The Newman Central Limit Theorem

The result in this lecture is a model for the type of result that the rest of this course will be elaborating on. It is a very sophisticated central limit theorem that characterises the long distance structure of fluctuations in a class (ferromagnetic) of statistical mechanical models which are not critical. The term critical will be defined later.

We say

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}$$

is <u>increasing</u> if, for $x, y \in \mathbb{R}^n$,

$$x_l \le y_l \quad \text{for } l = 1, 2, \cdots, n$$

$$\Longrightarrow$$

$$F(x) \le F(y).$$

<u>Defn 1</u>

A finite set $X = \{X_1, \cdots, X_n\}$ of random variables is <u>FKG</u> if

$$\mathrm{Cov}\big(F(X), G(X)\big) \ge 0$$

for all increasing F, G. An infinite set of random variables is FKG if every finite subset is FKG

FKG = Fortuin Kasteleyn Ginibre

Note that all increasing functions of FKG random variables are themselves FKG random variables

<u>Theorem 2</u> (FKG 1971, Thm 1.1 of Battle-Rosen 1980)

All ferromagnetic systems

$$d\mu(x) = \frac{1}{Z}\, d^n x\, e^{F(x)}$$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} \geq 0, \qquad 1 \leq i \neq j \leq n,$$

are FKG.

<u>Block spins</u>

For $x \in \mathbb{Z}^d$, $L \in \mathbb{N}$,

$$\Phi_L(x) = |B(x)|^{-1/2} \sum_{y \in B(x)} (\Phi_y - \langle \Phi_y \rangle)$$

where $B(x) \in \mathcal{B}_L$ is the block in $\mathcal{B}_L$ centred on Lx.

<u>Assume</u>

$\phi = \{\phi_x : x \in \mathbb{Z}^d\}$ is such that

(1) Probability law is $\mathbb{Z}^d$ translation invariant.

(2) $\langle \phi_x^2 \rangle < \infty$.

(3) $\sum_{y \in \mathbb{Z}^d} Cov(\phi_x, \phi_y) < \infty$

(4) ϕ is FKG.

<u>Theorem 3</u> (Newman 1980)

$$\{\phi_L(x) : x \in \mathbb{Z}^d\} \underset{L \to \infty}{\Longrightarrow} \text{i.i.d. Gaussian}$$

Lemma 4

If X, Y are EKG, $f, g \in C^1$,
THEN

$$\mathrm{Cov}\left(f(X), g(Y)\right)$$

$$\leq \|f'\|_\infty \, \|g'\|_\infty \, \mathrm{Cov}(X,Y)$$

Proof

For $f(s) \to 0$, as $s \to -\infty$,

$$\mathbb{E} f(X) = \int \mathbb{P}(X > s)\, f'(s)\, ds$$

Proof: insert $f(X) = \int_{s<X} f'(s)\, ds$. Similarly
for f, g with $f(-\infty) = g(-\infty) = 0$,

$$\mathrm{Cov}(f(X), g(Y))$$

$$= \iint \overset{(*)}{\left(\mathbb{P}\{X>s, Y>t\} - \mathbb{P}\{X>s\}\mathbb{P}\{Y>t\}\right)}$$

$$f'(s)\, g'(t)\, ds\, dt$$

In this formula we no longer need the condition $f(\infty) = g(-\infty) = 0$.

$$(*) = \mathrm{Cov}\left(\mathbb{1}_{X>s}, \mathbb{1}_{Y>t}\right) \geq 0$$

by choosing
$f(s) = s$,
$g(t) = t$

$$\leq \|f'\|_\infty \, \|g'\|_\infty \underbrace{\iint (*)\, ds\, dt}_{= \mathrm{Cov}(X,Y)}$$

Proposition 5

If $\{X_i, i=1,2,\cdots,n\}$ are FKG then

$$\left| \langle e^{i\sum r_j X_j} \rangle - \prod_{j=1}^{n} \langle e^{i r_j X_j} \rangle \right|$$

$$\leq \frac{1}{2} \sum_{\substack{j,k=1,\dots,n \\ j \neq k}} \mathrm{Cov}(X_j, X_k) \, |r_j r_k|$$

Proof. By induction on n. Lemma 4 starts the induction at $n=2$ and Lemma 4 also accomplishes the inductive step. For details see [Newman].

Without loss of generality assume

$$\langle \phi_x \rangle = 0$$

$$\sum_{x \in \mathbb{Z}^d} \mathrm{Cov}(\phi_x, \phi_y) = 1$$

Write

$$\Phi(X) = |X|^{-1/2} \sum_{x \in X} \phi_x \qquad X \subset \mathbb{Z}^d$$

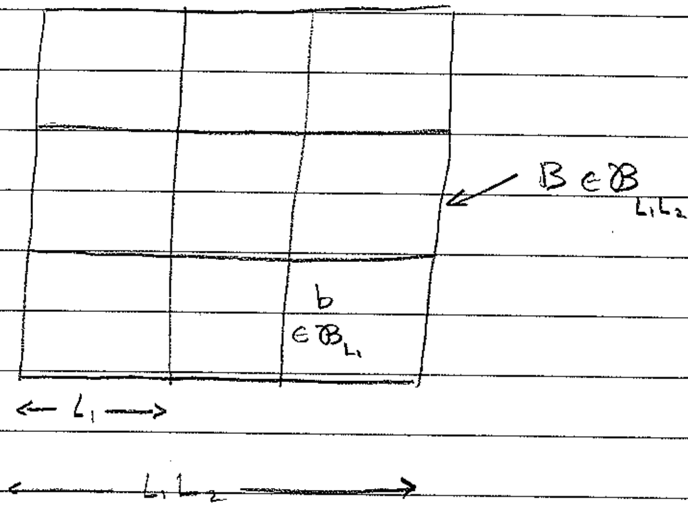
$$f_L(r) = \langle e^{ir\phi(B)} \rangle \qquad B \in \mathcal{B}_L$$

Lemma 6 If $g(r)$ is C^2 at $r=0$ and $g(0) = 1$, $g'(1) = 0$, then

$$\lim_{n \to \infty} \left(g\left(r/\sqrt{n} \right) \right)^n = e^{g''(0) \frac{r^2}{2}}$$

Proof Taylor expansion.

Idea



To prove that

$$f_L(r) \to e^{-\frac{1}{2} r^2}$$

and independence of $\{\phi(B(x))\}$ as $L \to \infty$.

① For $L_1 >> 1$, pairs (x, y)

$$x \in b, \ y \in b' \neq b$$

make negligible contribution to $\frac{1}{|B|} \sum_{x,y \in B} \mathrm{Cov}(\phi_x, \phi_y)$

② $$\phi(B) = \sqrt{\frac{|b|}{|B|}} \sum_{b \in \mathcal{B}_L(B)} \phi(b)$$

so Prop 4 $\Rightarrow$

$$\left| \langle e^{ir\phi(B)} \rangle - \left(\langle e^{ir \frac{1}{\sqrt{n}} \phi(b)} \rangle \right)^n \right| < \epsilon(L_1)$$

$$n = \frac{|B|}{|b|}$$

uniform in n

As $L_2 \to \infty$, $n \to \infty$, so Lemma 6 $\Rightarrow$

$$\left(\langle e^{ir \frac{1}{\sqrt{n}} \phi(b)} \rangle \right)^n \longrightarrow e^{-\frac{r^2}{2} \mathrm{Var}\, \phi(b)}$$

(3) Combining Lemma 6 with (2)

$$\lim_{L_1 \to \infty} \limsup_{L_2 \to \infty} \left| f_{L_1 L_2}(r) - e^{-\frac{1}{2} r^2} \right| = 0$$

(4) $$\lim_{L \to \infty} \left| f_L(r) - e^{-\frac{1}{2} r^2} \right| = 0$$

Lemma 7 (Step ①)

For $B \in \mathcal{B}_L$

$$\lim_{L \to \infty} \sum_{B' \in \mathcal{B}_L(B^c)} \mathrm{Cov}(\phi(B), \phi(B')) = 0$$

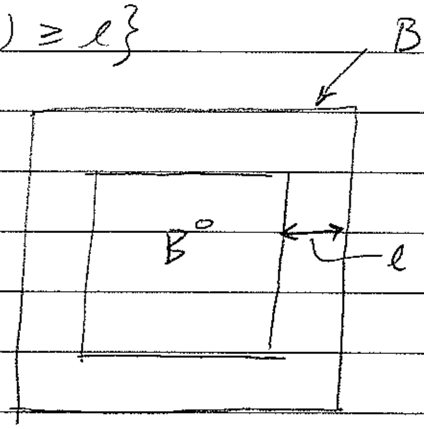
Proof

Let $\epsilon > 0$. Let ℓ:

$$\sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\| \geq \ell}} \mathrm{Cov}(\phi_x, \phi_y) < \epsilon$$

Let

$$B^o = \{x \in B : \mathrm{dist}(x, B^c) \geq \ell\}$$



$$\sum'_{B' \in \mathcal{B}(B^c)} \mathrm{Cov}\left(\phi(B), \phi(B')\right)$$

change to $x \in B \setminus B^\circ$

$$= \frac{1}{|B|} \sum_{x \in B}' \sum_{y \notin B} \mathrm{Cov}(\phi_x, \phi_y) \, \mathbb{1}_{\|x-y\| < l} \qquad (1)$$

$$+ \quad - \mathbb{1}_{\|x-y\| \geq l}$$

$$\leq \underbrace{\frac{|B \setminus B_0|}{|B|}}_{\to 0 \text{ as } L \to \infty} + \underbrace{\frac{1}{|B|} \sum_{x \in B} \epsilon}_{\leq \epsilon}$$

$$\leq 2\epsilon \quad \text{for all large } L$$

<u>Lemma 8</u>

$$f_L''(0) \longrightarrow 1 \qquad L \to \infty$$

<u>Proof</u> Like proof of Lemma 7.

(2) <u>Lemma 9</u>

$$\limsup_{L_1 \to \infty} \sup_{L_2} \left| f_{L_1 L_2}(r) - \left(f_{L_1}\left(\frac{r}{L_2^{d/2}}\right)\right)^{L_2^d} \right|$$

$$= 0$$

<u>Proof</u>

$$\Phi(B) = L_2^{-d/2} \sum_{b \in \mathcal{B}_{L_1}(B)} \phi(b)$$

By Prop. 5, for all L_2,

$$\left| f_{L_1 L_2}(r) - f_{L_1}\left(\frac{r}{L_2^{d/2}}\right)^{L_2^d} \right|$$

$$\leq L_2^{-d} \sum_{b \in \mathcal{B}_{L_1}(B)} \sum_{b' \in \mathcal{B}_{L_1}(b^c)} \mathrm{Cov}(\phi(b), \phi(b'))$$

$$\leq \sum_{b' \in \mathcal{B}_{L_1}(b^c)} \mathrm{Cov}\left(\phi(b), \phi(b')\right)$$ any $b \in \mathcal{B}_L(B)$.

$\longrightarrow 0$ as $L_1 \to \infty$

by Lemma 7

uniform in L_2.

▨

(3) Lemma 10 Let $\epsilon > 0$. $\exists L_1$ s.t.

$$\lim_{L_1 \to \infty} \limsup_{L_2 \to \infty} \left| f_{L_1 L_2}(r) - e^{-\frac{1}{2} r^2} \right| = 0$$

Proof

$$\left| f_{L_1 L_2}(r) - e^{-\frac{1}{2} r^2} \right|$$

$$\leq \sup_{L_2} \left| f_{L_1 L_2}(r) - \left(f_{L_1}\left(\frac{r}{L_2^{d/2}} \right) \right)^{L_2^d} \right| \quad (1)$$

$$+ \left| \left(f_{L_1}\left(\frac{r}{L_2^{d/2}} \right) \right)^{L_2^d} - e^{-\frac{1}{2} r^2} \right| \quad (2)$$

Taking $\limsup_{L_2 \to \infty}$, (2) $\to 0$

Taking $L_1 \to \infty$ (1) $\to 0$

$\blacksquare$

④ Lemma 11

For $L_1, L \in \mathbb{N}$ define

$$L_2 = \left\lfloor \frac{L}{L_1} \right\rfloor$$

Then

$$\lim_{L \to \infty} \left(f_L(r) - f_{L_1 L_2}(r) \right) = 0$$

Proof L_2 defined s.t.

$$L_1 L_2 < L < L_1 L_2 + L_1$$

Let

$$B \in \mathcal{B}_L, \quad \tilde{B} \in \mathcal{B}_{L_1 L_2}$$

centred on $x = 0$. Then

$$B = \tilde{B} \cup X$$

$$\frac{|X|}{|\tilde{B}|} \leq \frac{L^d - (L_1 L_2)^d}{(L_1 L_2)^d}$$

$$\leq \frac{(L_1 L_2 + L_1)^d - (L_1 L_2)^d}{(L_1 L_2)^d} = O\left(\frac{1}{L_2}\right)$$

so X becomes negligible relative to $\tilde{B}$

Prop 5 $\Rightarrow$ (skipping some details!)

$$\langle e^{ir\phi(B)} \rangle - \langle e^{ir\phi(\tilde{B})} \rangle \to 0$$

as $L \to \infty$

Then we obtain

$$\lim_{L \to \infty} f_L(r) = e^{-\frac{1}{2} r^2}$$

by

$$\left| f_L(r) - e^{-\frac{1}{2} r^2} \right|$$

$$\leq \underbrace{\left| f_L(r) - f_{L_1 L_2}(r) \right|}_{\substack{\to 0 \text{ as } L \to \infty \\ \text{by Lemma 11}}} + \underbrace{\left| f_{L_1 L_2}(r) - e^{-\frac{1}{2} r^2} \right|}_{\substack{\leq \epsilon \\ \text{by Lemma 10}}}$$

holds for all ϵ because L_1 is arbitrary

$\blacksquare$

Problems

① Look up and be prepared to present the proof of Theorem 1.1 (a version of FKG inequalities) in [Battle Rosen 1980].

② In the proof of Lemma 4 explain why the conditions $f(-\infty) = g(-\infty) = 0$ were dropped.

③ Prove Lemma 6

④ Prove Lemma 8

④ Fill in where it says "skipping some details" in the proof of Lemma 11.