DYNAMICAL ANOMALOUS SUBVARIETIES: STRUCTURE AND BOUNDED HEIGHT THEOREMS

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Abstract. According to Medvedev and Scanlon [MS14], a polynomial $f(x) \in \bar{\mathbb{Q}}[x]$ of degree $d \geq 2$ is called disintegrated if it is not conjugate to x^d or to $\pm C_d(x)$ (where C_d is the Chebyshev polynomial of degree d). Let $n \in \mathbb{N}$, let $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degrees at least 2, and let $\varphi = f_1 \times \cdots \times f_n$ be the corresponding coordinate-wise self-map of $(\mathbb{P}^1)^n$. Let X be an irreducible subvariety of $(\mathbb{P}^1)^n$ of dimension r defined over $\bar{\mathbb{Q}}$. We define the φ-*anomalous* locus of X which is related to the φ-*periodic* subvarieties of $(\mathbb{P}^1)^n$. We prove that the φ-anomalous locus of X is Zariski closed; this is a dynamical analogue of a theorem of Bombieri, Masser, and Zannier [BMZ07]. We also prove that the points in the intersection of X with the union of all irreducible φ-periodic subvarieties of $(\mathbb{P}^1)^n$ of codimension r have bounded height outside the φ-anomalous locus of X; this is a dynamical analogue of Habegger's theorem [Hab09a] which was previously conjectured in [BMZ07]. The slightly more general self-maps $\varphi = f_1 \times \cdots \times f_n$ where each $f_i \in \bar{\mathbb{Q}}(x)$ is a disintegrated rational function are also treated at the end of the paper.

1. Introduction

Throughout this paper, a variety is always over $\bar{\mathbb{Q}}$ and is defined as in [BG06, Appendix A.4]. In other words, a variety is the set of closed points of a (not necessarily irreducible) reduced and separated scheme of finite type over $\bar{\mathbb{Q}}$ equipped with the Zariski topology, sheaf of regular functions, etc. For a map μ from a set to itself and for every positive integer m, we let μ^m denote the m-fold iterate: $\mu \circ \ldots \circ \mu$; the notation μ^0 denotes the identity map. Let h denote the absolute logarithmic Weil height on $\mathbb{P}^1$ (see [BG06, Chapter 1] or [HS00, Part B]). Let n be a positive integer, we define the height function h_n on $(\mathbb{P}^1)^n$ by $h_n(a_1, \ldots, a_n) := h(a_1) + \ldots + h(a_n)$. When we say that a subset of $(\mathbb{P}^1)^n$ has bounded height, we mean boundedness with respect to h_n.

After a series of papers [BMZ06], [BMZ08] and [BMZ07] following the seminal work [BMZ99, Theorem 1], Bombieri, Masser and Zannier define anomalous subvarieties in $\mathbb{G}_{\mathrm{m}}^n$ as follows. By a *special* subvariety of $\mathbb{G}_{\mathrm{m}}^n$, we mean a translate of an irreducible algebraic subgroup. For any irreducible subvariety $X \subseteq \mathbb{G}_{\mathrm{m}}^n$ of dimension r, an irreducible subvariety Y of X is said to be *anomalous* (or better, X-anomalous) if there exists a special subvariety Z satisfying the following conditions:

$$Y \subseteq X \cap Z \text{ and } \dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}. \tag{1}$$

1991 *Mathematics Subject Classification.* 11G50, 37P15.

Key words and phrases. dynamics, Bounded Height, Structure Theorem, Medvedev-Scanlon theorem.

The first author is partially supported by an NSERC grant. The second author is partially supported by a fellowship from the Pacific Institute for the Mathematical Sciences.

We define $X^{\mathrm{oa}} := X \setminus \bigcup_Y Y$, where Y ranges over all anomalous subvarieties of X. We let $\mathbb{G}_m^{n,[r]}$ be the union of all algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$ of *codimension* r. The following has been established by Bombieri, Masser, Zannier [BMZ07, Theorem 1.4] and Habegger [Hab09a, Theorem 1.2] (after being previously conjectured in [BMZ07]):

Theorem 1.1. *Let X be an irreducible subvariety of $\mathbb{G}_m^n$ of dimension r (defined over $\bar{\mathbb{Q}}$, as always). We have:*

(a) (Bombieri-Masser-Zannier) Structure Theorem: the set X^{oa} is Zariski open in X. Moreover, there exists a finite collection $\mathcal{T}$ of subtori of $\mathbb{G}_{\mathrm{m}}^n$ (depending on X) such that the anomalous locus of X is the union of all anomalous subvarieties Y of X for which there exists a translate Z of a tori in $\mathcal{T}$ satisfying $Y \subseteq X \cap Z$ and $\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}$.

(b) (Habegger) Bounded Height Theorem: the set $X^{\mathrm{oa}} \cap \mathbb{G}_m^{n,[r]}$ has bounded height.

The Bounded Height Theorem is closely related to the problem of unlikely intersections in arithmetic geometry introduced in [BMZ99] whose motivation comes from the classical Manin-Mumford conjecture (which is Raynaud's theorem [Ray83a, Ray83b] for abelian varieties and Laurent's theorem [Lau84] for $\mathbb{G}_{\mathrm{m}}^n$). Moreover, Pink [Pin] and Zilber [Zil02] independently propose a similar problem to the unlikely intersection problem introduced in [BMZ99] in the more general context of semiabelian varieties and mixed Shimura varieties. For an excellent treatment of these topics, we refer the readers to Zannier's book [Zan12]. Both the Bounded Height Theorem and the Pink-Zilber problem are also considered in the context of function fields in [CGMM13]. On the other hand, very little is known in the context of arithmetic dynamics. Zhang [Zha06] proposed a dynamical analogue of the Manin-Mumford conjecture, which was later amended in [GTZ11] and more recently in [YZ].

This paper is the first to establish a dynamical analogue of Theorem 1.1. For $d \geq 2$, the Chebyshev polynomial C_d is the unique polynomial of degree d such that $C_d\left(x + \frac{1}{x}\right) = x^d + \frac{1}{x^d}$. Following the terminology in Medvedev-Scanlon [MS14], we say that a polynomial $f \in \bar{\mathbb{Q}}[x]$ of degree $d \geq 2$ is *disintegrated* if it is not linearly conjugate to x^d or $\pm C_d(x)$. Let $n \geq 2$ and let $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[x]$ be polynomials of degrees at least 2. Let $\varphi = f_1 \times \cdots \times f_n$ be the induced coordinate-wise self-map of $(\mathbb{P}^1)^n$. According to [MS14, Theorem 2.30], to study the arithmetic dynamics of φ, it suffices to study two cases: the case when none of the f_i's are disintegrated which reduces to diophantine questions on $\mathbb{G}_{\mathrm{m}}^n$ (as studied by Bombieri-Masser-Zannier [BMZ99, BMZ06, BMZ07, BMZ08]) and the case when all the f_i's are disintegrated.

It is the dynamics of φ in the case where f_i is disintegrated for $1 \leq i \leq n$ for which we prove an analogue of Theorem 1.1. In fact, one of the main theorems of [Ngu13] provides a bounded height result when we intersect a fixed *curve* with *periodic hypersurfaces*. Our main results in this paper (see Theorem 1.3 and Theorem 1.4) not only solve completely a bounded height problem for a general "dynamical complementary dimensional intersection" (similar to Theorem 1.1 (b)), but also establish a structure theorem similar to Theorem 1.1 (a).

An irreducible subvariety V of $(\mathbb{P}^1)^n$ is said to be *periodic* (or better, φ-periodic) if there exists an integer $m > 0$ such that $\varphi^m(V) = V$. If there exists $k \geq 0$ such that $\varphi^k(V)$ is periodic then we say that V is *preperiodic* (or better, φ-preperiodic).

While it is most natural to regard periodic subvarieties as a dynamical analogue of *irreducible* algebraic subgroups, the first major obstacle is to come up with an analogue of *arbitrary translates* of subgroups (which were the special subvarieties of $\mathbb{G}_{\rm m}^n$). Motivated by [Ngu13, Theorem 1.2], we let C be the curve $\zeta \times \mathbb{P}^1$ in $(\mathbb{P}^1)^2$ (endowed with the diagonal action of a polynomial $f \in \bar{\mathbb{Q}}[z]$) and we intersect C with periodic hypersurfaces defined by $y = f^\ell(x)$ for $\ell \geq 0$; then the resulting set will have unbounded height (if ζ is not preperiodic). Hence, in order to establish a dynamical analogue of Theorem 1.1, we have to exclude certain varieties having a constant projection to some factor $(\mathbb{P}^1)^m$ of $(\mathbb{P}^1)^n$.

Let $I_n := \{1, \ldots, n\}$. For every non-empty subset J of I_n, let φ^J be the coordinate-wise self-map of $(\mathbb{P}^1)^{|J|}$ induced by the polynomials f_j for $j \in J$. We say that an irreducible subvariety Z of $(\mathbb{P}^1)^n$ is φ-special if there is a subset J of I_n such that after a possible rearrangement of the factors of $(\mathbb{P}^1)^n$, Z has the form $\zeta \times V$, where $\zeta \in (\mathbb{P}^1)^{n-|J|}$ and $V \subseteq (\mathbb{P}^1)^{|J|}$ is φ^J-periodic. We refer the readers to Definition 2.1 for a formal definition of special subvarieties for the dynamics of φ. We remark that we got our inspiration for defining the φ-special subvarieties from the classical case of special subvarieties of $\mathbb{G}_m^n$ (see [BMZ99]) since the irreducible periodic subvarieties are the analogue of irreducible algebraic groups, while $\zeta \times V$ is considered to be the analogue of a *translation*. We also note that using the above definition for φ-special subvarieties when each $f_i(x)$ is the same power map (i.e. $f_i(x) = x^d$ for some $d \geq 2$), we get certain non-torsion translates of subtori (since ζ need not be a root of unity); yet some translates of tori are not φ-special as defined above. This should not be a surprise though since it is well-known that x^d (and $\pm C_d(x)$) have very different dynamical behavior compared to disintegrated polynomials. For example, while a subtorus of codimension 1 can be described by an equation involving every variable, a theorem of Medvedev-Scanlon (see Section 2) asserts that when each $f_i(x)$ is disintegrated, every φ-periodic hypersurface is described by an equation involving only at most two variables. Next we define a dynamical analogue of anomalous subvarieties and of the set $X^{\rm oa}$:

Definition 1.2. *Let $n \geq 2$, let $f_1(x), \ldots, f_n(x) \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degrees at least 2, and let $\varphi := f_1 \times \cdots \times f_n$ be as before. For an irreducible subvariety $X \subseteq (\mathbb{P}^1)^n$, an irreducible subvariety $Y \subseteq X$ is called φ-anomalous (with respect to X) if there exists an irreducible φ-special subvariety $Z \subseteq (\mathbb{P}^1)^n$ such that*

$$Y \subseteq X \cap Z \text{ and } \dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}. \tag{2}$$

Define $X_\varphi^{\rm oa}$ to be the complement in X of the union of all φ-anomalous subvarieties of X.

For each $0 \leq r \leq n$, we let $\mathrm{Per}_\varphi^{[r]}$ be the union of all irreducible φ-periodic subvarieties of $(\mathbb{P}^1)^n$ of *codimension* r. When the map φ (hence the collection $\{f_1, \ldots, f_n\}$) is clear from the context, we will use the notation $\mathrm{Per}^{[r]}$. Our first main result is the following dynamical analogue of the Bounded Height Theorem:

Theorem 1.3. *Let $n \geq 2$, let $f_1(x), \ldots, f_n(x) \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degrees at least 2, and let $\varphi := f_1 \times \cdots \times f_n$ be as before. Let X be an irreducible subvariety of $(\mathbb{P}^1)^n$ of dimension r. Then the set $X_\varphi^{\rm oa} \cap \mathrm{Per}^{[r]}$ has bounded height.*

We give examples at the end of Section 2 that it is necessary to omit the anomalous subvarieties of X. Theorem 1.3 generalizes the results in [Ngu13] which only treat the case that X is a curve. Our second main result is a dynamical analogue of the Structure Theorem [BMZ07, Theorem 1.4]:

Theorem 1.4. *With the notation as in Theorem 1.3, we have the following.*

(a) *Fix a subset J of $I_n := \{1, \dots, n\}$ and an irreducible φ^J-periodic subvariety Z of $(\mathbb{P}^1)^{|J|}$. Identify $(\mathbb{P}^1)^n = (\mathbb{P}^1)^{|I_n \setminus J|} \times (\mathbb{P}^1)^{|J|}$. Let $T(X, J, Z)$ be the union of subvarieties Y of X such that $Y \subseteq \zeta \times Z$ for some $\zeta \in (\mathbb{P}^1)^{|I_n\setminus J|}$ and $\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}$. Then $T(X, J, Z)$ is Zariski closed in X.*

(b) *There exists a collection (depending on φ and X) consisting of finitely many (not necessarily distinct) subsets $J_1, \dots, J_\ell$ of I_n together with irreducible φ^{J_k}-periodic subvarieties $Z_k \subseteq (\mathbb{P}^1)^{|J_k|}$ for $1 \le k \le \ell$ such that the φ-anomalous locus of X is $\bigcup_{k=1}^{\ell} T(X, J_k, Z_k)$.*

As a consequence, the set X_φ^{oa} is Zariski open in X.

Remark 1.5. As pointed out by the referee, the height bound in Theorem 1.3 and the equations defining the subvarieties Z_i in Theorem 1.4 are effectively computable since our arguments are effective.

It is also possible to ask a variant of the above theorems for preperiodic subvarieties. We define φ-pre-special subvarieties to be those of the form $\zeta \times Z$ where $\zeta \in (\mathbb{P}^1)^{n-|J|}$ and $Z \subseteq (\mathbb{P}^1)^{|J|}$ is φ^J-preperiodic for some subset J of $\{1, \dots, n\}$. For an irreducible subvariety $X \subseteq (\mathbb{P}^1)^n$, we define φ-pre-anomalous subvarieties as in Definition 1.2 where the only change is that Z is required to be φ-pre-special rather than φ-special. Similarly, we define $X_\varphi^{\mathrm{oa,pre}}$ to be the complement of the union of all φ-pre-anomalous subvarieties in X. Finally, we use the notation $\mathrm{Pre}_\varphi^{[r]}$ (or $\mathrm{Pre}^{[r]}$ if φ is clear) to denote the union of all φ-preperiodic subvarieties of codimension r. We expect the following to have an affirmative answer:

Question 1.6. *Let n, $f_1, \dots, f_n$, φ, X and r be as in Theorem 1.3.*

(a) *Does the set $X_\varphi^{\mathrm{oa,pre}} \cap \mathrm{Pre}^{[r]}$ have bounded height?*
(b) *Is the set $X_\varphi^{\mathrm{oa,pre}}$ Zariski open in X?*

Note that Question 1.6 is *neither stronger nor weaker* than our main theorems. The set $\mathrm{Pre}^{[r]}$ is larger than $\mathrm{Per}^{[r]}$, yet the set $X_\varphi^{\mathrm{oa,pre}}$ is smaller than X_φ^{oa}. Note that the largest intersection $X_\varphi^{\mathrm{oa}} \cap \mathrm{Pre}^{[r]}$ cannot have bounded height. As an easy example, we let $n = 2$, $f_1 = f_2 =: f$, and let X be a preperiodic but not periodic curve in $(\mathbb{P}^1)^2$ having non-constant projection to each factor $\mathbb{P}^1$ (for instance, an irreducible component of the curve $f(x) = f(y)$ other than the diagonal $x = y$). In this case, we have $X_\varphi^{\mathrm{oa}} = X \subset \mathrm{Pre}^{[1]}$.

The two main ingredients for the proof of Theorem 1.3 are the classification of φ-periodic subvarieties by Medvedev-Scanlon presented in the next section and an elementary height inequality (see Corollary 3.4). The proof of Theorem 1.4 also uses certain bounded height arguments that are similar to those in the proof of Theorem 1.3. One natural way to attack Question 1.6 is to relate it to results in the periodic case such as Theorem 1.3 and Theorem 1.4 since for every preperiodic subvariety V we have $\varphi^k(V)$ is periodic for some k. The difficulty is to obtain a good bound (perhaps uniform in the height of X) on the inequalities obtained in the proof of Theorems 1.3 and 1.4. For more details when X is a curve, we refer the readers to [Ngu13].

In this paper, we first provide all the details for the proof of Theorem 1.3 and Theorem 1.4 in the special yet essential case of the self-map $f \times \cdots \times f$ on $(\mathbb{P}^1)^n$ (i.e. when $f_1 = \ldots = f_n = f$). The general case $\varphi = f_1 \times \cdots f_n$ is explained in Section 7 and consists of two steps. First, we reduce φ to a map of the form $\psi = \psi_1 \times \cdots \times \psi_s$ for some $1 \leq s \leq n$ where ψ_i is a coordinate-wise self-map of $(\mathbb{P}^1)^{n_i}$ of the form $w_i \times \cdots \times w_i$ for some $1 \leq n_i \leq n$ and disintegrated $w_i(x) \in \bar{\mathbb{Q}}[x]$ such that $\sum_{i=1}^{s} n_i = n$ and w_i and w_j are "inequivalent" for $i \neq j$ (see Definition 7.1 and the discussion following Definition 7.1). Then maps ψ (as above) could be treated by a completely similar arguments used to settle the special case $\varphi = f \times \cdots \times f$ albeit with a more complicated system of notation for bookkeeping.

In fact, we can define the sets X_φ^{oa} and $X_\varphi^{\text{oa,pre}}$ and formulate the dynamical bounded height and structure theorems for the more general self-map of $(\mathbb{P}^1)^n$ of the form $\varphi = f_1 \times \cdots \times f_n$ where each $f_j(x) \in \bar{\mathbb{Q}}(x)$ is a "disintegrated *rational* function". The readers are referred to Question 7.3 and Proposition 7.4 for more details. In view of our main results, another possible direction of research is to formulate a weak dynamical form of the classical Pink-Zilber Conjecture by asking that if $X \subseteq (\mathbb{P}^1)^n$ is not contained in a φ-periodic (resp. φ-preperiodic) hypersurface, then $X \cap \text{Per}^{[r+1]}$ (resp. $X \cap \text{Pre}^{[r+1]}$) is not Zariski dense in X, where $r = \dim(X)$. In the case of hypersurfaces in $(\mathbb{P}^1)^n$, this restricts to the dynamical Manin-Mumford problem formulated in [Zha06]. The case of lines $X \subseteq (\mathbb{P}^1)^2$ was already proven in [GTZ11], and in light of the observations made in [YZ, Section 3] (especially the discussion following [YZ, Question 3.17]), one might expect that the general case of curves, and perhaps even the case of arbitrary subvarieties of $(\mathbb{P}^1)^n$ holds when each coordinate of $\mathbb{P}^1$ is acted on by a common *disintegrated polynomial*. However, the scarcity of positive results for the Dynamical Manin-Mumford Conjecture and also, the exotic behavior of certain examples produced in [GTZ11] when different *rational functions* f_i act on the coordinates of $(\mathbb{P}^1)^n$ prevent us from formally stating a conjecture for the unlikely intersection principle in dynamics.

For the rest of this paper, we do not refer to the notion of special and anomalous subvarieties of $\mathbb{G}_m^n$ given by Bombieri, Masser and Zannier. For simplicity, when the self-map φ is clear from the context, we use the terminology special (resp. pre-special, anomalous, pre-anomalous) subvarieties instead of φ-special (resp. φ-pre-special, φ-anomalous, φ-pre-anomalous) subvarieties.

The organization of this paper is as follows. In Section 2, following [MS14] and [Ngu13] we give a precise description of the φ-periodic subvarieties of $(\mathbb{P}^1)^n$ in the special case $f_1 = \ldots = f_n$. The proof of Theorems 1.3 and 1.4 in that case takes up the following four sections. This proof uses certain properties of height and canonical height in Section 3 coupled with some elementary geometric properties of the set X_φ^{oa} in Section 4. In the last section, we explain how to prove Theorem 1.3 and Theorem 1.4 in general and conclude the paper with a brief discussion on the dynamics of more general self-maps of $(\mathbb{P}^1)^n$ of the form $f_1 \times \cdots \times f_n$ where each $f_i(x) \in \bar{\mathbb{Q}}(x)$ is a "disintegrated rational function".

Acknowledgments. We thank Tom Scanlon and Tom Tucker for many useful conversations. We are grateful to the anonymous referee for many useful suggestions.

2. Structure of periodic subvarieties

Recall that a polynomial $f \in \bar{\mathbb{Q}}[x]$ of degree $d \geq 2$ is called disintegrated if it is not linearly conjugate to x^d or $\pm C_d(x)$. Let n be a positive integer and $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degrees at least 2. Let $\varphi = f_1 \times \cdots \times f_n$ be the corresponding coordinate-wise self-map of $(\mathbb{P}^1)^n$. Let $I_n := \{1, \ldots, n\}$. For each *ordered* subset J of I_n, we define:

$$(\mathbb{P}^1)^J := (\mathbb{P}^1)^{|J|}$$

equipped with the canonical projection $\pi^J : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^J$. Occasionally, we also work with the Zariski open subset $(\mathbb{A}^1)^n = \mathbb{A}^n$ of $(\mathbb{P}^1)^n$ and we use the notation $\mathbb{A}^J$ to denote the Zariski open subset $(\mathbb{A}^1)^{|J|}$ of $(\mathbb{P}^1)^J$. Obviously, $\mathbb{A}^J = \pi^J(\mathbb{A}^n)$. Let φ^J denote the coordinate-wise self-map of $(\mathbb{P}^1)^J$ induced by the polynomials f_j's for $j \in J$. In this paper, we will consider ordered subsets of I_n whose orders need not be induced from the usual order of the set of integers. If $J_1, \ldots, J_m$ are ordered subsets of I_n which partition I_n, then we have the canonical isomorphism:

$$(\pi^{J_1}, \ldots, \pi^{J_m}) : (\mathbb{P}^1)^n = (\mathbb{P}^1)^{J_1} \times \cdots \times (\mathbb{P}^1)^{J_m}.$$

For each irreducible subvariety V of $(\mathbb{P}^1)^n$, let J_V denote the set of all $j \in I_n$ such that the projection from V to the j^{th} coordinate $\mathbb{P}^1$ is constant. If $J_V \neq \emptyset$, we equip J_V with the natural order of the set of integers, and we let $a_V \in (\mathbb{P}^1)^{J_V}$ denote $\pi^{J_V}(V)$. Even when $J_V = \emptyset$, we will *vacuously* define $(\mathbb{P}^1)^{J_V}$ as the variety consisting of one point and define a_V to be that point. We have then the following formal definition for *special* subvarieties.

Definition 2.1. *Let Z be an irreducible subvariety of $(\mathbb{P}^1)^n$. Identify $(\mathbb{P}^1)^n = (\mathbb{P}^1)^{J_Z} \times (\mathbb{P}^1)^{I_n - J_Z}$. We say that Z is φ-special (respectively φ-pre-special) if it has the form $a_Z \times Z'$ where Z' is $\varphi^{I_n - J_Z}$-periodic (respectively preperiodic).*

We now present (a crucial case of) the Medvedev-Scanlon classification of φ-periodic subvarieties of $(\mathbb{P}^1)^n$ from [MS14] along with its refinement from [Ngu13]. The complete classification in [MS14] consists of two parts: the first part treats the special case $f_1 = \ldots = f_n$ which is given in this section while the second part explains why the general case reduces to this special case and is given in Section 7.

Assumption 2.2. For the rest of this section, assume that $f_1(x) = \ldots = f_n(x)$ and denote this common polynomial by $f(x)$. Since the map φ^J on $(\mathbb{P}^1)^J$ now depends only on $|J|$, for every positive integer m we usually use the notation $\varphi_m := \varphi_{m,f}$ to denote the self-map $f \times \cdots \times f$ on $(\mathbb{P}^1)^m$. In particular, $\varphi = \varphi_n$ and $\varphi^J = \varphi_{|J|}$; such notation also emphasizes the fact that we are assuming $f_1 = \ldots = f_n = f$.

Let $x_1, \ldots, x_n$ denote the coordinate functions of the factors $\mathbb{P}^1$ of $(\mathbb{P}^1)^n$. Medvedev and Scanlon prove the following important result [MS14, pp. 85]:

Theorem 2.3. *Let V be an irreducible φ_n-periodic subvariety of $(\mathbb{P}^1)^n$. Then V is given by a collection of equations of the following forms:*

(a) *$x_i = \zeta$ where ζ is a periodic point of f.*
(b) *$x_j = g(x_i)$ for some $1 \leq i \neq j \leq n$, where g is a polynomial commuting with an iterate of f.*

Next we describe all polynomials g commuting with an iterate of f mentioned in Theorem 2.3; the following result is [Ngu13, Proposition 2.3].

Proposition 2.4. *We have:*

(a) *If $g \in \bar{\mathbb{Q}}[x]$ has degree at least 2 such that g commutes with an iterate of f then g and f have a common iterate.*
(b) *Let $M(f^\infty)$ denote the collection of all linear polynomials commuting with an iterate of f. Then $M(f^\infty)$ is a finite cyclic group under composition.*
(c) *Let $\tilde{f} \in \bar{\mathbb{Q}}[x]$ be a polynomial of minimum degree $\tilde{d} \geq 2$ such that $\tilde{f}$ commutes with an iterate of f. Then there exists $D = D_f > 0$ relatively prime to the order of $M(f^\infty)$ such that $\tilde{f} \circ L = L^D \circ \tilde{f}$ for every $L \in M(f^\infty)$.*
(d) $\left\{\tilde{f}^m \circ L : \ m \geq 0, \ L \in M(f^\infty)\right\} = \left\{L \circ \tilde{f}^m : \ m \geq 0, \ L \in M(f^\infty)\right\}$*, and these sets describe exactly all polynomials g commuting with an iterate of f. As a consequence, there are only finitely many polynomials of bounded degree commuting with an iterate of f.*

From the discussion in [Ngu13, Proposition 2.3], we have the following more refined description of φ_n-periodic subvarieties:

Proposition 2.5. (a) *Let V be an irreducible φ_n-periodic subvariety of $(\mathbb{P}^1)^n$ of dimension $n-r$. Then there exists a partition of $I_n - J_V$ into $n-r$ non-empty subsets $J_1, \ldots, J_{n-r}$ such that the following hold. We fix an order on each $J_1, \ldots, J_{n-r}$, and identify:*

$$(\mathbb{P}^1)^n = (\mathbb{P}^1)^{J_V} \times (\mathbb{P}^1)^{J_1} \times \cdots \times (\mathbb{P}^1)^{J_{n-r}}.$$

For $1 \leq i \leq n-r$, there exists an irreducible $\varphi_{|J_i|}$-periodic curve C_i in $(\mathbb{P}^1)^{J_i}$ such that:

$$V = a_V \times C_1 \times \cdots \times C_{n-r}.$$

(b) *Let C be an irreducible φ_n-periodic curve in $(\mathbb{P}^1)^n$ and denote $m := |I_n - J_C| \geq 1$. Then there exist a permutation $(i_1, \ldots, i_m)$ of $I_n - J_C$ and non-constant polynomials $g_{i_2}, \ldots, g_{i_m} \in \bar{\mathbb{Q}}[x]$ such that C is given by the equations $x_{i_2} = g_{i_2}(x_{i_1}), \ldots, \ x_{i_m} = g_{i_m}(x_{i_{m-1}})$. Furthermore, the polynomials $g_{i_2}, \ldots, g_{i_m}$ commute with an iterate of f.*

Remark 2.6. The permutation $(i_1, \ldots, i_m)$ mentioned in part (b) of Proposition 2.5 induces the order $i_1 \prec \ldots \prec i_m$ on $I_n - J_C$. Such a permutation and its induced order are not uniquely determined by V. For example, let L be a *linear* polynomial commuting with an iterate of f. Let C be the periodic curve in $(\mathbb{P}^1)^2$ defined by the equation $x_2 = L(x_1)$. Then $I - J_C = \{1, 2\}$, and $1 \prec 2$ is an order satisfying the conclusion of part (b). However, we can also express C as $x_1 = L^{-1}(x_2)$. Then the order $2 \prec 1$ also satisfies part (b). Therefore, in part (a), the choice of an order on each J_i is not unique. Nevertheless, the partition of $I_n - J_V$ into the subsets $J_1, \ldots, J_r$ is unique (see [Ngu13, Section 2]).

Definition 2.7. *Let V be an irreducible φ_n-periodic subvariety of dimension r. Partition $I_n - J_V$ into $J_1, \ldots, J_r$, and let $C_1, \ldots, C_r$ be the periodic curves as in the conclusion of part (a) of Proposition 2.5. For $1 \leq i \leq r$, we equip J_i with an order discussed in Remark 2.6 viewing each C_i as $\varphi_{|J_i|}$-periodic in $(\mathbb{P}^1)^{|J_i|}$. Then the collection consisting of J_V and the ordered sets $J_1, \ldots, J_r$ is called a signature of V.*

Remark 2.8. Using the fact that there are finitely many signatures for periodic varieties, in Theorem 1.3 it suffices to show that for any given signature $\mathscr{S}$, the intersection of $X^{\mathrm{oa}}_{\varphi_n}$ with the union of all periodic subvarieties having signature $\mathscr{S}$ and codimension r has bounded height.

We finish this section by giving a couple of examples to show why it is necessary to remove the anomalous locus in Theorem 1.3.

Example 2.9. Let $X \subseteq (\mathbb{P}^1)^5$ be a 3-fold with the property that its intersection with the periodic 3-fold V given by the equations $x_2 = f(x_1)$ and $x_3 = f(x_2)$ contains a surface S. We claim that S should be removed from X in order for the points in the intersection with $\mathrm{Per}^{[3]}$ have bounded height. Indeed, for each positive integer m, we let V_m be the periodic surface given by the equations

$$x_2 = f(x_1),\ x_3 = f(x_2) \text{ and } x_4 = f^m(x_3).$$

Then $V_m \subseteq V$ and so, $H_m \cap S \subseteq V_m \cap X$ where $H_m \subseteq (\mathbb{P}^1)^5$ is the hypersurface given by $x_4 = f^m(x_3)$. In particular, there is a curve $C_m \subseteq H_m \cap S \subseteq V_m \cap X$, and obviously this curve contains points of arbitrarily large height. It is easy to see that each curve C_m is different as we vary m, and moreover, their union is Zariski dense in S.

Example 2.10. Let $X \subseteq (\mathbb{P}^1)^4$ be a surface with the property that its intersection with the surface $(a_1, a_2) \times (\mathbb{P}^1)^2$ (for $a_1, a_2 \in \mathbb{P}^1$) contains a curve C. Assume a_1 is not preperiodic, $a_2 = f(a_1)$, and also that C projects onto the third coordinate of $(\mathbb{P}^1)^4$. We show that the curve C must be removed from X in order for the points in the intersection with $\mathrm{Per}^{[2]}$ have bounded height. Indeed, for each positive integer m, we let V_m be the periodic surface given by the equations: $x_2 = f(x_1)$ and $x_3 = f^m(x_2)$. Because C projects onto the third coordinate of $(\mathbb{P}^1)^4$, there exists $a_4 \in \mathbb{P}^1$ such that $\big(a_1, f(a_1), f^{m+1}(a_1), a_4\big) \in C \cap V_m \subseteq X \cap V_m$, whose height grows to infinity as $m \to \infty$ (because a_1 is not preperiodic).

As an aside, we note that even though the above anomalous curves need to be removed, it is not clear whether one would also have to remove the curves which appear in the intersection of a surface $X \subseteq (\mathbb{P}^1)^4$ with $(a_1, a_2) \times (\mathbb{P}^1)^2$ if a_1 and a_2 are in different orbits under f. This phenomenon also occurs in the diophantine situation (see the discussion in [BMZ07, Section 5, pp. 24-25]).

3. Properties of the height

Recall that h denotes the absolute logarithmic Weil height on $\mathbb{P}^1$ (see [HS00, Part B] or [BG06, Chapter 1]). The following result is well-known:

Lemma 3.1. *For every $a, b \in \bar{\mathbb{Q}}$, we have:*

(a) $h(ab) \le h(a) + h(b)$.
(b) $h(a) - h(b) \le h(a/b)$ *if* $b \neq 0$.
(c) $h(a+b) \le h(a) + h(b) + \log 2$.
(d) $h(a) - h(b) - \log 2 \le h(a-b)$.
(e) $h(a^d) = |d| \cdot h(a)$ *for any integer d (where $a \neq 0$ if $d < 0$).*

We can use Lemma 3.1 to prove the following result (we note that part (b) of the following Lemma is instrumental in our proof of Theorem 1.3).

Lemma 3.2. *Let $n \ge 1$ and $F(X_1, \dots, X_n) \in \bar{\mathbb{Q}}[X_1, \dots, X_n]$ having degree $D \ge 1$ in X_n. Write:*

$$F(X_1, \dots, X_n) = F_D(X_1, \dots, X_{n-1})X_n^D + \ldots + F_0(X_1, \dots, X_{n-1}).$$

For each $i = 1, \dots, n$, let D_i be the degree in X_i of F (so $D_n = D$). The following hold.

(a) *Then there exists a positive constant C_1 depending only on $F(X_1, \dots, X_n)$ such that for every $a_1, \dots, a_n \in \bar{\mathbb{Q}}$, we have*
$$h(F(a_1, \dots, a_n)) \le \sum_{i=1}^{n} D_i h(a_i) + C_1.$$

(b) *There exists a positive constant C_2 depending only on $F(X_1, \dots, X_n)$ such that for every $a_1, \dots, a_n \in \bar{\mathbb{Q}}$ satisfying $F_i(a_1, \dots, a_{n-1}) \ne 0$ for some $1 \le i \le D$, we have:*
$$h(a_n) - 2D_1 h(a_1) - \dots - 2D_{n-1} h(a_{n-1}) - C_2 \le h(F(a_1, \dots, a_n)).$$

Proof. (a) This is a standard result which follows from triangle inequalities at each place (both archimedean and nonarchimedean).

(b) Given $a_1, \dots, a_n \in \bar{\mathbb{Q}}$, we let i be maximum with the property that $F_i(a_1, \dots, a_{n-1}) \ne 0$. Then, letting
$$Q_i(X_1, \dots, X_n) := F_{i-1} X_n^{i-1} + \dots + F_0$$
so that $F(a_1, \dots, a_n) = F_i(a_1, \dots, a_{n-1}) a_n^i + Q_i(a_1, \dots, a_n)$ we obtain the following inequalities from part (a) and Lemma 3.1:
$$\begin{aligned} h(F(a_1, \dots, a_n)) &\ge h(F_i(a_1, \dots, a_{n-1}) a_n^i) - h(Q_i(a_1, \dots, a_n)) - \log 2 \\ &\ge i h(a_n) - h(F_i(a_1, \dots, a_{n-1})) - 2\log 2 \\ &\quad - (i-1) h(a_n) - \sum_{j=1}^{n-1} D_j h(a_j) - C_3 \\ &\ge h(a_n) - \sum_{j=1}^{n-1} 2 D_j h(a_j) - C_2 \end{aligned}$$
where C_3 is the constant appearing when applying part (a) to the polynomial Q_i and C_2 is another constant (depending on C_3 and the constant appearing when applying part (a) to the polynomial F_i). Clearly, all these constants depend only on the coefficients of F and are effectively computable.

□

For every polynomial $f(x) \in \bar{\mathbb{Q}}[x]$ of degree $d \ge 2$, the canonical height $\widehat{h}_f$ can be defined using a well-known trick of Tate (see [Sil07, Chapter 3]):
$$\widehat{h}_f(a) := \lim_{m \to \infty} \frac{h(f^m(a))}{d^m}, \text{ for all } a \in \bar{\mathbb{Q}}.$$

We have the following properties:

Lemma 3.3. *Let $f(x) \in \bar{\mathbb{Q}}[x]$ of degree $d \ge 2$. We have:*

(a) *There is a constant C_4 depending only on f such that $|\widehat{h}_f(a) - h(a)| \le C_4$ for every $a \in \bar{\mathbb{Q}}$.*

(b) *$\widehat{h}_f(f(a)) = d\widehat{h}_f(a)$ for every $a \in \bar{\mathbb{Q}}$.*

(c) *a is f-preperiodic if and only if $\widehat{h}_f(a) = 0$.*

(d) *If f is disintegrated and $g(x) \in \bar{\mathbb{Q}}[x]$ commutes with an iterate of $f(x)$ then $\widehat{h}_f(g(a)) = \deg(g)\widehat{h}_f(a)$ for every $a \in \bar{\mathbb{Q}}$.*

Proof. For parts (a), (b) and (c): see [Sil07, Chapter 3]. For part (d), Proposition 2.4 gives that f and g have a common iterate. Hence $\widehat{h}_f = \widehat{h}_g$ and the desired result follows from part (b). $\square$

The following inequality will be the key ingredient for the proof of Theorem 1.3. It is essentially part (b) of Lemma 3.2 where we replace h by $\widehat{h}_f$:

Corollary 3.4. *Let $n \geq 1$ and $F(X_1, \ldots, X_n) \in \bar{\mathbb{Q}}[X_1, \ldots, X_n]$ having degree $D \geq 1$ in X_n. Let $f(x) \in \bar{\mathbb{Q}}[x]$ having degree at least 2. Write:*

$$F(X_1, \ldots, X_n) = F_D(X_1, \ldots, X_{n-1})X_n^D + \ldots + F_0(X_1, \ldots, X_{n-1}).$$

For $1 \leq i \leq n-1$, let D_i be the degree of F in X_i. There exists a positive constant C_5 depending only on F and f such that for every $a_1, \ldots, a_n \in \bar{\mathbb{Q}}$ satisfying $F(a_1, \ldots, a_n) = 0$ and $F_i(a_1, \ldots, a_{n-1}) \neq 0$ for some $1 \leq i \leq D$, we have:

$$\widehat{h}_f(a_n) \leq D_1\widehat{h}_f(a_1) + \ldots + D_{n-1}\widehat{h}_f(a_{n-1}) + C_5.$$

Proof. This follows from part (b) of Lemma 3.2 and part (a) of Lemma 3.3. $\square$

We finish this section with the following remark about Proposition 2.4:

Remark 3.5. Recall the group $M(f^\infty)$ and the choice of $\tilde{f}$ from Proposition 2.4. We explain that both $M(f^\infty)$ and $\tilde{f}$ could be determined effectively. In fact, after conjugating with an effectively determined linear polynomial, we may assume that f has the normal form:

$$f(x) = x^d + a_{d-r}x^{d-r} + \ldots$$

with $a_{d-r} \neq 0$ and $r \geq 2$. Then it is an easy exercise to show that $M(f^\infty)$ is a subgroup of the group of linear polynomials of the form μx where μ is an r^{th} root of unity.

To determine $\tilde{f}$, note that part (d) of Proposition 2.4 gives that d is a power of $\tilde{d} := \deg(\tilde{f})$ and there are at most $|M(f^\infty)|$ choices of $\tilde{f}$. Let K be the field obtained by adjoining all coefficients of f to $\mathbb{Q}$. Then let $\tilde{K}$ be the field obtained by adjoining all the coefficients of $\tilde{f}$ to K. We must have $[\tilde{K} : K] \leq |M(f^\infty)|$ since $\sigma(\tilde{f})$ is another choice for $\tilde{f}$ for every K-embedding σ of $\tilde{K}$ into $\bar{\mathbb{Q}}$. Together with $\widehat{h}_f(\tilde{f}(m)) = \tilde{d} \cdot \widehat{h}_f(m)$, we have that the degree and (Weil) height of $\tilde{f}(m)$ are bounded effectively in terms of f and m for every integer m. Hence $\tilde{f}$ is effectively determined.

4. Basic properties of the anomalous locus

Let $n \geq 2$, let $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degree at least 2, and let φ be the corresponding coordinate-wise self-map of $(\mathbb{P}^1)^n$. Let $I = I_n := \{1, \ldots, n\}$. For every (ordered) subset $J \subseteq I$, let φ^J be the coordinate-wise self-map of $(\mathbb{P}^1)^J := (\mathbb{P}^1)^{|J|}$ induced by f_j's for $j \in J$ and let π^J be the projection from $(\mathbb{P}^1)^n$ to $(\mathbb{P}^1)^J$ as in the beginning of Section 2. Fix an irreducible subvariety X of $(\mathbb{P}^1)^n$ of dimension r satisfying $1 \leq r \leq n-1$. Recall Definition 2.1 and Definition 1.2 of special and anomalous subvarieties with respect to φ and X. In this section we prove several geometric properties of the set X_φ^{oa} which will be used repeatedly in the proofs of Theorem 1.3 and Theorem 1.4. By the affine part of $(\mathbb{P}^1)^n$, we mean the Zariski open subset $(\mathbb{A}^1)^n = \mathbb{A}^n$. More generally the affine part of a subset of $(\mathbb{P}^1)^n$ is its intersection with $\mathbb{A}^n$.

Lemma 4.1. *Assume X is contained in a proper special subvariety Z of $(\mathbb{P}^1)^n$. Write $Z = \alpha \times Z_1$ where Z_1 is φ^J-periodic and $\alpha \in (\mathbb{P}^1)^{I\setminus J}$ for some subset J of I. Then $X = T(X, J, Z_1)$ is an anomalous subvariety of X itself, where $T(X, J, Z_1)$ is defined as in Theorem 1.4.*

Proof. This is immediate from the definition of anomalous subvarieties and the definition of $T(X, J, Z_1)$. □

Lemma 4.2. *Let $J \subseteq I$ with $|J| \geq r$. Denote $J' = I \setminus J$ and $Z = (\mathbb{P}^1)^{J'}$. If $\dim(\pi^J(X)) < r$ then $X = T(X, J', Z)$ is the anomalous locus of X.*

Proof. Identify $(\mathbb{P}^1)^n = (\mathbb{P}^1)^J \times (\mathbb{P}^1)^{J'}$ and write $\pi = \pi^J$. Since $\dim(\pi(X)) < r$, for any point $\alpha \in \pi(X)$ every irreducible component of $\pi^{-1}(\alpha) \cap X$ has dimension at least 1 (see Mumford's book [Mum99, pp. 48]). By intersecting X with the special variety $\alpha \times (\mathbb{P}^1)^{J'} = \alpha \times Z$, we conclude that every irreducible component of $\pi^{-1}(\alpha) \cap X$ is an anomalous subvariety of X and is contained in $T(X, J', Z)$. Since this holds for every $\alpha \in \pi(X)$, we have $X = T(X, J', Z)$. □

Assumption 4.3. Thanks to Lemma 4.1 and Lemma 4.2, for the rest of this section we make the extra assumption that the affine part of X is non-empty (otherwise X is contained in a special subvariety of the form $\infty \times (\mathbb{P}^1)^{n-1}$) and the image of X under the projection to $(\mathbb{P}^1)^J$ has dimension r for every $J \subseteq I$ having $|J| \geq r$.

Corollary 4.4. *Let J be any ordered subset of I of size $r + 1$ explicitly listed in increasing order as $(i_1, \ldots, i_{r+1})$. Then (under Assumption 4.3) there exists an irreducible polynomial $F^J(X_{i_1}, \ldots, X_{i_{r+1}})$ such that the affine part $\pi^J(X) \cap \mathbb{A}^J$ of $\pi^J(X)$ is defined by the equation $F^J(x_{i_1}, \ldots, x_{i_{r+1}}) = 0$. The polynomial F^J is unique up to multiplication by an element in $\bar{\mathbb{Q}}^*$.*

We now fix a choice of F^J for each ordered subset J of I as in Corollary 4.4. We have the following result which allows us to apply Corollary 3.4 later.

Proposition 4.5. *Assume that $X^{\text{oa}}_{\varphi} \cap \mathbb{A}^n$ is non-empty and pick any $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $X^{\text{oa}}_{\varphi} \cap \mathbb{A}^n$. Let J be any ordered set $(i_1, \ldots, i_{r+1})$ as in Corollary 4.4 and let $1 \leq \ell \leq r + 1$. Let $D \geq 0$ denote the degree of X_{i_ℓ} in F^J. Write:*

$$F^J = F^J_D X^D_{i_\ell} + \ldots + F^J_1 X_{i_\ell} + F^J_0$$

where F^J_k for $0 \leq k \leq D$ is a polynomial in the variables $X_{i_1}, \ldots, \widehat{X_{i_\ell}}, \ldots, X_{i_{r+1}}$. Then there exists $1 \leq k \leq D$ such that $F^J_k(\alpha_{i_1}, \ldots, \widehat{\alpha_{i_\ell}}, \ldots, \alpha_{i_{r+1}}) \neq 0$. In particular, this implies $D \geq 1$.

Proof. Without loss of generality, we may assume that $J = \{1, \ldots, r + 1\}$ with the usual order on the natural numbers and $\ell = r + 1$. Write $\pi := \pi^J$.

We now assume that $F^J_k(\alpha_1 \ldots, \alpha_r) = 0$ for every $1 \leq k \leq D$ (which implies $F^J_0(\alpha_1, \ldots, \alpha_r) = 0$ too) and we arrive at a contradiction. Write $\alpha' = (\alpha_1, \ldots, \alpha_r)$. Since the equation $F^J(x_1, \ldots, x_{r+1}) = 0$ defines (the affine part of) $\pi(X)$, we have that $\pi(X)$ contains the curve $\alpha' \times \mathbb{P}^1$. Consider the morphism $\pi\mid_X\colon\ X \to \pi(X)$; we have that some irreducible component Y of $(\pi\mid_X)^{-1}\left(\alpha' \times \mathbb{P}^1\right)$ has dimension at least 1 (see [Mum99, pp. 48]). In other words, we have:

- $Y \subseteq X \cap (\alpha' \times (\mathbb{P}^1)^{n-r})$;
- $\dim(Y) > 0$.

Therefore Y is an anomalous subvariety of X containing α, contradiction. □

Assumption 4.6. For the rest of this section, we adopt Assumption 2.2, namely $f_1 = \ldots = f_n = f$. In particular, the notation $\varphi_{|J|}$ is the same as φ^J (hence $\varphi_n = \varphi$).

Now let V be an irreducible φ_n-periodic hypersurface of $(\mathbb{P}^1)^n$. We define next an embedding $e_V : (\mathbb{P}^1)^{n-1} \to (\mathbb{P}^1)^n$ such that $e_V\left((\mathbb{P}^1)^{n-1}\right) = V$. According to Theorem 2.3, V is defined by $x_i = \zeta$ for some $1 \le i \le n$ and periodic ζ or $x_i = g(x_j)$ for some $1 \le i \ne j \le n$ and $g \in \bar{\mathbb{Q}}[x]$ commuting with an iterate of f. If V is given by $x_i = \zeta$, we define:

$$e_V(a_1, \ldots, a_{n-1}) := (a_1, \ldots, a_{i-1}, \zeta, a_i, \ldots, a_{n-1}).$$

If V is given by $x_i = g(x_j)$ and $i < j$, we define:

$$e_V(a_1, \ldots, a_{n-1}) = (a_1, \ldots, a_{i-1}, g(a_{j-1}), a_i, \ldots, a_{n-1}),$$

while if $j < i$, we define:

$$e_V(a_1, \ldots, a_{n-1}) = (a_1, \ldots, a_{i-1}, g(a_j), a_i, \ldots, a_{n-1}).$$

Consider the self-map φ_{n-1} on $(\mathbb{P}^1)^{n-1}$. Since the point ζ is periodic and g commutes with an iterate of f, we obtain that e_V maps φ_{n-1}-periodic subvarieties of $(\mathbb{P}^1)^{n-1}$ to φ_n-periodic subvarieties of $(\mathbb{P}^1)^n$ (contained in V). Then it is also easy to prove that e_V maps special subvarieties of $(\mathbb{P}^1)^{n-1}$ to special subvarieties of $(\mathbb{P}^1)^n$. Later on, in the proof of Theorem 1.3, we use e_V to identify irreducible components of $X \cap V$ with subvarieties of $(\mathbb{P}^1)^{n-1}$ so that we can use the induction hypothesis. In fact, we have the following:

Lemma 4.7. *Let W be an irreducible subvariety and let V be an irreducible φ_n-periodic hypersurface of $(\mathbb{P}^1)^n$. Let $\alpha \in W^{\mathrm{oa}}_{\varphi_n} \cap V$. Let W' be an irreducible component of $W \cap V$ containing α. Since $e_V^{-1}(W')$ is an irreducible subvariety of $(\mathbb{P}^1)^{n-1}$, we can define the set $e_V^{-1}(W')^{\mathrm{oa}}_{\varphi_{n-1}}$ as before; then $e_V^{-1}(\alpha) \in e_V^{-1}(W')^{\mathrm{oa}}_{\varphi_{n-1}}$.*

Proof. Since $W^{\mathrm{oa}}_{\varphi_n} \ne \emptyset$, we have that W is not contained in V. And since $W \cap V \ne \emptyset$, every irreducible component of $W \cap V$ has dimension $\dim(W) - 1$ by the Krull's Principal Ideal Theorem.

We prove the lemma by contradiction: assume there exists an anomalous subvariety Y of $e_V^{-1}(W')$ containing $e_V^{-1}(\alpha)$. Hence there exists a special subvariety Z of $(\mathbb{P}^1)^{n-1}$ satisfying the following condition:

(3) $Y \subseteq e_V^{-1}(W') \cap Z$ and $\dim(Y) > \max\{0, \dim(e_V^{-1}(W')) + \dim(Z) - (n-1)\}$.

We have $\dim(e_V^{-1}(W')) = \dim(W') = \dim(W) - 1$, $\dim(e_V(Y)) = \dim(Y)$ and $\dim(e_V(Z)) = \dim(Z)$. Together with the previous condition, we have:

(4) $e_V(Y) \subseteq W \cap e_V(Z)$ and $\dim(e_V(Y)) > \max\{0, \dim(W) + \dim(e_V(Z)) - n\}$.

By the discussion in the paragraph before this lemma, we have that $e_V(Z)$ is a special subvariety of $(\mathbb{P}^1)^n$. Therefore $e_V(Y)$ is an anomalous subvariety of W. Since $\alpha \in e_V(Y)$, we get a contradiction. ∎

We conclude this section with the following useful fact:

Corollary 4.8. *Let $W \subseteq (\mathbb{P}^1)^n$ be an irreducible subvariety, let $V \subseteq (\mathbb{P}^1)^n$ be an irreducible φ_n-periodic hypersurface intersecting W properly, and let W' be an irreducible component of $W \cap V$.*

(a) *If* $Y \subseteq (\mathbb{P}^1)^{n-1}$ *is a* φ_{n-1}*-anomalous subvariety of* $e_V^{-1}(W')$*, then* $e_V(Y) \subseteq (\mathbb{P}^1)^n$ *is a* φ_n*-anomalous subvariety of both* W' *and of* W*.*
(b) *If* $Z \subseteq V$ *is a* φ_n*-special subvariety and if* $\tilde{Y} \subseteq W' \cap Z$ *is a* φ_n*-anomalous subvariety of* W *satisfying* $\dim(\tilde{Y}) > \max\{0, \dim(W) + \dim(Z) - n\}$ *then* $e_V^{-1}(\tilde{Y})$ *is a* φ_{n-1}*-anomalous subvariety of* $e_V^{-1}(W')$*.*
(c) *If* V *is given by the equation* $x_i = \zeta$ *then part (b) holds without the assumption* $Z \subseteq V$*.*

Proof. Part (a) follows immediately from the proof of Lemma 4.7.

For part (b), the condition $Z \subseteq V$ makes sure that $e_V^{-1}(Z)$ remains a special subvariety of $(\mathbb{P}^1)^{n-1}$. We have $e_V^{-1}(\tilde{Y}) \subseteq e_V^{-1}(W') \cap e_V^{-1}(Z)$ and use $\dim(e_V^{-1}(W')) = \dim(W) - 1$ to obtain:

$$\dim\left(e_V^{-1}(\tilde{Y})\right) > \max\{0, \dim(e_V^{-1}(W')) + \dim(e_V^{-1}(Z)) - (n-1)\}.$$

This proves that $e_V^{-1}(\tilde{Y})$ is a special subvariety of $e_V^{-1}(W')$.

For part (c), note that if V is defined by $x_i = \zeta$ then each irreducible component of $V \cap Z$ is also special. Hence the conclusion follows by applying part (b) to the special variety $Z' \subseteq V$ which is an irreducible component of $Z \cap V$ containing $\tilde{Y}$. □

5. Proof of Theorem 1.3 for $f \times \cdots \times f$

Throughout this section, we fix a polynomial $f(x) \in \bar{\mathbb{Q}}[x]$ of degree $d \geq 2$ which is not conjugated to x^d or to $\pm C_d(x)$. Let n be a positive integer and let φ_n be the corresponding self-map of $(\mathbb{P}^1)^n$ as in Assumption 2.2. Let X be an irreducible subvariety of $(\mathbb{P}^1)^n$ of dimension r. Our goal is to prove Theorem 1.3 asserting that the set

$$X^{\mathrm{oa}}_{\varphi_n} \cap \bigcup_V V$$

has bounded height where V ranges among all irreducible φ_n-periodic subvarieties of dimension $n - r$. Note that this is obviously true when $r = 0$ (or when $r = n$); so, we now proceed by induction. Let $r \in \{1, \dots, n-1\}$ and assume Theorem 1.3 is valid for all varieties of dimension less than r for any n.

We may assume $X^{\mathrm{oa}}_{\varphi_n} \neq \emptyset$; in particular, X is not contained in any proper special subvariety of $(\mathbb{P}^1)^n$ (by Lemma 4.1). Furthermore, it suffices to replace $X^{\mathrm{oa}}_{\varphi_n}$ by its affine part. In other words, we only need to prove that the set $\left(X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n\right) \cap \mathrm{Per}^{[r]}$ has bounded height. The reason is that every point in the "non-affine part"

$$X^{\mathrm{oa}}_{\varphi_n} \setminus \mathbb{A}^n$$

is contained in $X \cap H$ where H is a φ_n-periodic hypersurface of the form (after a possible rearrangement of coordinates): $(\mathbb{P}^1)^{n-1} \times \infty$. We now use Lemma 4.7 and the embedding e_H introduced there to apply the induction hypothesis for the irreducible components of $e_H^{-1}(X \cap H) \subseteq (\mathbb{P}^1)^{n-1}$.

From now on, we assume $X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n \neq \emptyset$. Since there are only finitely many possible signatures (see Definition 2.7) for all irreducible φ_n-periodic subvarieties of dimension $n - r$, we fix a signature $\mathscr{S}$ consisting of the following data:

- A (possibly empty) subset J of $I_n := \{1, \dots, n\}$ such that $|I_n \setminus J| \geq n - r$.
- A partition of $I_n \setminus J$ into $n - r$ non-empty subsets $J_1, \dots, J_{n-r}$.

- For each $1 \leq k \leq n-r$, a choice of an order $\prec$ on J_{n-r}. So we can describe the ordered set J_k as:
$$i_{k,1} \prec i_{k,2} \prec \ldots \prec i_{k,m_k}$$
where $m_k := |J_k|$.

Convention 5.1. *From now on, to avoid triple subscripts we denote the coordinate functions $x_{i_{k,j}}$ as $x_{k,j}$; hence $x_{i_{k,m_k}}$ is denoted x_{k,m_k}.*

It suffices to prove that the set
$$X_{\mathscr{S}} := (X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n) \cap \bigcup_V V$$
has bounded height, where V ranges over all irreducible φ_n-periodic subvarieties of dimension $n-r$ *having signature* $\mathscr{S}$. Identify $(\mathbb{P}^1)^n = (\mathbb{P}^1)^J \times (\mathbb{P}^1)^{J_1} \times \cdots \times (\mathbb{P}^1)^{J_{n-r}}$. Such a V is described by the following equations:

- The equations $x_i = \zeta_i$ for $i \in J$, where ζ_i is f-periodic.
- For $1 \leq k \leq n-r$, the equations $x_{k,2} = g_{k,2}(x_{k,1}),\ldots, x_{k,m_k} = g_{k,m_k}(x_{k,m_k-1})$ where each $g_{k,i} \in \bar{\mathbb{Q}}[x]$ (for $1 \leq k \leq n-r$ and $2 \leq i \leq m_k$) is a polynomial commuting with an iterate of f.

If for some $1 \leq k \leq n-r$, we have $m_k \geq 2$ then we define:
$$D(V) := \min\{\deg(g_{k,m_k}) :\ 1 \leq k \leq n-r,\ m_k \geq 2\}.$$
If $m_k = 1$ for every k, then V is simply of the form $(\zeta_i)_{i\in J} \times (\mathbb{P}^1)^{n-r}$. If that is the case, we define $D(V) = +\infty$.

Proposition 5.2. *There exist positive constants c_1, c_2 depending only on X and $\mathscr{S}$ such that for every irreducible φ_n-periodic subvariety V of dimension $n-r$ having signature $\mathscr{S}$, if $D(V) > c_2$ then the height of points in $\left(X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n\right) \cap V$ is bounded above by c_1.*

Proof. Let V be defined by the equations $x_i = \zeta_i$ for $i \in J$ and $x_{k,i} = g_{k,i}(x_{k,i-1})$ for $1 \leq k \leq n-r$ and $2 \leq i \leq m_k$ as before.

We prove first the case when $m_k = 1$ for each k; this also gives insight to the general case. In this special case, without loss of generality we assume $J = \{1,\ldots,r\}$ so that $V = \zeta \times (\mathbb{P}^1)^{n-r}$, where $\zeta = (\zeta_1,\ldots,\zeta_r)$ such that each ζ_i is a periodic point for f. Let
$$\alpha := (\alpha_1,\ldots,\alpha_n) \in \left(X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n\right) \cap V.$$
In particular, $\alpha_j = \zeta_j$ for $j = 1,\ldots,r$. Now, for each $i = r+1,\ldots,n$, we let π_i be the projection of $(\mathbb{P}^1)^n$ onto $(\mathbb{P}^1)^{r+1}$ consisting of the $r+1$ coordinates $x_1,\ldots,x_r,x_i$. Using Proposition 4.5, Corollary 3.4 and Lemma 3.3, we obtain that $h(\alpha_i)$ is uniformly bounded, as desired.

From now on, we assume that $m_k \geq 2$ for some $k \in \{1,\ldots,n-r\}$. The idea of our proof is that on V, the $n-r$ coordinate functions x_{k,m_k} for $1 \leq k \leq n-r$ dominate the other r coordinate functions in terms of height. However, due to the equations defining X, the coordinate functions x_{k,m_k} for $1 \leq k \leq n-r$ cannot dominate the other r coordinates "too much". This will allow us to prove that for each periodic variety V of signature $\mathscr{S}$, and for each $(\alpha_1,\ldots,\alpha_n) \in \left(X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n\right) \cap V$ the heights of α_{k,m_k} for $k = 1,\ldots,n-r$ are uniformly bounded, and thus, in turn this would yield that the height of each α_i for $i = 1,\ldots,n$ is uniformly bounded.

Furthermore, using Lemma 3.3 (a), it suffices to prove that $\widehat{h}_f(\alpha_{k,m_k})$ is uniformly bounded independent of V. We formalize this idea as follows.

Define $\Gamma := I_n \setminus \{i_{1,m_1}, \ldots, i_{n-r,m_{n-r}}\}$ (i.e. the set of r indices that are "dominated by the other $n-r$ indices"). For $1 \le k \le n-r$, define:

$$\Gamma_k := \{i_{k,m_k}\} \cup \Gamma.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n) \cap V$. By the equations defining V, the definition of $D(V)$ and part (d) of Lemma 3.3, we have (see Convention 5.1):

(5) $\widehat{h}_f(\alpha_{k,m_k}) \ge D(V)\widehat{h}_f(\alpha_{k,i})$, for all $1 \le k \le n-r$ and $1 \le i \le m_k - 1$

Note that (5) is vacuously true when $m_k = 1$, so inequality (5) yields:

$$n\sum_{k=1}^{n-r} \widehat{h}_f(\alpha_{k,m_k}) \ge D(V)\sum_{k=1}^{n-r}\sum_{i=1}^{m_k-1} \widehat{h}_f(\alpha_{k,i}) \tag{6}$$

For $1 \le k \le n-r$, we consider the projection $\pi_k := \pi^{\Gamma_k}$ from $(\mathbb{P}^1)^n$ onto $(\mathbb{P}^1)^{\Gamma_k} = (\mathbb{P}^1)^{r+1}$. By Corollary 4.4, there is an irreducible polynomial $F_k := F^{\Gamma_k}$ in $r+1$ variables such that $\pi_k(X)$ is defined by the equation $F_k = 0$.

By Proposition 4.5 and Corollary 3.4, there exist positive constants c_3 and c_4 depending only on X and $\mathscr{S}$ such that:

$$\widehat{h}_f(\alpha_{k,m_k}) \le c_3 \sum_{i\in\Gamma} \widehat{h}_f(\alpha_i) + c_4. \tag{7}$$

for all $k = 1, \ldots, n-r$. By the equations defining V, we have $\alpha_i = \zeta_i$ is periodic for $i \in J$. Hence (7) and part (c) of Lemma 3.3 give:

$$\widehat{h}_f(\alpha_{k,m_k}) \le c_3 \sum_{\ell=1}^{n-r}\sum_{i=1}^{m_\ell-1} \widehat{h}_f(\alpha_{\ell,i}) + c_4. \tag{8}$$

for all $k = 1, \ldots, n-r$. This yields:

$$\sum_{k=1}^{n-r} \widehat{h}_f(\alpha_{k,m_k}) \le nc_3 \sum_{\ell=1}^{n-r}\sum_{i=1}^{m_\ell-1} \widehat{h}_f(\alpha_{\ell,i}) + nc_4 \tag{9}$$

From (6) and (9), we have that if $D(V) \ge 2n^2c_3$ then:

$$\sum_{k=1}^{n-r} \widehat{h}_f(\alpha_{k,m_k}) \le 2nc_4,$$

and more generally:

$$\sum_{i=1}^{n} \widehat{h}_f(\alpha_i) \le 2nc_4 + \frac{c_4}{c_3}.$$

This finishes the proof of the proposition. □

End of the proof of Theorem 1.3 for $f \times \cdots \times f$. We need to prove that the set $X_{\mathscr{S}}$ has bounded height. Let c_1 be the positive constant in the conclusion of Proposition 5.2, it suffices to show that the set:

$$X_{\mathscr{S},c_1} := (X^{\mathrm{oa}}_{\varphi_n} \cap \mathbb{A}^n) \cap \bigcup_V V$$

has bounded height, where V ranges over all irreducible φ_n-periodic subvarieties of dimension $n-r$ having signature $\mathscr{S}$ and $D(V) \le c_1$. For every such V, choose

$1 \le \ell \le n-r$ such that $\deg(g_{\ell,m_\ell}) = D(V) \le c_1$. By part (d) of Proposition 2.4, there are only finitely many such polynomials g_{ℓ,m_ℓ}. We let H be the periodic hypersurface defined by $x_{\ell,m_\ell} = g_{\ell,m_\ell}\left(x_{\ell,m_{\ell-1}}\right)$; therefore there are only *finitely many* possibilities for H. Since $V \subseteq H$, we can apply the induction hypothesis to each irreducible component of $e_H^{-1}(X \cap H)$ by using Lemma 4.7. Note that going from $e_H^{-1}(X \cap H)$ to $X \cap H$ can increase the height by the factor $\deg(g_{\ell,m_\ell})$, but this is fine since there are only finitely many possibilities for g_{ℓ,m_ℓ}. $\square$

6. Proof of Theorem 1.4 for $f \times \cdots \times f$

Throughout this section, let $f(x) \in \bar{\mathbb{Q}}[x]$ be a polynomial of degree $d \ge 2$ which is not conjugated to x^d or to $\pm C_d(x)$, let n be a positive integer, and let φ_n be the diagonal action of f on $(\mathbb{P}^1)^n$ as in Assumption 2.2. Let $X \subseteq (\mathbb{P}^1)^n$ be a given irreducible variety of dimension r as in the previous section.

6.1. **Part (a) of Theorem 1.4.** We proceed by induction on n; the case $n = 2$ is immediate. Now assume $n \ge 3$ and the conclusion holds for every smaller value of n. Without loss of generality, we may assume that $J = \{m+1, \ldots, n\}$ with $1 \le m < n$. There are two cases.

Case 1: $X \subseteq (\mathbb{P}^1)^m \times Z$. Let $\pi :\ X \to (\mathbb{P}^1)^m$ be the projection from X to the first m factors of $(\mathbb{P}^1)^n$. Let $\alpha \in T(X, J, Z)$. By the definition of $T(X, J, Z)$, there exists an anomalous subvariety Y of X satisfying the following conditions:

- $Y \subseteq X \cap (\pi(\alpha) \times Z)$.
- $\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}$.

Since $X \subseteq (\mathbb{P}^1)^m \times Z$, we have that $\pi^{-1}(\pi(\alpha)) = X \cap (\pi(\alpha) \times Z)$. Hence any irreducible component $\tilde{Y}$ of $\pi^{-1}(\pi(\alpha))$ that contains Y satisfies:

$$\dim(\tilde{Y}) \ge \dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}.$$

Conversely, let $\alpha \in X$ such that some irreducible component $\tilde{Y}$ of $\pi^{-1}(\pi(\alpha))$ containing α satisfies:

$$\dim(\tilde{Y}) > \max\{0, \dim(X) + \dim(Z) - n\}.$$

Then $\tilde{Y}$ is an anomalous subvariety of X cut out by $\pi(\alpha) \times Z$.

Thus our discussion so far proves that $T(X, J, Z)$ is exactly the set of points $\alpha \in X$ such that some irreducible component of $\pi^{-1}(\pi(\alpha))$ that contains α has dimension at least $\max\{1, \dim(X)+\dim(Z)-n+1\}$. By the Upper Semicontinuity Theorem [Mum99, pp. 51], $T(X, J, Z)$ is Zariski closed in X.

Case 2: $X \nsubseteq (\mathbb{P}^1)^m \times Z$. By the Medvedev-Scanlon description of periodic subvarieties, there exists a φ^J-periodic hypersurface H of $(\mathbb{P}^1)^J$ such that $X \nsubseteq (\mathbb{P}^1)^m \times H =: V$. If $X \cap V = \emptyset$ then $X \cap (\mathbb{P}^1)^m \times Z = \emptyset$, hence $T(X, J, Z) = \emptyset$ and we are done. We now assume that $X \cap V \neq \emptyset$. Let $X_1, \ldots, X_s$ be the irreducible components of $X \cap V$. By the Krull's Principal Ideal Theorem, X_i has dimension $r-1$ for $1 \le i \le s$.

By arranging coordinates, we may assume that V is defined by the equation $x_n = \zeta$ for some f-periodic $\zeta \in \mathbb{P}^1$ or $x_n = g(x_{n-1})$ for some $g \in \bar{\mathbb{Q}}[x]$ commuting with an iterate of f. We now consider the embedding $e_V :\ (\mathbb{P}^1)^{n-1} \to (\mathbb{P}^1)^n$ introduced in Section 4 and the self-map φ_{n-1} on $(\mathbb{P}^1)^{n-1}$. For each $1 \le i \le \ell$, we apply the induction hypothesis for the data consisting of the subvariety $e_V^{-1}(X_i)$ of $(\mathbb{P}^1)^{n-1}$, the subset $J' := J \setminus \{n\}$ of $\{1, \ldots, n-1\}$, and the φ_{n-1}-periodic subvariety

$e_V^{-1}(Z)$ to conclude that the resulting set $T(e_V^{-1}(X_i), J', e_V^{-1}(Z))$ is Zariski closed. The identity

$$T(X,J,Z) = \bigcup_{i=1}^{\ell} e_V(T(e_V^{-1}(X_i), J', e_V^{-1}(Z)))$$

concludes our proof of part (a) of Theorem 1.4.

6.2. **Part (b) of Theorem 1.4.** We proceed by induction on n; the case $n = 2$ is immediate. Now assume $n \geq 3$ and the conclusion holds for every smaller value of n. We may assume $1 \leq r \leq n-1$.

Let $U := U_X$ denote the union of all the anomalous subvarieties of X. By Lemma 4.1 and Lemma 4.2, we may assume that X is not contained in any special subvariety, and that for any choice of $j \geq r$ factors $\mathbb{P}^1$ the image of the projection from X to $(\mathbb{P}^1)^j$ has dimension r. Let

$$U_0 := \bigcup_{J \subseteq I_n} T(X, J, (\mathbb{P}^1)^J);$$

then U_0 is a closed subset of X since each $T(X, J, (\mathbb{P}^1)^J)$ is Zariski closed by part (a). We also let U_∞ denote the union of all anomalous subvarieties of X that are contained in a hypersurface of the form $\infty \times (\mathbb{P}^1)^{n-1}$ (after a possible rearrangement of coordinates). In other words:

$$U_\infty = \bigcup_{i=1}^{n} T(X, \{i\}, \{\infty\})$$

which is Zariski closed thanks to part (a).

Now we let Y be an anomalous subvariety of X that is not contained in $U_0 \cup U_\infty$. Let Z be a special subvariety of X such that $Y \subseteq X \cap Z$ and

$$\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}.$$

Write $\ell = \dim(Z)$. Without loss of generality, write $Z = \zeta \times Z_0$ where $\zeta \in (\mathbb{P}^1)^m$ (the first m coordinates) and Z_0 is a φ_{n-m}-periodic subvariety of $(\mathbb{P}^1)^{n-m}$ which projects dominantly onto each coordinate of $(\mathbb{P}^1)^{n-m}$; we allow for the possibility that $m = 0$, in which case Z is simply a φ_n-periodic subvariety of $(\mathbb{P}^1)^n$. Let $\delta := \dim(Y)$; then $\delta \geq \max\{1, r + \ell - n + 1\}$. There exist δ coordinates of $(\mathbb{P}^1)^n$ such that Y maps dominantly to $(\mathbb{P}^1)^\delta$. This fact follows from the Implicit Function Theorem by considering a smooth point of Y. Without loss of generality, we assume these δ coordinates are the last δ coordinates of $(\mathbb{P}^1)^n$. Furthermore, since the image of Y under the projection map $(\mathbb{P}^1)^n \mapsto (\mathbb{P}^1)^\delta$ is closed, it has to be the entire $(\mathbb{P}^1)^\delta$.

Partition $\{m+1, \dots, n\}$ into ℓ ordered subsets J_i such that, writing $(\mathbb{P}^1)^n := (\mathbb{P}^1)^m \times (\mathbb{P}^1)^{J_1} \times \cdots \times (\mathbb{P}^1)^{J_\ell}$, then $Z = \zeta \times C_1 \times \cdots \times C_\ell$, where each $C_i \subseteq (\mathbb{P}^1)^{J_i}$ is a periodic curve. For each $i = 1, \dots, \ell$ we list the elements of J_i as $j_{i,1} \prec \cdots \prec j_{i,m_i}$ (where $m_i := |J_i|$). As before, to avoid triple subscripts (such as $x_{j_{i,m_i}}$), we use $x_{i,s}$ to denote $x_{j_{i,s}}$. Each C_i is defined by the equations $x_{i,2} = g_{i,2}(x_{i,1}), \dots, x_{i,m_i} = g_{i,m_i}(x_{i,m_i-1})$ where each $g_{i,j}$ (for $1 \leq i \leq \ell$ and $2 \leq j \leq m_i$) commutes with an iterate of f.

Since Y projects onto the last δ coordinates $(\mathbb{P}^1)^\delta$ of $(\mathbb{P}^1)^n$, then each index $n - \delta + 1 \leq j \leq n$ is part of a different chain J_i. Furthermore, because for each $i = 1, \dots, \ell$, and for each $j = 2, \dots, m_i$ we have $x_{i,j} = g_{i,j}(x_{i,j-1})$ for some polynomial $g_{i,j}$ (commuting with f), we may assume

$$\{n-\delta+1, \dots, n\} \subseteq \{j_{1,m_1}, \dots, j_{\ell,m_\ell}\}.$$

Because Y projects onto the last δ coordinates $(\mathbb{P}^1)^\delta$, and $(Y \cap \mathbb{A}^n) \setminus U_0$ is non-empty open in Y, given any real number $B > 1$, there exists $(a_1, \dots, a_n) \in (Y \cap \mathbb{A}^n) \setminus U_0$ such that $h(a_{n-\delta+i}) < 1/B$ for each $i = 1, \dots, \delta - 1$, while $h(a_n) > B$. Let S_B denote the set of all such points (which is actually Zariski dense in Y).

Let $\Gamma_0 \subseteq I_n$ consist of:

- each $s = 1, \dots, m$;
- each $s \in \left(\cup_{i=1}^{\ell} J_i\right) \setminus \{j_{1,m_1}, \dots, j_{\ell,m_\ell}\}$;
- each $s = n - \delta + 1, ..., n - 1$.

In other words, x_s for $s \in \Gamma_0$ is either one of the first m "constant coordinates", or one of the "dominated coordinates" in the chains J_i, or one of the $\delta - 1$ coordinates $x_{n-\delta+1}, ..., x_{n-1}$ whose valuations on S_B have heights bounded by $1/B$. By construction, the above three sets are disjoint and therefore:

$$|\Gamma_0| = m + (n - m - \ell) + (\delta - 1) = n - \ell - 1 + \delta \geq r.$$

So we can fix Γ to be a subset of Γ_0 of cardinality r. For each $i \in I_n \setminus \Gamma_0$, let $\Gamma_i := \Gamma \cup \{i\}$ and consider the projection $\pi_i : X \longrightarrow (\mathbb{P}^1)^{\Gamma_i}$. Since $\dim(\pi_i(X)) = r$, then $\pi_i(X) \subset (\mathbb{P}^1)^{\Gamma_i}$ is defined by $F_i = 0$ where F_i is a polynomial in the variables x_k for $k \in \Gamma_i$. We write:

$$F_i = \sum_{j=0}^{D_i} F_{i,j} x_i^j,$$

where $D_i = \deg_{x_i} F_i$ and each $F_{i,j}$ is a polynomial in the variables x_k where $k \in \Gamma$.

Let $(a_1, \dots, a_n) \in S_B$ for some $B > 1$. We claim that for every $i \in I_n \setminus \Gamma_0$, there exists $j = 1, \dots, D_i$ for which $F_{i,j}(a_k)_{k\in\Gamma} \neq 0$. Otherwise, as explained in the proof of Proposition 4.5, the point $(a_1, \dots, a_n)$ would be contained in an anomalous subvariety obtained by intersecting X with $(a_k)_{k\in\Gamma} \times (\mathbb{P}^1)^{I_n\setminus\Gamma}$. This would imply $(a_1, \dots, a_n) \in U_0$, violating the above definition of S_B. Now we use arguments similar to those in the proof of Proposition 5.2. By Corollary 3.4, there exist positive constants c_5 and c_6 depending only on X such that that for every $i \in I_n \setminus \Gamma_0$,

$$\widehat{h}_f(a_i) \leq c_5 \sum_{j\in\Gamma} \widehat{h}_f(a_j) + c_6. \tag{10}$$

There exists $i \in \{1, \dots, \ell\}$ such that $m_i \geq 2$ since otherwise $Z = \zeta \times (\mathbb{P}^1)^{n-m}$ contradicting our assumption that Y is not contained in U_0. Let M be a positive integer to be chosen later. If $\deg(g_{i,m_i}) > M$ for each $i = 1, \dots, \ell$ such that $m_i \geq 2$, then

$$n\widehat{h}_f(a_{i,m_i}) > M \sum_{j=1}^{m_i-1} \widehat{h}_f(a_{i,j}).$$

Using the fact that $(a_1, \dots, a_m) = \zeta$ is constant, and also that $\widehat{h}(a_{n-\delta+1}), \dots, \widehat{h}_f(a_{n-1})$ are uniformly bounded by $1/B < 1$, we conclude that there exists a positive constant c_7 (depending only on n and ζ) such that:

$$n \sum_{i\in I_n\setminus\Gamma_0} \widehat{h}_f(a_i) > M \sum_{j\in\Gamma} \widehat{h}_f(a_j) - c_7. \tag{11}$$

Now fix $M > n^2c_5$, by (10) and (11), $\widehat{h}_f(a_i)$ for $i \in I_n \setminus \Gamma_0$ is bounded above solely in terms of M, n, c_5, c_6, c_7. This contradicts the fact that $\widehat{h}_f(a_n) > B$ once B is chosen to be sufficiently large. In conclusion, there exists $i = 1, \ldots, \ell$ such that $m_i \geq 2$ and $\deg(g_{i,m_i}) \leq M$.

By Proposition 2.4, there are at most finitely many polynomials g of degree bounded by M which commute with an iterate of f. For each such g there are $n(n-1)$ periodic hypersurfaces in $(\mathbb{P}^1)^n$ defined by $x_j = g(x_k)$ for $1 \leq k \neq j \leq n$. Denote the collection of all such hypersurfaces (for all choices of g) by $\mathcal{V}$. We conclude that for every special subvariety Z such that there exists a subvariety $Y \subseteq X \cap Z$ satisfying $\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}$ and $Y \not\subseteq U_0 \cup U_\infty$ then $Z \subseteq V$ for some $V \in \mathcal{V}$.

For every $V \in \mathcal{V}$, let $W_{V,i}$ for $1 \leq i \leq n(V)$ be all the irreducible components of $X \cap V$ and let e_V denote the embedding associated to V as in Section 4. Let U_V denote the union of the anomalous loci of $e_V^{-1}(W_{V,1}), \ldots, e_V^{-1}(W_{V,n(V)})$. By Corollary 4.8, we have that U is exactly the Zariski closed set:

$$U_0 \bigcup U_\infty \bigcup_{V \in \mathcal{V}} e_V(U_V). \tag{12}$$

By definition, both U_0 and U_∞ are a finite union of sets of the form $T(X, J_i, Z_i)$. We finish the proof by using the induction hypothesis for the sets U_V's and Corollary 4.8.

Remark 6.1. The referee points out that we can avoid the use of canonical height as follows. Instead of using part (d) of Lemma 3.3, we can obtain the results of this paper by using the inequality:

$$|h(g(a)) - \deg(g)h(a)| \leq C \deg(g)$$

for every non-constant $g \in \bar{\mathbb{Q}}[x]$ commuting with an iterate of f (which is still assumed to be disintegrated) and every $a \in \mathbb{P}^1$, where C is a constant depending only on f.

7. More general dynamical systems

7.1. **Proof of Theorem 1.3 and Theorem 1.4 in general.** Our proof consists of two steps:

(I) Reduce from φ to a map of the form $\psi = \psi_1 \times \cdots \times \psi_s$ where ψ_i is a coordinate-wise self-map of $(\mathbb{P}^1)^{n_i}$ of the form $w_i \times \cdots \times w_i$ for some $1 \leq n_i \leq n$ and disintegrated $w_i(x) \in \bar{\mathbb{Q}}[x]$ such that $\sum_{i=1}^s n_i = n$ and w_i and w_j are inequivalent for $i \neq j$ (see Definition 7.1).

(II) Consider maps having the same form like ψ above.

The arguments in Step (II) above are essentially the same as those used in the proof of Theorem 1.3 and Theorem 1.4 in the special case $f \times \cdots \times f$; one simply needs to introduce an extra layer of complexity in the notation used in the previous sections. Hence we will explain Step (I) in detail and only briefly sketch the arguments for Step (II). Again, the main ingredient in Step (I) comes from [MS14]. We need the following:

Definition 7.1. *For simplicity, an irreducible curve in $(\mathbb{P}^1)^2$ is called non-trivial if the projection to each factor $\mathbb{P}^1$ is non-constant. Let $A_1(x), A_2(x) \in \bar{\mathbb{Q}}[x]$ be disintegrated polynomials of degrees at least 2. We define $A_1 \approx A_2$ if the self-map $A_1 \times A_2$ of $(\mathbb{P}^1)^2$ admits a non-trivial irreducible periodic curve.*

By [MS14, Proposition 2.35], $A_1 \approx A_2$ if and only if there exist a positive integer N, non-constant $p_1(x), p_2(x) \in \bar{\mathbb{Q}}[x]$, and a disintegrated polynomial $w(x) \in \bar{\mathbb{Q}}[x]$ such that: $A_i^N \circ p_i = p_i \circ w$ for $i = 1, 2$. In other words, we have the commutative diagram:

$$\begin{tikzcd}
(\mathbb{P}^1)^2 \arrow[r, "w \times w"] \arrow[d, "p_1 \times p_2"'] & (\mathbb{P}^1)^2 \arrow[d, "p_1 \times p_2"] \\
(\mathbb{P}^1)^2 \arrow[r, "A_1^N \times A_2^N"'] & (\mathbb{P}^1)^2
\end{tikzcd}$$

We say that the dynamical system $w \times w$ covers $A_1^N \times A_2^N$ by the covering $p_1 \times p_2$. We can easily show that $\approx$ is an equivalence relation on the set of disintegrated polynomials of degrees as least 2 (see also [MS14, Proposition 2.11]) as follows. Suppose we also have $A_2 \approx A_3$, then $A_2^N \approx A_3^N$. Since $w \times A_3^N$ covers $A_2^N \times A_3^N$ thanks to the covering $p_2 \times \mathrm{id}$, we have that $w \approx A_3^N$. Applying the (existence of the) above commutative diagram for the pair (w, A_3^N) instead of the previous pair (A_1, A_2), there exists a positive integer K such that $w^K \times A_3^{NK}$ is covered by $W \times W$ for some disintegrated polynomial W. Hence the dynamical system $A_1^{NK} \times A_2^{NK} \times A_3^{NK}$ on $(\mathbb{P}^1)^3$ is covered by $W \times W \times W$. By using a non-trivial periodic curve under $W \times W$, we have that $A_1^{NK} \times A_3^{NK}$ admits a non-trivial periodic curve. Hence $A_1 \approx A_3$.

Applying the above arguments inductively, we can also show that given disintegrated polynomials $A_1, \ldots, A_\ell$ in the same equivalence class, there exist a positive integer N, non-constant polynomials $p_1, \ldots, p_\ell$ and a disintegrated polynomial w such that $A_i^N \circ p_i = p_i \circ w$ for every $1 \leq i \leq \ell$. In other words, the dynamical system $w \times \cdots \times w$ covers the system $A_1^N \times \cdots \times A_n^N$ by the covering $p_1 \times \cdots \times p_n$.

Now given disintegrated polynomials $f_1, \ldots, f_n$, let $\varphi = f_1 \times \cdots \times f_n$, and let s denote the number of (distinct) equivalence classes corresponding to $f_1, \ldots, f_n$ (under the equivalence relation $\approx$). Let $n_1, \ldots, n_s$ denote the number of distinct f_i's belonging in each of these classes (hence $n_1 + \ldots + n_s = n$). There exist a positive integer N, non-constant $p_1, \ldots, p_n \in \bar{\mathbb{Q}}[x]$ and disintegrated $w_1, \ldots, w_s \in \bar{\mathbb{Q}}[x]$ such that the following holds. For $1 \leq i \leq s$, let ψ_i be the self-map $w_i \times \cdots \times w_i$ on $(\mathbb{P}^1)^{n_i}$. Let $\eta = p_1 \times \cdots \times p_n$ as a self-map on $(\mathbb{P}^1)^n$. After a possible rearrangement of the polynomials $f_1, \ldots, f_n$, we have the commutative diagram:

$$\begin{tikzcd}
(\mathbb{P}^1)^{n_1} \times \cdots \times (\mathbb{P}^1)^{n_s} \arrow[r, "\psi_1 \times \cdots \times \psi_s"] \arrow[d, "\eta"'] & (\mathbb{P}^1)^{n_1} \times \cdots \times (\mathbb{P}^1)^{n_s} \arrow[d, "\eta"] \\
(\mathbb{P}^1)^n \arrow[r, "\varphi^N"'] & (\mathbb{P}^1)^n
\end{tikzcd}$$

Since the statements of Theorem 1.3 and Theorem 1.4 are unchanged when we replace f_i by f_i^N for $1 \leq i \leq n$, we may assume $N = 1$. Write $\psi := \psi_1 \times \cdots \times \psi_s$. By [MS14, Proposition 2.21], every irreducible ψ-periodic subvariety V of $(\mathbb{P}^1)^n = (\mathbb{P}^1)^{n_1} \times \cdots \times (\mathbb{P}^1)^{n_s}$ has the form $V_1 \times \cdots \times V_s$ where each V_i is a ψ_i-periodic subvariety of $(\mathbb{P}^1)^{n_i}$ for $1 \leq i \leq s$. Since η is a finite and coordinate-wise morphism, it satisfies the following properties:

(i) η maps ψ-periodic subvarieties to φ-periodic subvarieties. Some irreducible component of the inverse image of a φ-periodic subvariety under η is ψ-periodic.

(ii) The same conclusion in (i) remains valid for φ-special and ψ-special subvarieties.

(iii) As a consequence of (ii), for every irreducible subvariety X of $(\mathbb{P}^1)^n$ and every irreducible component W of $\eta^{-1}(X)$, η maps the ψ-anomalous locus of W into the φ-anomalous locus of X. Furthermore, we have:

$$\eta(\cup_W W_\psi^{\text{oa}}) = X_\varphi^{\text{oa}} \text{ and } \eta(\cup_W (W - W_\psi^{\text{oa}})) = X - X_\varphi^{\text{oa}}$$

where W ranges over all irreducible components of $\eta^{-1}(X)$.

(iv) Let $J \subseteq I_n$ and let Z be a φ^J-periodic subvariety of $(\mathbb{P}^1)^J$. We have the following:

$$\eta(\cup_W \cup_{Z'} T(W, J, Z')) = T(X, J, Z)$$

where W ranges over all the irreducible components of $\eta^{-1}(X)$ and Z' ranges over all the irreducible components of $(\eta^J)^{-1}(Z)$.

The above properties of η reduce φ to maps of the form $\psi_1 \times \cdots \times \psi_s$. This finishes Step (I). We briefly sketch Step (II).

For Bounded Height, we proceed as follows. We define a ψ-signature $\mathscr{S}$ to be a collection consisting of a signature $\mathscr{S}_i$ for each block $(\mathbb{P}^1)^{n_i}$ for $1 \leq i \leq s$. A ψ-periodic subvariety $V = V_1 \times \cdots \times V_s$ as above is said to have signature $\mathscr{S}$ if each V_i has signature $\mathscr{S}_i$. Note also that a periodic hypersurface V is still defined by $x_i = \zeta$ or by $x_j = g(x_i)$, so e_V is well-defined for ψ. Given an irreducible subvariety W of $(\mathbb{P}^1)^n$ having dimension r, it suffices to show that the set

$$W_\psi^{\text{oa}} \cap (\cup_V V)$$

has bounded height, where V ranges over all irreducible ψ-periodic subvarieties of dimension $n - r$ *having the fixed signature* $\mathscr{S}$.

By Lemma 4.2, once we assume $W_\psi^{\text{oa}} \neq \emptyset$, the projection from W to any $r + 1$ factors is a hypersurface. Proposition 5.2 remains valid with a similar proof (albeit with a more complicated system of notations in order to deal with the different blocks $(\mathbb{P}^1)^{n_i}$). Note that instead of the canonical height $\widehat{h}_f$ used in the proof of Proposition 5.2, here we use the canonical heights $\widehat{h}_{w_i}$ for the coordinates inside the block $(\mathbb{P}^1)^{n_i}$ for $1 \leq i \leq s$. Finally, we apply the induction hypothesis as in the end of Section 5. This finishes the proof of Theorem 1.3.

For part (a) of Theorem 1.4, let J be a subset of I_n and Z a ψ^J-periodic subvariety of $(\mathbb{P}^1)^J$. For $1 \leq i \leq s$, let $J_i = J \cap \{n_{i-1} + 1, \ldots, n_i\}$ (with the convention $n_0 = 0$); in other words, J_i is obtained from the block of variables in $(\mathbb{P}^1)^{n_i}$. Then Z has the form $Z_1 \times \cdots \times Z_s$ where each Z_i is a ψ^{J_i}-periodic subvariety of $(\mathbb{P}^1)^{J_i}$. This fact allows us to repeat the proof in Subsection 6.1 verbatim for the map ψ considered here.

For part (b) of Theorem 1.4, we proceed as follows. Let U denote the union of ψ-anomalous subvarieties of W. Define U_0 and U_∞ as in Subsection 6.2. Let Y be a ψ-anomalous subvariety of X and Z a ψ-special subvariety such that $Y \subseteq X \cap Z$, $\dim(Y) > \max\{0, \dim(X) + \dim(Z) - n\}$ and $Y \not\subseteq U_0 \cup U_\infty$. The same argument as in Section 6 (here we use the canonical heights $\widehat{h}_{w_i}$ for each block $(\mathbb{P}^1)^{n_i}$) shows that there is a finite collection of ψ-periodic hypersurfaces $\mathcal{V}$ such that for every Y

and Z as above, we have $Z \subseteq V$ for some $V \in \mathcal{V}$. Then we apply the induction hypothesis. This finishes the proof of Theorem 1.4.

7.2. **More general questions for rational functions.** We need the following:

Definition 7.2. *A rational function $f(x) \in \bar{\mathbb{Q}}(x)$ of degree $d \geq 2$ is said to be disintegrated if it is not linearly conjugate to x^d, $\pm C_d(x)$ or a Lattès map.*

For the definition of Lattès maps, we refer the readers to [Sil07, Chapter 6]. Questions about the arithmetic dynamics of $f_1 \times \cdots \times f_n$ where each f_i is Lattès reduce to diophantine questions on products of elliptic curves. The Bounded Height Conjecture has also been studied in this context (see, for example [Via03], [Hab09b]).

Now let n be a positive integer and let $f_1(x), \ldots, f_n(x)$ be *disintegrated* rational functions of degrees at least 2. Let φ denote the coordinate-wise self-map $f_1 \times \cdots \times f_n$ on $(\mathbb{P}^1)^n$. Let $I_n := \{1, \ldots, n\}$ as before. Recall that for an ordered subset J of I_n listed as $i_1 \prec \ldots \prec i_m$ where $m = |J|$, we let φ^J denote the self-map $f_{i_1} \times \cdots \times f_{i_m}$ on $(\mathbb{P}^1)^J$. Let X be an irreducible subvariety of $(\mathbb{P}^1)^n$. We can define φ-special, φ-pre-special, φ-anomalous and φ-pre-anomalous subvarieties and the sets X_φ^{oa} and $X_\varphi^{\mathrm{oa,pre}}$ as in Definition 2.1 and Definition 1.2. We expect an affirmative answer for the following:

Question 7.3. *Let n be a positive integer, let $f_1, \ldots, f_n \in \bar{\mathbb{Q}}(x)$ be disintegrated rational functions of degrees at least 2 and let φ be the associated self-map of $(\mathbb{P}^1)^n$. Let X be an irreducible subvariety of dimension r in $(\mathbb{P}^1)^n$.*

(a) *Is it true that the set X_φ^{oa} is Zariski open in X and $X_\varphi^{\mathrm{oa}} \cap \mathrm{Per}_\varphi^{[r]}$ has bounded height?*
(b) *Is it true that the set $X_\varphi^{\mathrm{oa,pre}}$ is Zariski open in X and $X_\varphi^{\mathrm{oa,pre}} \cap \mathrm{Pre}_\varphi^{[r]}$ has bounded height?*

Proposition 7.4. *We have the following:*

(a) *Assume the Medvedev-Scanlon classification (Theorem 2.3) is valid for a disintegrated rational function f. Then part (a) of Question 7.3 has an affirmative answer when $f_1 = \ldots = f_n = f$.*
(b) *Assume the Medvedev-Scanlon classification is valid for* every *disintegrated rational function. Then part (a) of Question 7.3 has an affirmative answer.*

Proof. Part (a) can be proved by essentially the same arguments in Section 5 and Section 6. Besides the Medvedev-Scanlon classification, the two key results needed in those proofs are: part (d) of Proposition 2.4 stating that for every given degree there are only finitely many g commuting with an iterate of f and part (b) of Lemma 3.2. In fact, we used the "affine part" of X (and the set U_∞ in Subsection 6.2) in the above proofs solely for simplifying the notation. In the more general case, we can instead work with the *multi-homogeneous* polynomial F^J defining the hypersurface $\pi^J(X)$ in $(\mathbb{P}^1)^J$ (for $|J| = r+1$). More details are given in Propositions 7.5 and 7.6.

For part (b), we consider the general case $\varphi = f_1 \times \cdots \times f_n$ for disintegrated $f_1, \ldots, f_n$. The arguments in Step (I) in the last subsection actually work when $f_1, \ldots, f_n$ are disintegrated *rational* functions. This boils down to [MS14, Definition 2.20, Fact 2.25]; although the statement given by Medvedev-Scanlon in [MS14, Fact 2.25] only treats the polynomial case, it is also known to be true for rational functions thanks to Medvedev's PhD thesis [Med07, Theorem 10]. In other words, we reduce φ to a map of the form $\psi_1 \times \cdots \times \psi_s$ where each ψ_i is a coordinate-wise self-map of $(\mathbb{P}^1)^{n_i}$ of the form $w_i \times \cdots w_i$ for some disintegrated rational function $w_i \in \bar{\mathbb{Q}}(x)$. After that, Step (II) could be done as in the previous subsection; this requires the Medvedev-Scanlon classification for each w_i. □

We need a counterpart of Proposition 2.4 for rational functions:

Proposition 7.5. *Let $f \in \bar{\mathbb{Q}}(x)$ be a disintegrated rational function. Then the following hold:*

(a) *If $g \in \bar{\mathbb{Q}}(x)$ has degree at least 2 and commutes with an iterate of f then g and f have a common iterate.*
(b) *The group $\mathrm{Aut}(f^\infty)$ of all Möbius maps commuting with an iterate of f is finite.*
(c) *Assume that the Medvedev-Scanlon classification (Theorem 2.3) is valid for f. In the collection of rational functions of degrees at least 2 commuting with an iterate of f, choose $\tilde{f}$ that has the smallest degree. We have:*
$$\left\{\tilde{f}^m \circ L:\ m \geq 0,\ L \in \mathrm{Aut}(f^\infty)\right\} = \left\{L \circ \tilde{f}^m:\ m \geq 0,\ L \in \mathrm{Aut}(f^\infty)\right\}$$
and this is exactly the set of all rational functions commuting with an iterate of f.

Proof. Part (a) is a well-known theorem of Ritt [Rit23] while part (c) could be proved by the same arguments used in [Ngu13, Proposition 2.3(d)]. For part (b), Levin's theorem [Lev90] implies that for a given degree there are at most finitely many rational functions having the same iterate with f. Together with part (a), we have that the set $\{L \circ f:\ L \in \mathrm{Aut}(f^\infty)\}$ is finite. Hence $\mathrm{Aut}(f^\infty)$ is finite. □

We now let $[x_i : y_i]$ be the homogeneous coordinate on the i^{th} factor of $(\mathbb{P}^1)^n$ for $1 \leq i \leq n$. For every $P \in \mathbb{P}^1$, we choose the homogeneous coordinate $[a : b]$ so that $b = 1$ or $(a, b) = (1, 0)$. Hence given any homogeneous polynomial $F(x, y)$, we have a well-defined value $F(P)$. Now let $F(X_1, Y_1, \ldots, X_n, Y_n)$ be a multi-homogeneous polynomial that is homogeneous in each $[X_1 : Y_1], \ldots, [X_n : Y_n]$. Let D be the homogeneous degree of F in $[X_n : Y_n]$. Assume $D \geq 1$ and write:
$$F = F_D X_n^D + F_{D-1} X_n^{D-1} Y_n + \ldots + F_0 Y_n^D$$
where each F_j is a multi-homogeneous polynomial in $X_1, \ldots, Y_{n-1}$. We have the following:

Proposition 7.6. *There exist positive constants C_8 and C_9 depending only on F such that for every $(P_1, \ldots, P_n) \in (\mathbb{P}^1)^n$ satisfying $F_j(P_1, \ldots, P_{n-1}) \neq 0$ for some $1 \leq j \leq D$, we have:*
$$h(P_n) - C_8(h(P_1) + \ldots + h(P_{n-1})) - C_9 \leq h(F(P_1, \ldots, P_n)).$$

Proof. The inequality is obvious (for any positive C_8 and C_9) when $P_n = \infty$. When $P_n \neq \infty$, we can use completely similar arguments used in the proof of part (b) of Lemma 3.2. □

Remark 7.7. In an ongoing joint work of the second author with Michael Zieve, there are given reasons to expect that the Medvedev-Scanlon classification should hold for every disintegrated rational function.

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