Lecture 9

With Newman's theorem as motivation we introduce the idea of scaling limits. Scaling limits are a way to focus only on the long distance fluctuations of a statistical mechanical model. Many different models can have the same scaling limit. When two different models have the same scaling limit we say they are in the same universality class. The grand goal of equilibrium statistical mechanics is to classify scaling limits. A starting point is to ask which models are in the universality class of the massless free field.

The renormalization group is one way to answer this question. We will get used to the main ideas in the context of hierarchical models.

White noise

White noise

$$W = \{ W(X) : X \subset \mathbb{R}^d, \ |X| < \infty \}$$

is a collection of Gaussian random variables such that

$$\mathrm{Cov}\left(W(X), W(Y) \right) = |X \cap Y|$$

$$W(\cup X_i) \overset{a.s}{=} \sum W(X_i), \quad \{X_i\} \text{ disjoint}$$

For $X \subset \mathbb{R}^d$, $[\phi] > 0$, let

$$\phi(L, X) = \sum_{y \in LX \cap \mathbb{Z}^d} L^{-[\phi]} \left(\phi_y - \langle \phi_y \rangle \right)$$

The conclusion of Newman's theorem can be restated as, for $X \in \mathcal{P}_{L=1}$,

$$\phi(L, X) \Longrightarrow W(X)$$

$[\phi] = d/2$

We say that W is the scaling limit of ϕ.

$[\phi]$ is called the dimension of ϕ.
Choosing "the wrong" value for $[\phi]$ will give either no limit or a trivial limit concentrated on the zero field.

We say that two models are in the same universality class if they have the same scaling limit. Thus Newman's theorem is saying that all non-critical ferromagnetic models are in the same universality class, where non-critical means

$$\sum_y \mathrm{Cov}(\phi_x, \phi_y) < \infty.$$

The grand goal of equilibrium statistical mechanics is to classify the universality classes for models which are critical,

$$\sum_y \mathrm{Cov}(\phi_x, \phi_y) = \infty.$$

Example 1

Recall that the ∞ vol limit of the massless Gaussian on $\mathbb{Z}^d$ has

$$\langle \phi_x \, \phi_y \rangle_\infty = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \phi_x \, \phi_y \rangle_\Lambda$$

$$= (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{1}{\sum_{\substack{u \in \mathbb{Z}^d \\ \|u\|=1}} (e^{ik \cdot u} - 1)} e^{ik \cdot (x-y)} \, dk$$

$(d > 2)$

Calculation (problem 2) shows, for $[\phi] = \frac{d-2}{2}$,

$$\langle \phi(L,x) \, \phi(L,y) \rangle$$

$$\xrightarrow[L \to \infty]{} (2\pi)^{-d} \int_{\mathbb{R}^d} (1_X)^\wedge(k) \, \frac{1}{k^2} \, (1_Y)^\wedge(k) \, dk$$

$$= c_d \int_X \int_Y \frac{1}{\|x-y\|^{d-2}} \, dx \, dy \qquad \text{(e: ex1)}$$

Is there a Gaussian field with covariance $\|x-y\|^{-(d-2)}$?

Let

$$[\phi] \in (0, d/2)$$

<u>Proposition 2</u> There exists a probability space with

$$\phi = \{\phi(X) : X \subset \mathbb{R}^d \quad \int_X \int_X \|x-y\|^{-2[\phi]} \, dx\, dy < \infty\},$$

which are gaussian random variables such that

$$\mathrm{Cov}(\phi(X), \phi(Y))$$

$$= \int_X \int_Y \|x-y\|^{-2[\phi]} \, dx\, dy$$

and $\phi(X \cup Y) = \phi(X) + \phi(Y)$ a.s

holds for $X \cap Y = \emptyset$.

The case $[\phi] = \frac{d-2}{2}$ is called the <u>massless continuum free field</u>.

<u>Proof</u>

The next Proposition constructs ϕ with these properties.

$\boxtimes$

Proposition 3

Let $L \geq 1$, There exists a Gaussian

$$\zeta = \{ \zeta(x) \in C^\infty, \ x \in \mathbb{R}^d \}$$

s.t.

(1) $\mathrm{Cov}(\zeta(x), \zeta(y)) = 0 \qquad \|x-y\| \geq L/2$

(2) With

$$\zeta_j(x) \overset{\mathcal{D}}{=} L^{-j[\phi]} \zeta\left(\frac{x}{L^j}\right),$$

independent scaled copies of ζ,

$$\phi(X) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \int_X \zeta_j(x)\, dx \qquad \text{(prop3:1)}$$

converges a.s., and ϕ satisfies the conclusions of Prop. 2.

To prove this we use

Lemma 4

Let $u(x) = u(\|x\|) \in C_0(\mathbb{R}^d)$

$\exists\ c > 0$ s.t. for $x \neq 0$

$$\|x\|^{-2[\phi]} = \int_0^\infty \frac{d\ell}{\ell}\, \ell^{-2[\phi]}\, c\, u\left(\frac{x}{\ell}\right)$$

Proof

Let $\ell = \|x\|\,\ell'$. Then

$$\text{RHS} = \|x\|^{-2[\phi]} \int_0^\infty \frac{d\ell'}{\ell'}\, \ell'^{-2[\phi]}\, c\, u(1/\ell)$$

$$= \|x\|^{-2[\phi]}$$

by choice of c

$\blacksquare$

Part of proof
of Prop. 3

In Lemma 4
choose $u \in C_0^\infty$ and absorb c into u.
We can also assume $\hat{u}(k) \geq 0$ because
we can replace u by $u * u$ which is still
C^∞ and of compact support. We can choose
the support so that $u(x) = 0$ for $|x| \geq \frac{1}{2}$.
Let

$$C(x) = \int_1^L \frac{dl}{l}\, l^{-2[\phi]}\, u(x/l)$$

Then $\hat{C}(k) \geq 0$ and $C(x) = 0$ for $|x| \geq L/2$.

Standard theory of Gaussian processes
$\Longrightarrow$

$\exists\ \zeta \in C^\infty$ covariance $C(x-y)$

By Lemma 4,

$$\|x-y\|^{-2[\phi]} = \sum_{j \in \mathbb{Z}} L^{-2j[\phi]}\, C\left(\frac{x-y}{L^{2j}}\right). \qquad (*)$$

We construct
a probability space carrying independent
"increments" ζ_j with covariance $L^{-2j[\phi]} C\left(\frac{x-y}{L^j}\right)$.
Define $\phi(x)$ by (prop3:1). This series
converges a.s. by Thm 8.3 of Durrett and (problem 1)
$\phi(x)$ defined this way has the properties
claimed in Proposition 2 because $(*)$ makes
the covariances match. ▨

This construction has created the scaling limit which labels the universality class of the lattice massless free field.

What other models are in this universality class?

<u>Theorem 5</u> (Aizenman 1981, Fröhlich 1981)

In $d \geq 5$ the scaling limit of the nearest neighbour ferromagnetic Ising model, if it exists, is Gaussian.

This is also true for the ϕ^4 field lattice field theory (which we have not yet defined).

This result was proved by random walk representation related to lecture 6. There is another possible way to prove this type of result. It is weaker in that it requires a small parameter and stronger in that it applies to a much wider class of models and also proves existence of the scaling limit.

Being much more complicated I want to first exhibit the ideas for hierarchical models.

Hierarchical models

These were invented by (Dyson, 1969), but not quite in the form I am about to describe, which is inspired by (Gallavotti et al 1978) and (Evans, 1989). The d-dimensional hierarchical lattice Λ_∞ with parameter $L > 1 \in \mathbb{N}$, is a countable Abelian group with the following properties

1. There is an ultrametric defined by a norm

$$|x+y| \le |x| \vee |y|$$

2. There is a map $L^{-1} : \Lambda_\infty \to \Lambda_\infty$ such that

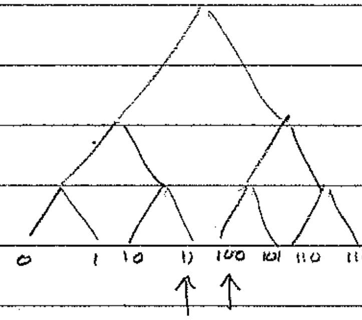
$$|L^{-1}x| = \frac{|x|}{L} \quad \text{if } L^{-1}x \neq 0$$

3. The ball $\{x : |x-y| \le L^p\}$ has L^{dp} points.

Example 6 $(L=2,\ d=1)$

$\Lambda_\omega = \{\text{all finite binary sequences}\}$

Group structure $\oplus \mathbb{Z}_2$



$|100-11| = 2^3$

$$\begin{array}{r} 100 \\ -\ \ 11 \\ \hline 111 \end{array}$$

The map $2^{-1}: \Lambda_\omega \to \Lambda_\omega$ is right shift $\equiv$ collapse ball.

$(\overset{\neq 0}{x_n} \cdots x_2, x_1) \longmapsto (y_{n-1}, y_{n-2}, \cdots, y_1)$

$y_1 = x_2$

$y_2 = x_3$

$\vdots$

$y_{n-1} = x_n$

The metric $|x| = \begin{cases} 2^{-n}, & x = (\overset{\neq 0}{x_n}, \cdots, x_1), \quad n \geq 1 \\ 0 \end{cases}$

satisfies

$$|2^{-1}x| = \frac{|x|}{2} \quad \text{if } |2^{-1}x| \neq 0$$

$$|x+y| \leq |x| \vee |y|$$

Ultrametric $\iff$ no balls overlap: $B \cap B' \neq \emptyset$

$\Rightarrow B \subset B'$ or $B' \subset B$

Dimension $d=1$

There are 2^p points in the ball $|x| \leq 2^p$

The hierarchical free field

We construct the hierarchical gaussian free field

$$\Phi = \{\Phi_x : x \in \Lambda_\infty\}$$

by creating the same structure as in Proposition 3.

Let

$$\zeta = \{\zeta_x : x \in \Lambda_\infty\}$$

be gaussian such that

$$\mathrm{Cov}(\zeta_x, \zeta_y) = 0$$

if $|x-y| > L$

Then define independent scaled copies

$$\zeta_j(x) \overset{\mathcal{D}}{=} L^{-j[\Phi]} \zeta(L^{-j} x)$$

where

$$L^{-j} = (L^{-1})^j : \Lambda_\infty \to \Lambda_\infty$$

Then we set

$$\phi(x) = \sum_{j \geq 1} \zeta_j(x)$$

a.s. convergence on a big probability space carrying all the increments ζ_j.

This means that

$$\phi = \zeta_1 + \phi'$$

$$\phi' \overset{\mathcal{D}}{=} L^{-[\phi]} \phi(L^{-1} x)$$

and, (problem 4),

$$\phi'_x = \phi'_y \quad \text{a.s.} \qquad \text{(e: phiprime)}$$

for $|x-y| \leq L$

Since this is an ultrametric no balls overlap and balls are the same as blocks $B \in \mathcal{B}_L$.

Problems

1. Prove that $\phi(X)$ defined by (prop3:1) has the properties claimed in Proposition 2.

2. Prove (e:ex1).

3. Construct a d-dimensional hierarchical lattice

4. Prove (e:phiprime).