not contained in any proper algebraic subgroup of  $A$ , or even by giving an asymptotic count on the number of points  $x \in A$  defined over any finitely generated subfield of  $K_0$ ). However, it is condition (b) that is much more difficult to deal with in order to obtain a quantitative statement about the number of points  $x \in A$  with a Zariski dense  $S$ -orbit. Our proof of Claim 5.3 yields that  $[K_0 : K_1] = \infty$ , and moreover, one can find *many* points  $x \in A$  satisfying condition (b). But a priori (at least based on our proof) there could be infinitely many finitely generated subfields  $L_1$  of  $K_0$  such that no point  $x \in A(L_1)$  has a Zariski dense  $S$ -orbit.

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