A DYNAMICAL VARIANT OF THE PINK-ZILBER CONJECTURE

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ABSTRACT. Let $f_1, \dots, f_n \in \overline{\mathbb{Q}}[x]$ be polynomials of degree $d > 1$ such that no f_i is conjugated to x^d or to $\pm C_d(x)$, where $C_d(x)$ is the Chebyshev polynomial of degree d. We let φ be their coordinatewise action on $\mathbb{A}^n$, i.e., $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is given by $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$. We prove a dynamical version of the Pink-Zilber conjecture for subvarieties V of $\mathbb{A}^n$ with respect to the dynamical system $(\mathbb{A}^n, \phi)$, if $\min\{\dim(V), \text{codim}(V) - 1\} \le 1$.

1. Introduction

1.1. **Notation.** As always in dynamics, we write φ^m for the m-th compositional power of the self-map φ for any $m \in \mathbb{N}_0$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$); also, φ^0 is the identity map. The orbit of some point α under φ is denoted by $\mathcal{O}_\varphi(\alpha)$ and it consists of all $\varphi^m(\alpha)$ for $m \in \mathbb{N}_0$. For a subvariety $Y \subset \mathbb{A}^n$ under the action of an endomorphism φ, we say that Y is periodic if there exists a positive integer m such that $Y = \varphi^m(Y)$; similarly, we say that Y is preperiodic under the action of φ if there exists $m \in \mathbb{N}_0$ such that $\varphi^m(Y)$ is periodic.

For every $d \ge 2$, the Chebyshev polynomial of degree d, denoted $C_d(x)$, is the polynomial of degree d satisfying the functional equation $C_d(x + \frac{1}{x}) = x^d + \frac{1}{x^d}$. Following [MS14], a *disintegrated polynomial* is a polynomial of degree $d \ge 2$ that is not linearly conjugated to x^d or $\pm C_d(x)$.

1.2. **The Dynamical Manin-Mumford and the Dynamical Bogomolov Conjectures.** The following theorem proven in [GNY] is a special case of the more general Dynamical Manin-Mumford Conjecture and of the Dynamical Bogomolov Conjecture proposed by Zhang [Zha06].

Theorem 1.1 ([GNY]). *Let $f_1, \dots, f_n \in \overline{\mathbb{Q}}[x]$ be disintegrated polynomials of degree $d > 1$ and we let φ be their coordinatewise action on $\mathbb{A}^n$, i.e., $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is given by $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_r))$. For any irreducible $\overline{\mathbb{Q}}$-subvariety $X \subset \mathbb{A}^n$, if X contains a Zariski dense set of preperiodic points, then X is preperiodic. Furthermore, if for each $\epsilon > 0$,*

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the set of points $(a_1, \dots, a_n) \in X(\overline{\mathbb{Q}})$ *such that*

$$\widehat{h}_{f_1}(a_1) + \cdots + \widehat{h}_{f_n}(a_n) < \epsilon$$

is Zariski dense in X*, then* X *is a preperiodic subvariety.*

In Theorem 1.1, given a polynomial $f \in \overline{\mathbb{Q}}[x]$ of degree larger than 1, $\widehat{h}_f(\cdot)$ is the canonical height defined as $\widehat{h}_f(a) := \lim_{n\to\infty} \frac{h(f^n(a))}{\deg(f)^n}$ for any $a \in \overline{\mathbb{Q}}$, where $h(\cdot)$ is the usual Weil height. For more details regarding heights, see [BG06].

Actually, in [GNY, Theorem 1.1], the above result was proven for polarizable endomorphisms of $(\mathbb{P}^1)^n$, i.e., maps of the form $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$ where each $f_i \in \overline{\mathbb{Q}}(x)$ is a rational function of degree $d \geq 2$ (which is not conjugated to a monomial, a $\pm$Chebyshev polynomial, or a Lattés map). We will prove in Theorem 3.2 a slightly more precise version of Theorem 1.1 for any subvariety of $\mathbb{A}^n$ which contains a Zariski dense set of periodic points.

In Theorem 1.1, if each polynomial f_i is conjugated with either a monomial or a $\pm$Chebyshev polynomial, then we recover the classical conjectures of Manin-Mumford and Bogomolov for algebraic tori. Actually, those conjectures (including in their version for abelian varieties) motivated Zhang to formulate in early 1990's a far-reaching dynamical conjecture for polarizable algebraic dynamical systems generalizing both these classical diophantine problems and Theorem 1.1 (see also [Zha06]).

In Theorem 1.1, since the coordinatewise action of φ on $\mathbb{A}^n$ is given by polynomials, one does not encounter the counterexamples (see [GTZ11]) to the original formulation of the Dynamical Manin-Mumford Conjecture (and of the Dynamical Bogomolov Conjecture), and hence one is not expected to require the stronger hypothesis for the reformulation from [GTZ11, Conjecture 1.4] of the Dynamical Manin-Mumford Conjecture. We note that Theorem 1.1 was initially proven when $X \subset \mathbb{A}^n$ is a curve in [GNY16].

1.3. **The Dynamical Pink-Zilber Conjecture.** In [GN16], a dynamical version of the Bounded Height Conjecture (see [BMZ07] for the formulation of this classical conjecture in the context of algebraic tori) was proven for endomorphisms of $\mathbb{A}^n$ given by coordinatewise action of disintegrated polynomials. The results of [GN16] suggest the following variant of the Pink-Zilber Conjecture in a dynamical setting; see [BMZ99, Zil02, Pin] for the statement of this conjecture in the classical setting of algebraic tori, or more generally, of semiabelian schemes.

Conjecture 1.2. *Let* $f_1, \dots, f_n \in \overline{\mathbb{Q}}[x]$ *be disintegrated polynomials of degree* $d \geq 2$*. We let* φ *be their coordinatewise action on* $\mathbb{A}^n$*, i.e.,* $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ *is given by* $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$*. For each positive integer* $s \leq n$*, we let* $\mathrm{Per}^{[s]}$ *be the union of all irreducible periodic subvarieties* *of $\mathbb{A}^n$ of codimension s; similarly, we let $\mathrm{Prep}^{[s]}$ be the union of all irreducible preperiodic subvarieties of $\mathbb{A}^n$ of codimension s. Let $X \subset \mathbb{A}^n$ be an irreducible subvariety of dimension m.*

(1) *If $X \cap \mathrm{Per}^{[m+1]}$ is Zariski dense in X, then X is contained in a proper, irreducible subvariety of $\mathbb{A}^n$, which is periodic under the action of φ.*
(2) *If $X \cap \mathrm{Prep}^{[m+1]}$ is Zariski dense in X, then X is contained in a proper, irreducible subvariety of $\mathbb{A}^n$, which is preperiodic under the action of φ.*

It makes sense to restrict in Conjecture 1.2 to polynomials which are not conjugated to monomials or Chebyshev polynomials since otherwise we would encounter the classical Pink-Zilber Conjecture (see [Zan12] for a comprehensive discussion). Also, we note that if X is contained in a proper, irreducible (pre)periodic subvariety Y of $\mathbb{A}^n$, then (simply, by geometric considerations of counting the dimensions) X intersects nontrivially each (pre)periodic subvariety of relative codimension in Y equal to $\dim(X)$, and thus, X has a Zariski dense intersection with $\mathrm{Per}^{[\dim(X)+1]}$ (respectively, $\mathrm{Prep}^{[\dim(X)+1]}$); this scenario is exactly similar to the classical case when a subvariety $X \subset \mathbb{G}_m^n$ contained in a proper algebraic subtorus would have a Zariski dense intersection with the union of all subtori in $\mathbb{G}_m^n$ of codimension equal to $\dim(X) + 1$.

We also note that the two parts of Conjecture 1.2 are *independent*, neither one implying the other one. Furthermore, it is likely that the methods one would need to employ in proving the above two conjectures might differ slightly. For example, we would expect that some of the p-adic techniques developed for proving the Dynamical Mordell-Lang Conjecture (for more details, see [BGT16, Chapter 4]) could prove useful in treating Conjecture 1.2 (1) in full generality. On the other hand, in attacking Conjecture 1.2 (2), one might need to develop generalizations of the arguments employed in [GNY]. Also, Conjecture 1.2 (2) is particularly challenging since one lacks a corresponding Dynamical Bounded Height Conjecture for preperiodic subvarieties, in the spirit of the one proven in [GN16] (which is valid only for periodic subvarieties). Attempting to prove a variant of the Bounded Height Conjecture for preperiodic subvarieties of $\mathbb{A}^n$ leads to subtle diophantine questions similar to the ones encountered in [DGKNTY].

Finally, it is important to observe that if we did not impose the condition that the polynomials have the same degree, then there would be simple counterexamples (similar to a naive formulation of the Dynamical Manin-Mumford Conjecture, which does not require the polarizability of the given endomorphism). Indeed, if $f \in \overline{\mathbb{Q}}[x]$ has degree $d \geq 2$, then its graph $y = f(x)$ is a (rational) plane curve containing infinitely many points which are periodic under the coordinatewise action of $(x, y) \mapsto (f(x), f^2(y))$; however, this graph is *not periodic* under the action of (f, f^2).

1.4. **Our results.** We prove the following results.

Theorem 1.3. *Let $f_1, \dots, f_n \in \overline{\mathbb{Q}}[x]$ be disintegrated polynomials of degree $d > 1$ and let φ be their coordinatewise action on $\mathbb{A}^n$, i.e., $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is given by $\varphi(x_1, \dots, x_n) = (f_1(x_1), \dots, f_n(x_n))$. Let $X \subset \mathbb{A}^n$ be an irreducible subvariety defined over $\overline{\mathbb{Q}}$ such that $\min\{\dim(X), \mathrm{codim}(X) - 1\} \le 1$. If $X \cap \mathrm{Per}^{[\dim(X)+1]}$ is Zariski dense, then X is contained in a proper, irreducible, periodic subvariety of $\mathbb{A}^n$.*

Therefore Theorem 1.3 provides a proof for Conjecture 1.2(1) in the following 3 *nontrivial* cases:

(I) X is a hypersurface (see Theorem 3.2, which proves a more general result).

(II) X is a curve (see Theorem 4.1).

(III) $X \subset \mathbb{A}^n$ has codimension 2 (see Theorem 5.1).

Clearly, if X is a point (i.e., $\dim(X) = 0$), or if $X = \mathbb{A}^n$ (i.e., $\mathrm{codim}(X) = 0$), Conjecture 1.2 holds.

It is *difficult* to extend any of our results to dynamical systems given by the coordinatewise action of rational functions due to the absence of Medvedev-Scanlon's classification [MS14] of periodic subvarieties in that case (see also [GN16]). Also, it is difficult to extend Theorem 1.3 to subvarieties $X \subset \mathbb{A}^n$ of dimension either larger than 1, or codimension larger than 2; see the following Example, which can be generalized to any subvariety of $\mathbb{A}^n$ of dimension in the range $\{2, \dots, n-3\}$.

Example 1.4. Let $f \in \overline{\mathbb{Q}}[x]$ be a polynomial of degree $d \ge 2$ and let φ be its coordinatewise action on $\mathbb{A}^6$. Let $X \subset \mathbb{A}^3$ be a surface which projects to a non-preperiodic point to each of the first 3 coordinates, i.e., $X = \zeta \times X_1$, where $\zeta \in \mathbb{A}^3(\overline{\mathbb{Q}})$ and $X_1 \subset \mathbb{A}^3$ is a surface defined over $\overline{\mathbb{Q}}$. We also assume X_1 is not a periodic surface, while ζ is not contained in a proper periodic subvariety of $\mathbb{A}^3$; this last assumption can be achieved (see Section 2) by assuming the coordinates of $\zeta := (\zeta_1, \zeta_2, \zeta_3)$ belong to different orbits under f, i.e., there is no $i, j \in \{1, 2, 3\}$ and no $m, n \in \mathbb{N}$ such that $f^m(\zeta_i) = f^n(\zeta_j)$. Then X is not contained in a proper periodic subvariety of $\mathbb{A}^3$ and therefore, Conjecture 1.2 predicts that $X \cap \mathrm{Per}^{[3]}$ is not Zariski dense in X. In particular, this yields that

$$X_1 \cap (\mathcal{O}_f(\zeta_1) \times \mathcal{O}_f(\zeta_2) \times \mathcal{O}_f(\zeta_3)) \tag{1.5}$$

is not Zariski dense in X_1. However, understanding the intersection from (1.5) is equivalent with solving a stronger form of the Dynamical Mordell-Lang Conjecture for hypersurfaces in $\mathbb{A}^3$ and at the present moment, this problem seems very difficult; for a comprehensive discussion about the Dynamical Mordell-Lang Conjecture, see [BGT16].

As shown in a series of papers by Bombieri-Masser-Zannier (see [BMZ99, BMZ06]), even the classical Pink-Zilber conjecture in the context of algebraic tori is very difficult and initially, only the case of curves was established; for more details, see the beautiful book of Zannier [Zan12]. In the dynamical context, the fact that we do not even know the validity of the Dynamical Mordell-Lang conjecture makes Conjecture 1.2 particularly challenging.

1.5. **Plan for our paper.** We prove Theorem 1.3 by splitting it into its 3 nontrivial cases (I)-(III), i.e., X is a hypersurface (Theorem 3.2), X is a curve (Theorem 4.1) and finally, X has codimension 2 (Theorem 5.1). The common ingredients for proving these results are the classification of periodic subvarieties of $\mathbb{A}^n$ under the coordinatewise action of n one-variable polynomials (as obtained by Medvedev-Scanlon [MS14], along with some further refinements obtained by the authors in [GN16]) and also the proof of the Dynamical Manin-Mumford and of the Dynamical Bogomolov conjectures for endomorphisms of $(\mathbb{P}^1)^n$ (see theorem 1.1 and [GNY16, GNY]). In the case of curves $X \subset \mathbb{A}^n$, we also need to employ the recent result of Xie [Xie], who proved the dynamical Mordell-Lang Conjecture for plane curves.

In Section 2, using [MS14] (along with its refinements from [Ngu13, GN16]) we introduce the structure of periodic subvarieties of $\mathbb{A}^n$ under the coordinatewise action of n one-variable polynomials. In Section 3 we prove Theorem 1.3 for hypersurfaces $X \subset \mathbb{A}^n$ (see Theorem 3.2, which actually proves that *any* subvariety of $\mathbb{A}^n$ containing a Zariski dense set of periodic points must be periodic itself). Then we continue by proving Theorem 1.3 when X is a curve (see Theorem 4.1) in Section 4, and we conclude our paper by proving Theorem 1.3 when $\operatorname{codim}(X) = 2$ (see Theorem 5.1) in Section 5.

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2. Structure of preperiodic subvarieties

Most of this section is taken from [GN16, GN17] which, in turn, follows from [MS14, Ngu13]. Throughout this section, let $n \geq 2$, and let $f_1, \ldots, f_n$ be disintegrated polynomials in $\mathbb{C}[x]$. For $m \geq 2$, an irreducible curve (or more generally, a higher dimensional subvariety) in $\mathbb{A}^m$ is said to be fibered if its projection to one of the coordinate axes is constant, otherwise the curve (or the subvariety) is called non-fibered. For any two disintegrated polynomials $f(x)$ and $g(x)$, write $f \approx g$ if the self-map $(x, y) \mapsto (f(x), g(y))$ of $\mathbb{A}^2$ admits an irreducible non-fibered periodic curve. The relation $\approx$ is an equivalence relation in the set of disintegrated polynomials (see [GN16, Section 7]).

Let $\varphi = f_1 \times \ldots \times f_n$ be the self-map of $\mathbb{A}^n$ given by $\varphi(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$. Let s denote the number of equivalence classes arisen from $f_1, \ldots, f_n$ (under $\approx$). Let $n_1, \ldots, n_s$ denote the sizes of these classes (hence $n_1 + \ldots + n_s = n$). We relabel the polynomials $f_1, \ldots, f_n$ as $f_{i,j}$ for $1 \leq i \leq s$ and $1 \leq j \leq n_i$ according to the equivalence classes. After rearranging the polynomials $f_1, \ldots, f_n$ so that equivalence polynomials stay in blocks, we have $\varphi = \varphi_1 \times \ldots \times \varphi_s$ where φ_i is the self-map $f_{i,1} \times \ldots \times f_{i,n_i}$ of $\mathbb{A}^{n_i}$. There exist a positive integer N, non-constant $p_{i,j} \in \mathbb{C}[x]$ for $1 \leq i \leq s$

and $1 \leq j \leq n_i$ and disintegrated $w_1, \dots, w_s \in \mathbb{C}[x]$ in s different equivalence classes such that the following holds. For $1 \leq i \leq s$, let ψ_i be the self-map $w_i \times \dots \times w_i$ on $\mathbb{A}^{n_i}$, and let $\psi = \psi_1 \times \dots \times \psi_s$. Let η_i be the self-map $p_{i,1} \times \dots \times p_{i,n_i}$ of $\mathbb{A}^{n_i}$ and let $\eta = \eta_1 \times \dots \times \eta_s$. We have the commutative diagram:

$$\begin{array}{ccc} \mathbb{A}^{n_1} \times \dots \times \mathbb{A}^{n_s} & \xrightarrow{\psi} & \mathbb{A}^{n_1} \times \dots \times \mathbb{A}^{n_s} \\ \eta \downarrow & & \downarrow \eta \\ \mathbb{A}^{n_1} \times \dots \times \mathbb{A}^{n_s} & \xrightarrow[\varphi^N]{} & \mathbb{A}^{n_1} \times \dots \times \mathbb{A}^{n_s} \end{array} \tag{2.1}$$

We have the following simple observation:

Lemma 2.2. *Let V be an irreducible φ-preperiodic subvariety of dimension r. We have:*

(a) *Every irreducible component of $\eta^{-1}(V)$ is ψ-preperiodic and has dimension r.*
(b) *If V is φ-periodic then some irreducible component of $\eta^{-1}(V)$ is ψ-periodic.*
(c) *Let X be an irreducible subvariety in $\mathbb{A}^n$ and let $\mathrm{Per}_{\varphi}^{[r]}$ (respectively $\mathrm{Per}_{\psi}^{[r]}$) be the union of φ-periodic (respectively ψ-periodic) subvarieties of codimension r. If $X \cap \mathrm{Per}_{\varphi}^{[r]}$ is Zariski dense in X then there is an irreducible component X' of $\eta^{-1}(X)$ such that $X' \cap \mathrm{Per}_{\psi}^{[r]}$ is Zariski dense in X'.*

Proof. Part (a) follows from the commutative diagram (2.1) and the fact that η is finite. For part (b), if $\varphi^{M_0}(V) = V$ then ψ^{M_0} maps the set of irreducible components of $\eta^{-1}(V)$ to itself; hence at least one element in this set is a ψ-periodic subvariety.

For part (c), we have a collection of points $\{P_i : \ i \in \mathcal{S}\}$ that is Zariski dense in X and satisfies the property that for each $i \in \mathcal{S}$, there is an irreducible φ-subvariety V_i of codimension r such that $P_i \in X \cap V_i$. For each $i \in \mathcal{S}$, there is an irreducible component W_i of $\eta^{-1}(V_i)$ that is ψ-periodic and there is a point $Q_i \in W_i$ such that $\eta(Q_i) = P_i$. Let $X_1, \dots, X_M$ denote all the irreducible components of $\eta^{-1}(X)$. We partition $\mathcal{S}$ into $\mathcal{S}_1, \dots, \mathcal{S}_M$ such that $i \in \mathcal{S}_j$ implies $Q_i \in X_j$ for every $1 \leq j \leq M$. We claim that there exists some $j \in \{1, \dots, M\}$ such that $\{Q_i : \ i \in \mathcal{S}_j\}$ is Zariski dense in X_j; consequently $X_j \cap \mathrm{Per}_{\psi}^{[r]}$ is Zariski dense in X_j. To prove this claim, assume that the Zariski closure of $\{Q_i : \ i \in \mathcal{S}_j\}$ is strictly smaller than X_j for every $j \in \{1, \dots, M\}$. Then the image under η of the union of these M Zariski closures contains $\{P_i : \ i \in \mathcal{S}\}$ and is strictly smaller than X, contradiction. □

Remark 2.3. We will also use the following simple observation which can be proved by arguments which are similar to the ones employed in the proof of part (c) above. If X is an irreducible subvariety of $\mathbb{A}^n$ and $\{V_i:\ i\in\mathcal{S}\}$ is a collection of irreducible subvarieties of $\mathbb{A}^n$ such that $X\cap\bigcup_{i\in\mathcal{S}}V_i$ is Zariski dense in X and $\mathcal{S}_1,\ldots,\mathcal{S}_M$ is a partition of $\mathcal{S}$ then there exists j such that $X\cap\bigcup_{i\in\mathcal{S}_j}V_i$ is Zariski dense in X.

Each irreducible φ-preperiodic subvariety V of $\mathbb{A}^n$ has the form $V_1\times\ldots\times V_s$ where each V_i is an irreducible φ_i-preperiodic subvariety of $\mathbb{A}^{n_i}$. Let W be an arbitrary irreducible component of $\eta^{-1}(V)$. Then W is ψ-preperiodic and has the form $W_1\times\ldots\times W_s$ where each W_i is an irreducible component of $\psi_i^{-1}(V_i)$ and it is ψ_i-preperiodic. Note that ψ_i is the coordinate-wise self-map of $\mathbb{A}^{n_i}$ induced by the *common* polynomial w_i.

Let f be a disintegrated polynomial and let $\Phi=f\times\ldots\times f$ be the corresponding self-map of $\mathbb{A}^n$. We recall the structure of Φ-periodic subvarieties of $\mathbb{A}^n$ given in [GN16, Section 2]. Write $I_n=\{1,\ldots,n\}$. For each *ordered* subset J of I_n, we define:

$$\mathbb{A}^J:=\mathbb{A}^{|J|}$$

equipped with the canonical projection $\pi_J:\ \mathbb{A}^n\to\mathbb{A}^J$. In this paper, we will consider ordered subsets of I_n whose orders need not be induced from the usual order of the set of integers. If $J_1,\ldots,J_m$ are ordered subsets of I_n which partition I_n, then we have the canonical isomorphism:

$$(\pi_{J_1},\ldots,\pi_{J_m}):\ \mathbb{A}^n=\mathbb{A}^{J_1}\times\ldots\times\mathbb{A}^{J_m}.$$

For each irreducible subvariety V of $\mathbb{A}^n$, let J_V denote the set of all $j\in I_n$ such that the projection from V to the j^{th} coordinate axis is constant. If $J_V\neq\emptyset$, we equip J_V with the natural order of the set of integers, and we let $a_V\in\mathbb{A}^{J_V}(\mathbb{C})$ denote $\pi_{J_V}(V)$. Even when $J_V=\emptyset$, we will *vacuously* define $(\mathbb{A}^1)^{J_V}$ as the variety consisting of one point and define a_V to be that point. We have the following:

Proposition 2.4. (a) *Let V be an irreducible Φ-periodic subvariety of $\mathbb{A}^n$ of dimension r. Then there exists a partition of I_n-J_V into r non-empty subsets $J_1,\ldots,J_r$ such that the following hold. We fix an order on each $J_1,\ldots,J_r$, and identify:*

$$\mathbb{A}^n=\mathbb{A}^{J_V}\times\mathbb{A}^{J_1}\times\ldots\times\mathbb{A}^{J_r}.$$

For $1\leq i\leq r$, let Φ_i denote the coordinatewise self-map of $\mathbb{A}^{J_i}$ induced by f. For $1\leq i\leq r$, there exists an irreducible Φ_i-periodic curve C_i in $\mathbb{A}^{J_i}$ such that:

$$V=\{a_V\}\times C_1\times\ldots\times C_r.$$

(b) *Let C be an irreducible Φ-periodic curve in $\mathbb{A}^n$ and denote $m:=|I_n-J_C|\geq 1$. Then there exist a permutation $(i_1,\ldots,i_m)$ of I_n-J_C and non-constant polynomials $g_2,\ldots,g_m\in\overline{\mathbb{Q}}[x]$ such that C is given by the equations $x_{i_2}=g_2(x_{i_1}),\ldots,\ x_{i_m}=g_m(x_{i_{m-1}})$. Furthermore, the polynomials $g_2,\ldots,g_m$ commute with an iterate of f.*

Remark 2.5. Let C be a non-fibered irreducible preperiodic curve in $\mathbb{A}^2$ under the map $\Phi(x,y)=(f(x),f(y))$. Then $\Phi^r(C)$ is periodic for some r. So we know that C satisfies an equation of the form $f^r(x_2)=g(f^r(x_1))$ or $f^r(x_1)=g(f^r(x_2))$ where g commutes with an iterate of f. We can express both cases by an equation of the form $g(x_1)=G(x_2)$ where both g and G commute with an iterate of f.

Remark 2.6. The above discussion gives a very precise description of irreducible φ-preperiodic subvarieties of $\mathbb{A}^n$ (recall that $\varphi = f_1\times\ldots\times f_n$). Occasionally, the following simpler observation is sufficient for our purpose. Let $V\subsetneq\mathbb{A}^n$ be an irreducible φ-periodic subvariety. Then there exist $1\le i<j\le n$ and an irreducible curve C in $\mathbb{A}^2$ which is preperiodic under $(x,y)\mapsto(f_i(x),f_j(y))$ such that $V\subseteq\pi^{-1}(C)$ where π is the projection from $\mathbb{A}^n$ to the i-th and j-th coordinates $\mathbb{A}^2$.

Remark 2.7. The permutation $(i_1,\ldots,i_m)$ mentioned in part (b) of Proposition 2.4 induces the order $i_1\prec\ldots\prec i_m$ on I_n-J_C. Such a permutation and its induced order are not uniquely determined by V. For example, let L be a *linear* polynomial commuting with an iterate of f. Let C be the periodic curve in $\mathbb{A}^2$ defined by the equation $x_2=L(x_1)$. Then $I-J_C=\{1,2\}$, and $1\prec 2$ is an order satisfying the conclusion of part (b). However, we can also express C as $x_1=L^{-1}(x_2)$. Then the order $2\prec 1$ also satisfies part (b). Therefore, in part (a), the choice of an order on each J_i is not unique. Nevertheless, the partition of I_n-J_V into the subsets $J_1,\ldots,J_r$ is unique (see [Ngu13, Section 2]).

Next we describe all polynomials g commuting with an iterate of f:

Proposition 2.8. *Let $f\in\mathbb{C}[x]$ be a disintegrated polynomial of degree greater than 1. We have:*

(a) *If $g\in\mathbb{C}[x]$ has degree at least 2 such that g commutes with an iterate of f then g and f have a common iterate.*

(b) *Let $M(f^\infty)$ denote the collection of all linear polynomials commuting with an iterate of f. Then $M(f^\infty)$ is a finite cyclic group under composition.*

(c) *Let $\tilde f\in\mathbb{C}[x]$ be a polynomial of minimum degree $\tilde d\ge 2$ such that $\tilde f$ commutes with an iterate of f. Then there exists $D=D_f>0$ relatively prime to the order of $M(f^\infty)$ such that $\tilde f\circ L=L^D\circ\tilde f$ for every $L\in M(f^\infty)$.*

(d) $\left\{\tilde f^m\circ L:\ m\ge 0,\ L\in M(f^\infty)\right\}=\left\{L\circ\tilde f^m:\ m\ge 0,\ L\in M(f^\infty)\right\}$, *and these sets describe exactly all polynomials g commuting with an iterate of f. As a consequence, there are only finitely many polynomials of bounded degree commuting with an iterate of f.*

Remark 2.9. In the diagram (2.1), if $f_1,\ldots,f_n$ are in $\overline{\mathbb{Q}}[x]$ then the polynomials w_i's and $p_{i,j}$'s can be chosen to be in $\overline{\mathbb{Q}}[x]$. In Proposition 2.8, if $f(x)\in\overline{\mathbb{Q}}[x]$ then $\tilde f\in\overline{\mathbb{Q}}[x]$ and elements of $M(f^\infty)$ are in $\overline{\mathbb{Q}}[x]$.

We will use the following immediate corollary to recognize when a point is f-periodic:

Corollary 2.10. *Let $f \in \mathbb{C}[x]$ be a disintegrated polynomial of degree greater than* 1*.*

(a) *Let $g(x) \in \mathbb{C}[x]$ such that $\deg(g) \geq 2$ and g commutes with an iterate of f. Then $\alpha \in \mathbb{C}$ is g-periodic if and only if it is f-periodic.*
(b) *Let $p(x) \in \mathbb{C}[x]$ such that $\deg(p) \geq 1$ and p commutes with an iterate of f. Let $\alpha \in \mathbb{C}$ be f-periodic. Then $p(\alpha)$ is also f-periodic.*
(c) *If α is f-preperiodic then for any polynomial g that commutes with an iterate of f and $\deg(g)$ is sufficiently large, $g(\alpha)$ is f-periodic.*
(d) *If α is f-preperiodic then the set*

$$\{g(\alpha) : \ g \text{ commutes with an iterate of } f\}$$

is finite.

Proof. Part (a) is obvious since g and f have a common iterate. For part (b), choose m such that f^m commutes with p and $\alpha = f^m(\alpha)$. Then $f^m(p(\alpha)) = p(f^m(\alpha)) = p(\alpha)$. For part (c), let $r \geq 0$ such that $f^r(\alpha)$ is f-periodic, then if $\deg(g) \geq \deg(f)^r$, we can write $g = g_1 \circ f^r$ where g_1 commutes with an iterate of f by Proposition 2.8(d). Now $g(\alpha) = g_1(f^r(\alpha))$ is f-periodic by part (b). For part (d), let $\tilde{f}$ be as in Proposition 2.8, we can write g as $L \circ \tilde{f}^m$ for some $m \geq 0$ and $L \in M(f^\infty)$. Since α is $\tilde{f}$-preperiodic and $M(f^\infty)$ is finite, there are only finitely many possibilities for $g(\alpha)$. ∎

We now consider the more general self-map $\varphi = f_1 \times \ldots \times f_n$ as in the beginning of this section. Let V be an irreducible φ-preperiodic subvariety of $\mathbb{A}^n$ with $r := \dim(V)$. As before, J_V denotes the set of $i \in I_n$ such that the projection from V to the i-th coordinate $\mathbb{A}^1$ is constant and $a_V \in \mathbb{A}^{J_V}(\mathbb{C})$ is the image $\pi^{J_V}(V)$. By Proposition 2.4 and the diagram 2.1, we can partition the set $I_n \setminus J_V$ into r non-empty subsets $J_1, \ldots, J_r$ such that after identifying

$$\mathbb{A}^n = \mathbb{A}^{J_V} \times \mathbb{A}^{J_1} \times \ldots \times \mathbb{A}^{J_r},$$

we have:

$$V = \{a_V\} \times C_1 \times \ldots \times C_r$$

where each C_j is a preperiodic curve in $\mathbb{A}^{J_j}$ with respect to the coordinate-wise self-map induced by the polynomials f_i's for $i \in J_j$. Moreover, if V is periodic then a_V and each C_j are periodic. Since each C_i is necessarily non-fibered thanks to the definition of J_V, we have that $f \approx g$ for $f, g \in J_j$ for $1 \leq j \leq r$. We have the following:

Definition 2.11. *The weak signature of V is the collection consisting of the set J_V and the partition of $I_n \setminus J_V$ into the sets $J_1, \ldots, J_r$.*

3. Proof of Theorem 1.3 for hypersurfaces

We start by proving a more precise version of [GNY, Theorem 1.1] for plane curves containing infinitely many periodic points.

Theorem 3.1. *Let $f_1, f_2 \in \overline{\mathbb{Q}}[x]$ be disintegrated polynomials of degree $d \geq 2$ and let $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ be their coordinatewise action, i.e., $(x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$. If $X \subset \mathbb{A}^2$ is an irreducible curve containing infinitely many periodic points, then X must be periodic under the action of φ.*

Proof. This theorem follows from the more technical result [GNY16, Theorem 5.1]. Here we include a different proof illustrating the sort of arguments used in the later sections. First of all, by Theorem 1.1, we have that X is preperiodic since it contains infinitely many (pre)periodic points. We now recall the equivalence relation $\approx$ in Section 2. If f_1 and f_2 are not in the same equivalence class then the only φ-preperiodic curves are of the form $\{\zeta_1\} \times \mathbb{A}^1$ or $\mathbb{A}^1 \times \{\zeta_2\}$ where ζ_i is f_i-preperiodic for $i = 1, 2$. However, since X contains infinitely many periodic points, we conclude that X is periodic (i.e., ζ_1, or respectively, ζ_2 must be periodic).

We now assume that $f_1 \approx f_2$. By Lemma 2.2, we may assume that $f_1 = f_2 =: f$. The arguments in the previous paragraph can be used to treat the case when X is fibered. We reduce to the case that X is a non-fibered preperiodic curve. Let $\tilde{f}$ be as in Proposition 2.8; since f and $\tilde{f}$ have a common iterate, we may assume that $f = \tilde{f}$ without changing the content of Theorem 3.1. By Proposition 2.4 and without loss of generality, we may assume that X satisfies the equation $f^k(x) = g(f^k(y))$ for some $k \geq 0$ and $g(x) \in \mathbb{C}[x]$ that commutes with an iterate of f. By Proposition 2.8, we can write $g = L \circ f^\ell$ for some $L \in M(f^\infty)$ and $\ell \geq 0$.

Let (a_1, a_2) be a φ-periodic point on X and let $\rho \geq k$ such that $f^\rho(a_1) = a_1$ and $f^\rho(a_2) = a_2$. By using Proposition 2.8 and applying $f^{\rho-k}$ to both sides of the equation

$$f^k(a_1) = g(f^k(a_2))$$

and thus using that there exists some $L_1 \in M(f^\infty)$ such that $f^{\rho-k} \circ L = L_1 \circ f^{\rho-k}$, we get

$$f^\rho(a_1) = L_1(f^{\rho+\ell}(a_2)).$$

Then using that both a_1 and a_2 are fixed by f^ρ yields that

$$a_1 = L_1(f^\ell(a_2)).$$

Now, because $M(f^\infty)$ is finite and there are infinitely many periodic points (a_1, a_2) on X, there exists $\tilde{L} \in M(f^\infty)$ such that $\alpha = \tilde{L}(f^\ell(\beta))$ for infinitely many (periodic) points (α, β) in X. Therefore X is given by the equation $x = \tilde{L}(f^\ell(y))$, thus proving that X must be periodic, as claimed. □

The above theorem is the key step to obtaining the following more general result:

Theorem 3.2. *Let $f_1, \dots, f_n$, d, and φ be as in Theorem 1.3. Let X be an irreducible subvariety of $\mathbb{A}^n$ such that X contains a Zariski dense set of φ-periodic points, then X is periodic. Consequently, Theorem 1.3 holds when* $\operatorname{codim}(X) = 1$.

Proof. Since X contains a Zariski dense set of preperiodic points, Theorem 1.1 yields that X must be preperiodic. Let $J \subset \{1, \dots, n\}$ denote the factors $\mathbb{A}^1$ of $\mathbb{A}^n$ for which the projection from X is constant and let $a \in \mathbb{A}^J$ be the constant image of X (under the aforementioned projection). After rearranging the coordinates of $\mathbb{A}^n$, the discussion from Section 2 gives that X has the form

$$\{a\} \times X_1 \times \ldots \times X_r$$

where $r = \dim(X)$ and each X_i is a preperiodic non-fibered curve. By the assumption on X, the point a must be periodic. Therefore, by working with each X_i, we now reduce Theorem 3.2 to the case when X is a non-fibered curve of $\mathbb{A}^n$. In particular $f_1, \dots, f_n$ belong to the same equivalence class. By Lemma 2.2, we may assume that $f_1 = \ldots = f_n =: f$.

For each $i \neq j$ in $\{1, \dots, n\}$, let $\pi_{i,j}$ denote the projection from X to the i-th and j-th coordinates $\mathbb{A}^2$ and let $X_{i,j}$ denote the Zariski closure of $\pi_{i,j}(X)$. By Theorem 3.1 and Proposition 2.4, $X_{i,j}$ is given by the equation $x_i = g(x_j)$ or $x_j = g(x_i)$ where g commutes with an iterate of f. Doing this for all pairs (i, j), we have a permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$ such that X satisfies the equations $x_{i_2} = g_2(x_{i_1}), \dots, x_{i_n} = g_n(x_{i_{n-1}})$ where $g_2, \dots, g_n$ commute with an iterate of f. Since such equations describe a periodic curve, we have that X is periodic. □

4. Proof of Theorem 1.3 for curves

In this Section we prove the following result

Theorem 4.1. *Theorem 1.3 holds when $X \subset \mathbb{A}^n$ is a curve.*

Proof. The case $n = 2$ follows from Theorem 3.2 (or equivalently, from Theorem 3.1). We will prove next the result for $n \in \{3, 4\}$ and proceed by induction for $n \geq 5$. We recall the notation and terminology from Section 2. We are given that the curve X has a Zariski dense (i.e. infinite) set of points each of which is contained in a periodic subvariety V of codimension 2. Since there are only finitely many possibilities for the weak signature, by Remark 2.3, we may assume that all of the above periodic subvarieties have a common weak signature consisting of a (possibly empty) subset $\mathcal{J} = J_V$ of I_n and a partition of $I_n \setminus \mathcal{J}$ into $n - 2$ non-empty subsets $J_1, \dots, J_{n-2}$. Let h denote the absolute logarithmic Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. We also let h denote the height on $\mathbb{A}^n(\overline{\mathbb{Q}}) \subset (\mathbb{P}^1)^n(\overline{\mathbb{Q}})$ given by

$$h(x_1, \dots, x_n) = h(x_1) + \ldots + h(x_n).$$

For each f_i, let $\widehat{h}_{f_i}$ denote the canonical height on $\mathbb{P}^1(\overline{\mathbb{Q}})$ associated to f_i, and let $\widehat{h}$ denote the function on $\mathbb{A}^n \subset (\mathbb{P}^1)^n(\overline{\mathbb{Q}})$ given by:

$$\widehat{h}(x_1, \dots, x_n) = \widehat{h}_{f_1}(x_1) + \dots + \widehat{h}_{f_n}(x_n).$$

Note that $\widehat{h}$ is the canonical height associated to φ (which is the coordinate-wise action of the polynomials f_i on $\mathbb{A}^n$). We refer the readers to [BG06] and [Sil07, Chapter 3] for more details on height and canonical height functions.

4.1. **The case when the ambient space has dimension 3.** Without loss of generality, we have the following possibilities for the weak signature $(\mathcal{J}, J_1)$:

Case A: $\mathcal{J} = \emptyset$ and $J_1 = \{1, 2, 3\}$. By part (c) of Lemma 2.2, we may assume that $f_1 = f_2 = f_3 =: f$. By Proposition 2.4 and Remark 2.3, we may assume that there are infinitely many points $\{P_i\}_{i=1}^{\infty}$ such that for each i, there is a periodic curve V_i defined by the equations $x_2 = g_{i,2}(x_1)$ and $x_3 = g_{i,3}(x_2)$ such that $P_i \in X \cap V_i$ where $g_{i,2}$ and $g_{i,3}$ are polynomials commuting with an iterate of f. If $\{\deg(g_{i,2})\}_{i\geq 1}$ has a bounded subsequence then Proposition 2.8(d) yields that there exists a polynomial g such that $g_{i,2} = g$ for infinitely many i. Hence X is contained in the periodic surface defined by $x_2 = g(x_1)$ because it is a curve containing infinitely many points from this surface. The case when $\{\deg(g_{i,3})\}_{i\geq 1}$ has a bounded subsequence is treated similarly. We now assume that

$$\lim_{i\to\infty} \deg(g_{i,2}) = \lim_{i\to\infty} \deg(g_{i,3}) = \infty.$$

Write $P_i = (a_i, b_i, c_i)$. Let $\pi_{1,2}$ denote the projection from $\mathbb{A}^3$ to the first two coordinates $\mathbb{A}^2$ and let Y be the Zariski closure of $\pi_{1,2}(X)$.

We consider the case when $\pi_{1,2}$ is non-constant on X, in other words Y is a curve in $\mathbb{A}^2$. Then there exist positive constants C_1 and C_2 depending only on the curve X such that for every point $(a, b, c) \in X(\overline{\mathbb{Q}})$, we have:

$$h(c) \leq C_1 \max\{h(a), h(b)\} + C_2. \tag{4.2}$$

Inequality (4.2) is a special case of [GN16, Lemma 3.2 (b)] (see also [GN16, Corollary 3.4]). Since $|h - \widehat{h}_f| = O(1)$, there exist positive constants C_3 and C_4 depending on X and f such that:

$$\widehat{h}_f(c) \leq C_3 \max\left\{\widehat{h}_f(a), \widehat{h}_f(b)\right\} + C_4$$

for every $(a, b, c) \in X(\overline{\mathbb{Q}})$ (see also [GN16, Corollary 3.4]). In particular, this inequality holds for the points $P_i = (a_i, b_i, c_i)$. On the other hand, we have:

$$\widehat{h}_f(c_i) = \deg(g_{i,3})\widehat{h}_f(b_i) \text{ and } \widehat{h}_f(b_i) = \deg(g_{i,2})\widehat{h}_f(a_i).$$

Overall, we have:

$$\deg(g_{i,3}) \max\left\{\widehat{h}_f(a_i), \widehat{h}_f(b_i)\right\} \leq \widehat{h}_f(c_i) \leq C_3 \max\left\{\widehat{h}_f(a_i), \widehat{h}_f(b_i)\right\} + C_4.$$

Since $\lim \deg(g_{i,3}) = \infty$, we get $\lim_{i\to\infty}(\max\left\{\widehat{h}_f(a_i), \widehat{h}_f(b_i)\right\}) = 0$ and so, Theorem 1.1 yields that the curve Y is preperiodic.

A more careful analysis shows that X is contained in a *periodic* surface, as follows. First, consider the case when the projection from X to the first or second coordinate $\mathbb{A}^1$ is constant, then this constant, denoted γ, is necessarily preperiodic since Y is preperiodic. From $c_i = g_{i,3}(b_i) = g_{i,3}(g_{i,2}(a_i))$ and Corollary 2.10, we have that c_i is periodic for all sufficiently large i and the sequence $\{c_i\}_{i\geq 1}$ consists of only finitely many points. Hence there is a periodic point ζ such that $c_i = \zeta$ for infinitely many i. We conclude that X is contained in the periodic surface $\mathbb{A}^2 \times \{\zeta\}$.

When the projection from X to neither the first nor second $\mathbb{A}^1$ is constant, by Proposition 2.4 and Remark 2.5, the preperiodic curve Y satisfies an equation of the form $g(x_1) = G(x_2)$ where g and G commute with an iterate of f. Therefore the point (a_i, b_i) satisfies both $g(a_i) = G(b_i)$ and $b_i = g_{i,2}(a_i)$.

The following observation will be used repeatedly throughout our proof.

Lemma 4.3. *With the above notation, for all i sufficiently large, we have that b_i is periodic.*

Proof of Lemma 4.3. When i is sufficiently large so that $\deg(g_{i,2}) \geq \deg(g)$, from Proposition 2.8(d), we can write $g_{i,2} = u_i \circ g$ where u_i is a polynomial commuting with an iterate of f. Therefore

$$b_i = u_i(g(a_i)) = u_i(G(b_i))$$

and Corollary 2.10(a) implies that b_i is f-periodic. □

Using that b_i is periodic along with the fact that $c_i = g_{i,3}(b_i)$, we obtain that c_i is also f-periodic (by Corollary 2.10(b)). Let Y' be the Zariski closure of the projection from X to the second and third coordinates $\mathbb{A}^2$. Since (b_i, c_i) is periodic for all sufficiently large i, we have that Y' is periodic (according to Theorem 3.1). Hence X is contained in the periodic subvariety $\mathbb{A}^1 \times Y'$.

The case when $\pi_{1,2}$ is constant on X is obvious. Indeed, $X = \{(a, b)\} \times \mathbb{A}^1$ and since $X \cap V_1 \neq \emptyset$, we have $b = g_{1,2}(a)$ and $g_{1,2}$ commutes with an iterate of f. Hence X is contained in the periodic surface defined by $x_2 = g_{1,2}(x_1)$.

Case B: $\mathcal{J} = \{1\}$ and $J_1 = \{2, 3\}$. As in Case A, we may assume that $f_2 = f_3 =: f$ and there are infinitely many points $\{P_i = (a_i, b_i, c_i)\}_{i\geq 1}$ such that for each i, there is a periodic curve V_i defined by $x_1 = \zeta_i$ and $x_3 = g_i(x_2)$ such that $P_i \in X \cap V_i$ where ζ_i is f_1-preperiodic and g_i commutes with an iterate of f. By similar arguments in Case A, we may assume $\lim_{i\to\infty} \deg(g_i) = \infty$.

When $\pi_{1,2}$ is non-constant on X, we can use similar arguments as in Case A. This time, we have an inequality of the form

$$\widehat{h}_f(c) \leq C_5 \max\left\{\widehat{h}_{f_1}(a), \widehat{h}_f(b)\right\} + C_6 \tag{4.4}$$

for every $(a,b,c) \in X(\overline{\mathbb{Q}})$ where C_5 and C_6 are constants depending only on X, f_1, and f. So we can conclude that $\lim_{i\to\infty} \widehat{h}_f(b_i) = 0$. Since Y contains the Zariski dense set $\{(a_i = \zeta_i, b_i)\}_i$, we have that Y is preperiodic, by Theorem 1.1. If the projection π_1 from X (and Y) to the first $\mathbb{A}^1$ is constant then we have $a_i = \zeta_1$ for every i and X is contained in the periodic surface $\{\zeta_1\}\times\mathbb{A}^2$. If the projection π_2 from X (and Y) to the second $\mathbb{A}^1$ is constant, then inequality (4.4) combined with the fact that $a_i = \zeta_i$ is periodic and the fact that $\lim_{i\to\infty}\deg(g_i) = \infty$ yields that b_i must be preperiodic. But then, because b_i is constant as we vary i and $\deg(g_i) \to \infty$, Corollary 2.10(c) yields that c_i must be constant *and* periodic, thus providing the desired conclusion in Theorem 4.1. If π_1 and π_2 are non-constant then Y satisfies an equation $g(x_1) = G(x_2)$, where g and G commute with an iterate of f. In particular $g(\zeta_i) = g(a_i) = G(b_i)$; so, by Corollary 2.10, $G(b_i)$ is f-periodic (note that ζ_i is periodic). When $\deg(g_i) \geq \deg(G)$, by (the proof of) part (c) of Corollary 2.10, we have that $c_i = g_i(b_i)$ is also periodic. Now the Zariski closure of the projection from X to the first and third coordinates $\mathbb{A}^2$ contains the Zariski dense set $\{(a_i, c_i) :\ i \text{ is large}\}$ of periodic points, it must be periodic thanks to Theorem 3.1. Hence X is contained in a periodic surface.

The case $\pi_{1,2}$ is constant on X is also obvious. Indeed, $X = \{(a,b)\}\times\mathbb{A}^1$ and since $X \cap V_1 \neq \emptyset$, we have that $a = \zeta_1$. Hence X is contained in the periodic surface $\{\zeta_1\}\times\mathbb{A}^2$.

Case C: $\mathcal{J} = \{1,2\}$ and $J_1 = \{3\}$. This time, each periodic curve V_i has the form $\{(\alpha_i,\beta_i)\}\times\mathbb{A}^1$ where α_i is f_1-periodic and β_i is f_2-periodic. If $\pi_{1,2}$ is non-constant on X then Theorem 3.1 implies that Y is a periodic curve in $\mathbb{A}^2$, hence X is contained in the periodic surface $Y\times\mathbb{A}^1$. If $\pi_{1,2}$ is constant on X, since $X\cap V_1 \neq \emptyset$, we have $X = V_1$ is periodic.

4.2. **The case when the ambient space has dimension 4.** We will need the following result:

Proposition 4.5. *Let $f(x), g(x) \in \overline{\mathbb{Q}}[x]$ with $\deg(f) = \deg(g) =: d \geq 2$. Let $C \subset \mathbb{A}^2$ be an irreducible $\overline{\mathbb{Q}}$-curve with the following properties:*

- *C is non-fibered.*
- *There exist $\alpha,\beta \in \overline{\mathbb{Q}}$ such that $C \cap (\mathcal{O}_f(\alpha)\times\mathcal{O}_g(\beta))$ is infinite.*

Then C is periodic under the action $(x_1,x_2)\mapsto(f(x_1),g(x_2))$.

Proof. As in Cases A and B (see also [GN16, Corollary 3.4]), since C is non-fibered there exist positive constants C_7 and C_8 depending on C, f, and g such that for each $(a_1,a_2) \in C(\overline{\mathbb{Q}})$, we have

$$\max\{\widehat{h}_f(a_1), \widehat{h}_g(a_2)\} \leq C_7 \min\{\widehat{h}_f(a_1), \widehat{h}_g(a_2)\} + C_8. \tag{4.6}$$

Now, since $C\cap(\mathcal{O}_f(\alpha)\times\mathcal{O}_g(\beta))$ is infinite and C projects dominantly to both coordinates, we get that α (respectively β) is not f-preperiodic (respectively g-preperiodic). Hence $\widehat{h}_f(\alpha) > 0$ and $\widehat{h}_g(\beta) > 0$. From this observation, inequality (4.6) for each point $(f^m(\alpha), g^n(\beta)) \in C(\overline{\mathbb{Q}})$, and the fact that

$\widehat{h}_f(f^m(\alpha)) = d^m\widehat{h}_f(a)$ and $\widehat{h}_g(g^n(\beta)) = d^n\widehat{h}_g(\beta)$, we conclude that $|m-n|$ is uniformly bounded as we vary among all points $(f^m(\alpha), g^n(\beta)) \in C(\overline{\mathbb{Q}})$. Therefore, there exists an integer ℓ such that there exist infinitely many $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$ with the property that $(f^m(\alpha), g^n(\beta)) \in C(\overline{\mathbb{Q}})$ and also $m-n=\ell$. Without loss of generality, we assume that $\ell \geq 0$, and therefore get that C contains infinitely many points from the orbit of $\left(f^\ell(\alpha), \beta\right)$ under the action of $(x_1,x_2) \mapsto (f(x_1), g(x_2))$. Since the Dynamical Mordell-Lang Conjecture (see [BGT16, Chapter 3]) is known in the case of endomorphisms of $\mathbb{A}^2$ (as proven in [Xie]), we conclude that C is periodic under the action of $(x_1,x_2) \mapsto (f(x_1), g(x_2))$, as desired. ☐

We now return to the proof of Theorem 4.1. We have the following cases for the weak signature $(\mathcal{J}, J_1, J_2)$:

Case D: $|J_1| = 1$ or $|J_2| = 1$. Without loss of generality, assume $|J_2| = 1$, more specifically $J_2 = \{4\}$. Now there are infinitely many points $\{P_i = (a_i, b_i, c_i, d_i)\}_{i\geq 1}$ such that for each i, there is a periodic surface V_i such that $P_i \in X \cap V_i$. Moreover, we have that $V_i = W_i \times \mathbb{A}^1$ where W_i is a periodic curve under the self-map $f_1 \times f_2 \times f_3$ of $\mathbb{A}^3$.

Let $\pi_{1,2,3}$ denote the projection from $\mathbb{A}^4$ to the first three coordinates $\mathbb{A}^3$. If $\pi_{1,2,3}$ is non-constant on X, then the Zariski closure Y of $\pi_{1,2,3}(X)$ in $\mathbb{A}^3$ is a curve and we can apply Theorem 4.1 to the data $(n=3, f_1, f_2, f_3, Y)$ to conclude that Y is contained in a periodic surface S in $\mathbb{A}^3$. Hence X is contained in the periodic hypersurface $S \times \mathbb{A}^1$. The case $\pi_{1,2,3}$ is constant on X is obvious. We have that $X = \{(a,b,c)\} \times \mathbb{A}^1$. Since $P_i = (a_i, b_i, c_i, d_i) = (a,b,c,d_i)$ lies in $V_i = W_i \times \mathbb{A}^1$, we have that X itself is contained in the periodic subvariety V_i (for every i).

Case E: $|J_1| = |J_2| = 2$. Without loss of generality, assume $J_1 = \{1,2\}$ and $J_2 = \{3,4\}$. As in Case A, we may assume $f_1 = f_2 =: f$ and $f_3 = f_4 =: g$. By Proposition 2.4 and without loss of generality, we may assume that there are infinitely many points $\{P_i = (a_i, b_i, c_i, d_i)\}_{i\geq 1}$ such that for each i, there is a periodic surface V_i defined by $x_2 = U_i(x_1)$ and $x_4 = T_i(x_3)$ such that $P_i \in X \cap V_i$ and $U_i(x)$ (respectively $T_i(x)$) commutes with an iterate of $f(x)$ (respectively $g(x)$). For such polynomials $U_i(x)$ and $T_i(x)$, and for any $a \in \overline{\mathbb{Q}}$, we have (see [Ngu13, Lemma 3.3]):

$$\widehat{h}_f(U_i(a)) = \deg(U_i)\widehat{h}_f(a),\ \widehat{h}_g(T_i(a)) = \deg(T_i)\widehat{h}_g(a). \tag{4.7}$$

As in Case A, we may assume that $\lim_{i\to\infty}\deg(U_i) = \lim_{i\to\infty}\deg(T_i) = \infty$. Let $\pi_{1,3}$ denote the projection from $\mathbb{A}^4$ to the first and third coordinates $\mathbb{A}^2$ and let Y denote the Zariski closure of $\pi_{1,3}(X)$.

We consider first the case when $\pi_{1,3}$ is non-constant on X, in other words Y is a curve in $\mathbb{A}^2$.

As in Case A, there are positive constants C_9 and C_{10} depending only on X and f such that for every point $(a,b,c,d) \in X(\overline{\mathbb{Q}})$, we have:

$$\widehat{h}_f(b) + \widehat{h}_g(d) \leq C_9(\widehat{h}_f(a) + \widehat{h}_g(c)) + C_{10}.$$

Combining with (4.7) and the fact that $P_i = (a_i, b_i, c_i, d_i) \in X \cap V_i$, we have:

$$(\deg(U_i) - C_9)\widehat{h}_f(a_i) + (\deg(T_i) - C_9)\widehat{h}_g(b_i) \leq C_{10}.$$

Since $\lim_{i\to\infty} \deg(U_i) = \lim_{i\to\infty} \deg(T_i) = \infty$, we get that $\lim_{i\to\infty} \widehat{h}_f(a_i) = \lim_{i\to\infty} \widehat{h}_g(c_i) = 0$. By Theorem 1.1, the curve Y is preperiodic under the map $(x_1, x_3) \mapsto (f(x_1), g(x_3))$. A more carefuly analysis shows that X is contained in a periodic subvariety as follows.

When the projection from X to the first (or respectively the third) coordinate is constant, then this constant is necessarily preperiodic since Y is preperiodic. Since $b_i = U_i(a_i)$ (respectively $d_i = T_i(c_i)$), we can argue as in Case A to conclude that there is an f-periodic point (respectively g-periodic point) ζ such that $b_i = \zeta$ (respectively $d_i = \zeta$) for infinitely many i. Hence X is contained in the periodic surface $\mathbb{A}^1 \times \{\zeta\} \times \mathbb{A}^2$ (respectively $\mathbb{A}^3 \times \{\zeta\}$).

Now consider the case when the projection from X to both the first and third coordinates is non-constant, or equivalently Y is a non-fibered curve in $\mathbb{A}^2$. This implies $f \approx g$. By Lemma 2.2, we may assume that $f = g$ (i.e. $f_1 = f_2 = f_3 = f_4 = f$). Remark 2.5 gives that Y satisfies an equation of the form $g(x_1) = G(x_3)$ where g and G commute with an iterate of f. In particular $b_i = U_i(a_i)$, $d_i = T_i(c_i)$, and $g(a_i) = G(c_i)$. When i is sufficiently large so that $\deg(U_i) \geq \deg(g)$ and $\deg(T_i) \geq \deg(G)$, we can write:

$$U_i = U_i^* \circ g \text{ and } T_i = T_i^* \circ G$$

where U_i^* and T_i^* commute with an iterate of f. Obviously, either $\deg(U_i^*) \geq \deg(T_i^*)$ or $\deg(T_i^*) \geq \deg(U_i^*)$. By restricting to an infinite subsequence of $\{P_i\}$ and without loss of generality, we may assume that $\deg(T_i^*) \geq \deg(U_i^*)$ for every i. From Proposition 2.8, we can write $T_i^* = S_i \circ U_i^*$ where S_i commutes with an iterate of f. We have:

$$d_i = T_i(c_i) = T_i^*(G(c_i)) = T_i^*(g(a_i)) = S_i(U_i^*(g(a_i))) = S_i(U_i(a_i)) = S_i(b_i).$$

If $\{\deg(S_i)\}_i$ has a bounded subsequence then by similar arguments as before, X would be contained in a periodic surface of the form $x_4 = S(x_2)$ and we are done. Now assume $\lim_{i\to\infty} \deg(S_i) = \infty$. Since the projection from X to the first 3 coordinates is non-constant, there exist C_{11} and C_{12} such that:

$$\widehat{h}_f(d_i) \leq C_{11} \max\{\widehat{h}_f(a_i), \widehat{h}_f(b_i), \widehat{h}_f(c_i)\} + C_{12}.$$

On the other hand:

$$\widehat{h}_f(d_i) = \deg(T_i)\widehat{h}_f(c_i),$$
$$\widehat{h}_f(d_i) = \deg(S_i)\widehat{h}_f(b_i) = \deg(S_i)\deg(U_i)\widehat{h}_f(a_i)$$

and $\{\deg(S_i)\}_i$, $\{\deg(T_i)\}_i$, and $\{\deg(U_i)\}_i$ become arbitrarily large; so, we conclude that

$$\lim_{i\to\infty} \widehat{h}_f(a_i) = \lim_{i\to\infty} \widehat{h}_f(b_i) = \lim_{i\to\infty} \widehat{h}_f(c_i) = 0.$$

By Theorem 1.1, the Zariski closure Z of the projection from X to the first 2 coordinates $\mathbb{A}^2$ is preperiodic. We are assuming that the projection from

X to the first coordinate is non-constant. If the projection to the second coordinate is constant then it must be preperiodic (since Z is preperiodic), denoted γ. Now $d_i = S_i(b_i) = S_i(\gamma)$ and we can argue as in Case A to conclude that X is contained in a periodic hypersurface of the form $\mathbb{A}^3 \times \{\zeta\}$. It remains to treat the case when the projection to the second coordinate is non-constant. Then Z satisfies an equation $g^*(x_1) = G^*(x_2)$ where g^* and G^* commute with an iterate of f. By similar arguments as in Case A (see Lemma 4.3), we conclude that b_i is f-periodic when i is sufficiently large, and so, $d_i = S_i(b_i)$ is also f-periodic. Then Theorem 3.1 implies that the projection from X to the second and fourth coordinates axes is a periodic curve and we are done since we obtain that X is contained in the periodic (irreducible) hypersurface in $\mathbb{A}^4$, which is the pullback of the aforementioned periodic plane curve under the projection map $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_4)$.

Finally, we treat the case when $\pi_{1,3}$ is constant on X.

Write $\{(\alpha, \gamma)\} = \pi_{1,3}(X)$, hence $(a_i, c_i) = (\alpha, \gamma)$ for every i. If α is f-preperiodic then for all i sufficiently large, we get that $b_i = U_i(a_i) = U_i(\gamma)$ must be some given periodic point β and thus, X is contained in the periodic hypersurface $\mathbb{A}^1 \times \{\beta\} \times \mathbb{A}^2$ and hence, we are done. Therefore we may assume that α (respectively γ) is not f-preperiodic (respectively g-preperiodic). Hence $\widehat{h}_f(\alpha) > 0$ and $\widehat{h}_g(\gamma) > 0$. From (4.7) and the fact that

$$\lim_{i\to\infty} \deg(U_i) = \lim_{i\to\infty} \deg(T_i) = \infty,$$

we conclude that $\lim_{i\to\infty} \widehat{h}_f(b_i) = \lim_{i\to\infty} \widehat{h}_g(d_i) = \infty$. Consequently, X projects dominantly to both the second and fourth coordinates of $\mathbb{A}^4$. Let X' be the curve in $\mathbb{A}^2$ which is the Zariski closure of the image of X under the projection to the second and fourth coordinates.

From Proposition 2.8, we can write

$$U_i = f^{m_i} \circ u_i, \; T_i = g^{n_i} \circ t_i$$

where $m_i, n_i \in \mathbb{N}_0$, u_i (respectively t_i) commutes with an iterate of f (respectively g) and $\max\{\deg(u_i), \deg(t_i)\} \leq \deg(f) = \deg(g)$. From Proposition 2.8 again, there are only finitely many possibilities for the pair (u_i, t_i). Hence there exists $u(x)$ and $t(x)$ such that $(u_i, t_i) = (u, t)$ for infinitely many i. Overall, the curve X' in $\mathbb{A}^2$ satisfies the following properties:

- X' is non-fibered.
- $X' \cap (\mathcal{O}_f(u(\alpha)) \times \mathcal{O}_g(t(\beta)))$ is infinite.

By Proposition 4.5, X' is periodic under the map $(x_2, x_4) \mapsto (f(x_2), g(x_4))$. Therefore X is contained in the periodic hypersurface

$$\{(x_1, x_2, x_3, x_4) : \; (x_2, x_4) \in X'\}$$

and we finish the proof of this case.

4.3. **The case when the ambient space has dimension larger than 4.** Let $N \geq 5$, assume Theorem 4.1 holds for $n \leq N-1$. We now consider $n = N$. Note that the common weak signature $(\mathcal{J}, J_1, \dots, J_{n-2})$ of the V_i's is a partition of $\{1, \dots, n\}$ for which $\mathcal{J}$ could possibly be empty while each J_j is non-empty. Since $2(n-2) > n$, there must be some j such that $|J_j| = 1$. Without loss of generality, assume $J_{n-2} = \{n\}$. We can now proceed as in Case D: if the projection from X to the first $(n-1)$ coordinates is non-constant then we reduce to $n = N-1$ and apply the induction hypothesis, otherwise we can easily conclude that X is contained in V_i for every i. This finishes the proof of Theorem 4.1. $\square$

5. Proof of Theorem 1.3 for subvarieties of codimension 2

Theorem 1.3 is proven once we deal with the last case of it, which is contained in the following result:

Theorem 5.1. *Theorem 1.3 holds if $X \subset \mathbb{A}^n$ has codimension 2.*

Proof. Here we are assuming that the intersection between X and the union of all periodic curves is Zariski dense in X and we need to prove that X is contained in a periodic hypersurface of $\mathbb{A}^n$. We argue by induction on n; the case $n = 2$ is trivial while the case $n = 3$ was proven in Theorem 3.1. We assume $n \geq 4$ from now on.

By using Remark 2.3 as in the proof of Theorem 4.1, we can assume that all of the above periodic curves have a common weak signature J_1 which is assumed to be $\{1, \dots, s\}$ where $1 \leq s \leq n$. By Lemma 2.2, Remark 2.3, and Proposition 2.4, we may assume that $f_1 = \ldots = f_s =: f$ and there are periodic curves $\{V_m\}_{m\geq 1}$ (in $\mathbb{A}^n$) such that the following hold:

(a) there is a Zariski dense set of points $\{P_m\}_{m\geq 1}$ in X such that $P_m \in X \cap V_m$ for every m, and
(b) each V_m is defined by equations $x_2 = g_{m,1}(x_1), \dots, x_s = g_{m,s-1}(x_{s-1})$ where the $g_{m,i}$'s are polynomials commuting with an iterate of f, along with equations $x_{s+1} = a_{m,s+1}, \dots, x_n = a_{m,n}$ where each $a_{m,i}$ is f_i-periodic for $s+1 \leq i \leq n$.

Write

$$P_m = (b_{m,1}, \dots, b_{m,n})$$

with $b_{m,j+1} = g_{m,j}(b_{m,j})$ for $1 \leq j \leq s-1$ and $b_{m,j} = a_{m,j}$ for $s+1 \leq j \leq n$.

By restricting to a subsequence, we may assume that $\{P_m\}_{m\geq 1}$ is generic which means that every subsequence is Zariski dense in X. This is possible, as follows. First we enumerate all the countably many strictly proper irreducible $\overline{\mathbb{Q}}$-subvarieties of X as $\{Z_1, Z_2, \dots\}$. Then we let $m_0 := 0$, let P_{m_1} be the first point in the sequence $\{P_m\}_{m>m_0}$ which is not contained in Z_1, let P_{m_2} be the first point in the sequence $\{P_m\}_{m>m_1}$ that is not contained in $Z_1 \cup Z_2$, and so on. The subsequence $\{P_{m_k}\}_{k\geq 1}$ is generic in X.

If for some $i \in \{s+1, \dots, n\}$, the projection from X to the i-th coordinate axis $\mathbb{A}^1$ is constant, then X is contained in the periodic hypersurface $x_i = a_{1,i}$ and we are done.

So, from now on, we may assume that each projection of X on the coordinate axes $x_{s+1}, \dots, x_n$ is not constant.

In particular, this means that for every $i \in \{s+1, \dots, n\}$ and any f_i-periodic point ζ, there are at most finitely many m's such that $a_{m,i} = \zeta$; otherwise an infinite subsequence of $\{P_m\}$ is contained in the hypersurface $x_i = \zeta$. Since $\{P_m\}_m$ is generic, X is also contained in $x_i = \zeta$, violating our assumption.

Claim 5.2. *Theorem 5.1 holds when $s = 1$.*

Proof. Since $s = 1$, each V_m is of the form

$$\mathbb{A}^1 \times (a_{m,2}, \dots, a_{m,n}).$$

We project X to the last $n-1$ coordinate axes and thus obtain a subvariety $X_1 \subset \mathbb{A}^{n-1}$ of codimension 1 or 2. Furthermore, according to our hypothesis, X_1 contains a Zariski dense set of periodic points $(a_{i,2}, \dots, a_{i,n})$; thus Theorem 3.2 yields that X_1 is periodic, hence X is contained in a periodic subvariety. □

From now on, we assume $2 \le s \le n$. Furthermore, as argued in the proof of Theorem 4.1, we may assume that for $j = 1, \dots, s-1$, we have $\deg(g_{m,j}) \to \infty$ as $m \to \infty$.

Claim 5.3. *Theorem 5.1 holds if X does not project dominantly onto the s-th coordinate $\mathbb{A}^1$ of $\mathbb{A}^n$.*

Proof of Claim 5.3. Let b_s be the image of the constant projection from X to the s-th coordinate $\mathbb{A}^1$ and let $\pi_{(s)}$ be the projection from X to the remaining $n-1$ coordinates $\mathbb{A}^{n-1}$. Let $X_{(s)}$ be the Zariski closure of $\pi_{(s)}(X)$.

For each m we have that $V_m \cap X$ contains some point $(b_{m,1}, \dots, b_{m,n})$ such that for $i = 1, \dots, s-1$, we have

$$\widehat{h}_f(b_{m,i}) = \frac{\widehat{h}_f(b_s)}{\prod_{j=i}^{s-1} \deg(g_{m,j})} \to 0 \text{ as } m \to \infty.$$

Since also for $i = s+1, \dots, n$ we have $\widehat{h}_f(b_{m,i}) = \widehat{h}_f(a_{m,i}) = 0$, we conclude that $X_{(s)}$ contains a Zariski dense set of points of canonical height converging to 0. Thus Theorem 1.1 yields that $X_{(s)}$ is preperiodic. A more careful analysis shows that X is contained in a proper *periodic* subvariety, as follows.

Since $\dim(X_{(s)}) = \dim(X)$, we have that $X_{(s)}$ is a hypersurface in $\mathbb{A}^{n-1}$. By Remark 2.6, there exist $i < j$ in $\{1, \dots, s-1, s+1, \dots, n\}$ and an irreducible curve C in $\mathbb{A}^2$ that is preperiodic under $(x_i, x_j) \mapsto (f_i(x_i), f_j(x_j))$ such that $X_{(s)} = \pi^{-1}(C)$ where π is the projection to the i-th and j-th coordinate axes, i.e.,

$$(x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_n) \longrightarrow (x_i, x_j). \tag{5.4}$$

We have several cases (note that the projection from X to each of the ℓ-th coordinate $\mathbb{A}^1$ for $\ell \in \{s+1, \dots, n\}$ is non-constant):

(i) $i, j \in \{s+1, \dots, n\}$. Then the curve C contains the Zariski dense set of periodic points $\{(a_{m,i}, a_{m,j})\}_m$. By Theorem 3.1, C is periodic. Hence $X_{(s)}$ is periodic and X is contained in the periodic hypersurface $\pi_{(s)}^{-1}(X_{(s)})$.

(ii) $i, j \in \{1, \dots, s-1\}$ and the curve C is fibered. Hence there exists an f-preperiodic point γ such that X is contained in the hypersurface $x_i = \gamma$, say. From $b_s = b_{m,s} = g_{m,s-1} \circ \dots \circ g_{m,i}(\gamma)$ and Corollary 2.10, by choosing sufficiently large m, we have that b_s is f-periodic. Hence X is contained in the periodic hypersurface $x_s = b_s$.

(iii) $i, j \in \{1, \dots, s-1\}$ and the curve C is non-fibered. By Remark 2.5, C satisfies an equation $g(x_i) = G(x_j)$ where g and G commute with an iterate of f. As in Case A in Section 4 (see Lemma 4.3), we have that $b_{m,j}$ is f-periodic when m is sufficiently large (see Lemma 4.3). Then $b_s = b_{m,s} = g_{m,s-1} \circ \dots \circ g_{m,j}(b_{m,j})$ is f-periodic and we are done.

(iv) $i \in \{1, \dots, s-1\}$, $j \in \{s+1, \dots, n\}$, and the curve C is fibered. We can use the same arguments as in Case (ii) above since we know C must project dominantly onto the x_j coordinate axis and therefore, we must have that the curve C is given by an equation of the form $x_i = \gamma$, for a preperiodic point γ.

(v) $i \in \{1, \dots, s-1\}$, $j \in \{s+1, \dots, n\}$, and the curve C is non-fibered. Then $f_i \approx f_j$. By Lemma 2.2, we may assume that $f_j = f_i = f$. Now C satisfies an equation $g(x_i) = G(x_j)$ as in Case (iii). Hence $g(b_{m,i}) = G(a_{m,j})$ is f-periodic. By choosing m sufficiently large such that $\deg(g_{m,s-1} \circ \dots \circ g_{m,i}) \geq \deg(g)$, we conclude that $b_s = b_{m,s} = g_{m,s-1} \circ \dots \circ g_{m,i}(b_{m,i})$ is periodic.

This finishes the proof of Claim 5.3. □

From now on, in the proof of Theorem 5.1 we assume that X projects dominantly onto the s-th axis.

Let $\pi_{(s)}$ and $X_{(s)}$ be as in the proof of Claim 5.2. We still have 2 more cases: $\dim(X_{(s)}) = n-3$ or $\dim(X_{(s)}) = n-2$.

Claim 5.5. *Theorem 5.1 holds if* $\dim(X_{(s)}) = n-3$

Proof of Claim 5.5. In this case, we have that $X = X_{(s)} \times \mathbb{A}^1$ (where the factor $\mathbb{A}^1$ comes from the s-th coordinate). Furthermore, by our assumption, we know that $X_{(s)}$ has a Zariski dense intersection with periodic curves of $\mathbb{A}^{n-1}$ given by the equations:

$$x_2 = g_{m,1}(x_1),\ x_3 = g_{m,2}(x_2), \cdots, x_{s-1} = g_{m,s-1}(x_{s-2})$$

and the equations

$$x_{s+1} = a_{m,s+1},\ x_{s+2} = a_{m,s+2}, \cdots, x_n = a_{m,n}.$$

In other words, $X_{(s)}$ has a dense intesection with $\mathrm{Per}^{[n-2]} \subset \mathbb{A}^{n-1}$. By the inductive hypothesis, we conclude that $X_{(s)}$ is contained in a strictly proper periodic subvariety of $\mathbb{A}^{n-1}$, and so is X. □

From now on, in the proof of Theorem 5.1 we may assume $\dim(X_{(s)}) = n - 2 = \dim(X)$.

Then there is a strictly smaller Zariski closed subset $Y_{(s)}$ of $X_{(s)}$ such that for $Y := \pi^{-1}(Y_{(s)})$, the induced morphism from $X \setminus Y$ to $X_{(s)} \setminus Y_{(s)}$ is finite. At the expense of removing finitely many pairs (P_m, V_m)'s for which $P_m \in Y$, we may assume that $P_m \in V_m \cap (X \setminus Y)$ for every m (note that the sequence of points $\{P_m\}$ is generic in X).

Since the map from $X \setminus Y$ to $X_{(s)} \setminus Y_{(s)}$ is finite, by [GN16, Corollary 3.4] there are constants $c_0, \dots, c_{s-1}, c_{s+1}, \dots c_n$ such that for each $m \in \mathbb{N}$ we have the inequality:

$$\widehat{h}_f(b_{m,s}) \le c_0 + \sum_{\substack{1\le i\le n\\ i\ne s}} c_i \widehat{h}_f(b_{m,i}). \tag{5.6}$$

Using the fact that for each $i = 1, \dots, s-1$, we have

$$\widehat{h}_f(b_{m,i}) = \frac{\widehat{h}_f(b_{m,s})}{\prod_{j=i}^{s-1} \deg(g_{m,j})}, \tag{5.7}$$

while for each $i = s+1, \dots, n$, we have that $\widehat{h}_f(b_{m,i}) = \widehat{h}_f(a_{m,i}) = 0$. Combining (5.7) with (5.6) and with the fact that $\deg(g_{m,i}) \to \infty$ as $m \to \infty$ for each $i = 1, \dots, s-1$, we conclude that

$$\lim_{m\to\infty} \widehat{h}_f(b_{m,i}) = 0 \text{ for each } i = 1, \dots, s-1. \tag{5.8}$$

So, $X_{(s)}$ contains a Zariski dense set of points of small height, i.e., the points $(b_{m,1}, \dots, b_{m,s-1}, b_{m,s+1}, \dots, b_{m,n})$. Then Theorem 1.1 yields that $X_{(s)}$ is preperiodic.

As in the proof of Claim 5.3, there exist $i < j$ in $\{1, \dots, s-1, s+1, \dots, n\}$ and a preperiodic curve C in $\mathbb{A}^2$ such that $X_{(s)} = \pi^{-1}(C)$ where π is the projection to the i-th and j-th coordinate axes, as in (5.4). We have cases (i)-(v) as in the proof of Claim 5.3. Case (i) can be handled by the exact same arguments. On the other hand, cases (ii) and (iv) cannot occur under the hypothesis that X projects dominantly onto the s-th coordinate axis. Indeed, in both those two cases (ii) and (iv) we would have that C is fibered, given by some equation $x_i = \gamma$ (or $x_j = \gamma$) for some i (or j) in $\{1, \dots, s-1\}$ and some preperiodic point γ. But then (without loss of generality) $b_{m,i} = \gamma$ for each m and so,

$$b_{m,s} = (g_{m,s-1} \circ \cdots \circ g_{m,i})(b_{m,i}) = (g_{m,s-1} \circ \cdots \circ g_{m,i})(\gamma)$$

takes only finitely many values as we vary m by Corollary 2.10. However, the points $\{P_m\}$ are dense in X and X projects dominantly onto the s-th coordinate axis, contradiction. Therefore, we are left to analyze only cases (iii) and (v) appearing in the proof of Claim 5.3.

In cases (iii) and (v), we have that $b_{m,s}$ is periodic when m is large; by removing finitely many m's, we may assume that $b_{m,s}$ is periodic for every m. For any $k \in \{1, \dots, s-1\}$, from $b_{m,s} = g_{m,s-1} \circ \dots \circ g_{m,k}(b_{m,k})$, we have that $b_{m,k}$ is f-preperiodic. Therefore, using again that each $b_{m,k} = a_{m,k}$ is periodic for $k > s$, Theorem 1.1 yields that X is preperiodic becaue it contains a Zariski dense set of preperiodic points. From the discussion in Section 2, we know that X is a product of preperiodic curves. Since $\dim(X) = n-2$ and $X_{(s)} = C \times \mathbb{A}^{n-3}$ (the factor $\mathbb{A}^{n-3}$ comes from all the ℓ-axes where $\ell \in \{1, \dots, n\} \setminus \{i, j, s\}$), we only have two possibilities.

Case F: The first possibility is that $X = C' \times \mathbb{A}^{n-3}$ where C' is a preperiodic curve in $\mathbb{A}^3$ which is also the projection from X to the i-th, j-th, and s-th axes (hence C is the projection from C' to the i-th and j-th axes $\mathbb{A}^2$). Now in both cases (iii) and (v) from the proof of Claim 5.3, we have that $b_{m,j}$ is periodic for all (sufficiently large) m. Consequently, the projection from X to the j-th axis together with the s-th axis is a curve containing the Zariski dense set of periodic points $(b_{m,j}, b_{m,s})_m$. Therefore this projection is a periodic curve by Theorem 3.1. Hence X lies in the periodic hypersurface which is the inverse image in $\mathbb{A}^n$ of this periodic plane curve under the projection map $(x_1, \dots, x_n) \mapsto (x_j, x_s)$.

Case G: The second possibility is that there exist $\ell \in \{1, \dots, n\} \setminus \{i, j, s\}$ such that $X = C \times C'' \times \mathbb{A}^{n-4}$ where C'' is a preperiodic curve in $\mathbb{A}^2$ which is also the projection from X to the s-th and ℓ-th axes and the factor $\mathbb{A}^{n-4}$ comes from the k-th axes for $k \in \{1, \dots, n\} \setminus \{i, j, s, \ell\}$. Now if $\ell \in \{s+1, \dots, n\}$ then we have $b_{m,\ell} = a_{m,\ell}$ is periodic, hence the curve C'' contains the Zariski dense set of periodic points $(b_{m,s}, b_{m,\ell})_m$. From Theorem 3.1, we have that C'' is periodic and we are done since then X is contained in the periodic hypersurface $\mathbb{A}^2 \times C'' \times \mathbb{A}^{n-4}$.

From now on, in the proof of Theorem 5.1 we assume that $\ell \in \{1, \dots, s\}$.

If the projection from C'' to the ℓ-th coordinate is constant then we derive a contradiction. Indeed, then $x_\ell = \gamma$ where γ is f-preperiodic. From $b_{m,s} = g_{m,s-1} \circ \dots \circ g_{m,\ell}(\gamma)$, we obtain that the s-th coordinates $b_{m,s}$ of the points P_m must belong to a finite set, contradicting thus the fact that these points are dense in X, which is a variety projecting dominantly onto the s-th coordinate axis.

So, from now on, we may assume that C'' is non-fibered (note that we are already working under the assumption that X projects dominantly onto the s-th coordinate axis).

Therefore C'' satisfies an equation $U(x_s) = T(x_\ell)$ where U and T commute with an iterate of f. It remains to treat case (iii) or case (v) in the proof of Claim 5.3. In either case, we may assume that $f_j = f$ and C satisfies an equation $g(x_i) = G(x_j)$ where g and G commute with an iterate of f. As in the proof of Claim 5.3, we have that $b_{m,j}$ is f-periodic for all large m. Hence both $T(b_{m,\ell}) = U(b_{m,s})$ and $g(b_{m,i}) = G(b_{m,j})$ are f-periodic for all large m.

If $i < \ell$, we have $b_{m,\ell} = g_{m,\ell-1} \circ \dots \circ g_{m,i}(b_{m,i})$. Therefore when m is large enough so that $\deg(g_{m,\ell-1} \circ \dots \circ g_{m,i}) \geq \deg(g)$, we have that $b_{m,\ell}$ is periodic (see Lemma 4.3). Consequently, the curve C'' is periodic since it contains a Zariski dense set of periodic points $(b_{m,\ell}, b_{m,s})$. Similarly, if $\ell < i$, when m is large so that $\deg(g_{m,i-1} \circ \dots \circ g_{m,\ell}) \geq \deg(T)$, we have $b_{m,i}$ is periodic (again using Lemma 4.3), hence C is periodic because it contains a Zariski dense set of periodic points $(b_{m,i}, b_{m,j})$. This finishes the proof of Theorem 5.1. □

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