Broken Symmetry

So far we have encountered different models and relations between them. All those models boil down to integrals of the form

$$Z = \int e^{-\alpha S(\phi)} d^{\Lambda}\phi$$

and the associated measure

$$\frac{1}{Z} e^{-\alpha S(\phi)} d^{\Lambda}\phi$$

Then there is the idea of mean field theory: when $\alpha >> 1$, the measure concentrates onto the minima of $S(\phi)$. In discussions we have also encountered the enemy of this idea, which is in the infinite volume limit fluctuations around the minima may cause the model to forget which minimum was selected by the boundary condition (or conditioning on the tail algebra $\mathcal{T}$). We have seen that the massless Gaussian in $\mathbb{Z}^2$ does forget the Dirichlet b.c at ∞, but in $\mathbb{Z}^{d\geq 3}$ does not. Fluctuations around the minima are modelled by Gaussians because at a minimum ϕ_0 $S(\phi) = S(\phi_0) + \frac{1}{2}(\phi-\phi_0) S''(\phi-\phi_0) + O(\phi-\phi_0)^3$, but there are corrections from $O(\phi-\phi_0)^3$. Can we still use the Gaussian intuition? In this lecture we see a proof of existence of phase transitions that relies on "Gaussian bounds" that capture this intuition

Consider models of the form

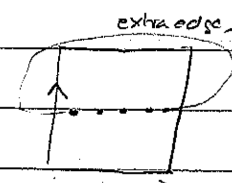
$$Z = \int \prod_{x \in \Lambda} d\rho(\phi_x) \; e^{-\frac{1}{2}(\phi, -\Delta_\Lambda \phi)}$$

$\phi : \Lambda \to \mathbb{R}^N$ vector-valued

$d\rho$ $O(N)$ invariant

Periodic b.c.'s: Λ is a torus.

$$(\phi, -\Delta_\Lambda \phi) = \sum_{xy \in \text{Edges}(\Lambda)} \| \phi_x - \phi_y \|^2$$



Edges (Λ): y is a nearest neighbour to x if $y = x + e \mod (\text{side}(\Lambda))$, $\|e\| = 1$

Examples

① $N = 1$, $d\rho(\phi_x) = \delta(\phi_x - \sqrt{\beta}) + \delta(\phi_x - \sqrt{\beta})$ gives Ising model at $(\text{temperature})^{-1} = \beta$

② $N > 1$, $d\rho$ is surface Lebesgue measure on sphere of radius $\sqrt{\beta}$. This is called the N-vector model or $O(N-1)$ model.

The joint distributions of $\phi = (\phi_x, x \in \Lambda)$ are $O(N)$ invariant and translation invariant:

$$\mathbb{P}_\Lambda \{ \phi \in E \} = \mathbb{P}_\Lambda \{ \phi \in RE \} \qquad R \in O(N-1)$$

$$\mathbb{P}_\Lambda \{ (\phi_x)_{x \in X} \in E \} =$$

$$= \mathbb{P}_\Lambda \{ (\phi_x)_{x-a \in X} \in E \} \qquad a \in \mathbb{Z}^d$$

Therefore any ∞ vol limit $\mathbb{P}_\infty$ also has these properties. Let $\langle \; \rangle_\infty$ be the expectation for an ∞ vol limit.

<u>Theorem 1</u> (Fröhlich-Simon-Spencer 1976)

For $d \geq 3$, $\beta >> 1$, $\exists$ $c(\beta) > 0$.

$$\lim_{y \to \infty} \langle \phi_x \cdot \phi_y \rangle_\infty = c(\beta) \qquad \text{(LRO)}$$

This result implies the algebra $\mathcal{T}$ is non trivial as follows

(1) $\lim_{X} \frac{1}{|X|} \sum_{x \in X} \phi_x$ exists a.s. $\mathbb{P}_\infty$ by ergodic theorem

This limit defines a random variable $Y \in m\mathcal{T}$

(2) This random variable Y is not a.s. constant because

$$\mathrm{Var}\, Y = \langle Y^2 \rangle_\infty - \underbrace{\langle Y \rangle_\infty^2}_{=0 \text{ by } O(N-1) \text{ symmetry}}$$

$$= \lim \frac{1}{|X|^2} \sum_{x,y \in X} \langle \phi_x \cdot \phi_y \rangle_\infty \qquad \text{Dominated convergence}$$

$$= c(\beta) > 0 \qquad \text{(Homework problem 1)}$$

Therefore $\{Y \in E\}$ is a non trivial event in $\mathcal{T}$.

High Temperature expansion proves that for $\beta \ll 1$ $\mathcal{T}$ is trivial, so $\exists\, \beta_c$ where phase transition takes place.

Physically: for $\beta > \beta_c$ there is long range order, a boundary condition that selects a preferred direction for Φ will be "remembered" by Φ_0 no matter how far away the boundary is. This is called broken O(N) symmetry. For $\beta < \beta_c$ the boundary condition is not remembered, all correlations decay exponentially.

For the Ising model $(N=1)$, the hypothesis $d \geq 3$ is misleading in the sense that there is also a phase transition in $d=2$.

Proposition 2 (Infra - red bound)

For $f : \Lambda \to \mathbb{R}^N$, $f \perp$ to all constant fields,

$$< (\phi, f) \cdot (\phi, f) >_\Lambda \quad \le \quad (f, (-\Delta_\Lambda)^{-1} f) \tag{IR}$$

Proof of Theorem 1

Let $|\Lambda| \to \infty$.

For f compact support, $(f, 1) = 0$,

$$< (\phi, f) \; (\phi, f) >_\infty \quad \le \quad \int |\hat{f}(k)|^2 \text{ "} \frac{1}{k^2} \text{" } dk$$

where "$\frac{1}{k^2}$" is $\dfrac{1}{\sum_{\substack{x \\ \|x\|=1}} (1 - e^{ik\cdot x})}$, $\int dk = \int_{[-\pi, \pi]^d} dk$

$< \phi_x \cdot \phi_y >_\infty$ is a positive definite function $F(x-y)$. Therefore, by Bochner's thm, there exists a measure* $d\omega(k)$ s.t.

$$< \phi_x \cdot \phi_y >_\infty \; = \; \int e^{ik(x-y)} \, d\omega(k) \tag{Bochner}$$

* positive measure

In terms of $d\omega$, (IR) says

$$\int |f(k)|^2 \, d\omega(k) \leq \int |\hat{f}(k)|^2 \, \text{"}\frac{1}{k^2}\text{"} \, dk$$

From this, recalling $f \perp$ constant fields $\Rightarrow$ $\hat{f}(0) = 0$,

$$d\omega(k) = c\,\delta(k) + \underbrace{g(k)}_{\geq 0}\, dk \qquad \text{(IR2)}$$

$$g(k) \leq \text{"}\frac{1}{k^2}\text{"}$$

For $d \geq 3$ "$\frac{1}{k^2}$" is integrable so $\int g(k)\,dk \leq$ Const. (IR3)
Riemann Lebesgue Lemma $\Rightarrow$ $g(x-y) \longrightarrow 0$ as $y \to \infty$.
Therefore

$$\langle \phi_x \phi_y \rangle_\infty \longrightarrow c \qquad y \to \infty.$$

To prove $c > 0$:

$$\langle \phi_x \cdot \phi_x \rangle = \beta$$ since $d\rho$ is measure on sphere radius $\sqrt{\beta}$

Set $x = y$ in (Bochner)

$$\int d\omega(k) = \beta$$

Integrate both sides of (IR2)

$$\beta = c + \underbrace{\int g(k)\,dk}_{O(\beta^0) \text{ by (IR3)}}$$

$\beta \to \infty \Rightarrow c \to \infty$. $\therefore$ $c > 0$ for $\beta >> 1$.

∎

Proposition 3

Let $Z(h) = \int \prod_{x\in\Lambda} d\rho(\phi_x - h_x)\ e^{-\frac{1}{2}(\phi, -\Delta_\Lambda \phi)}$, $h : \Lambda \to \mathbb{R}^N$,

THEN

$$Z(h) \le Z(0) = Z$$

Corollary 4 For $f \perp$ constant fields,

(i) $\langle e^{-(\phi, f)} \rangle_\Lambda \le e^{\frac{1}{2}(f, (-\Delta_\Lambda)^{-1} f)}$

(ii) (IR) holds

Proof (i) $\Rightarrow$ (ii) by replacing f by tf subtract 1 from both sides, $t \downarrow 1$.

To prove (i)

$$\langle e^{-(\phi, f)} \rangle_\Lambda = \frac{1}{Z} \int \prod d\rho\ e^{-\frac{1}{2}(\phi, -\Delta\phi)}\ e^{-(\phi, f)}$$

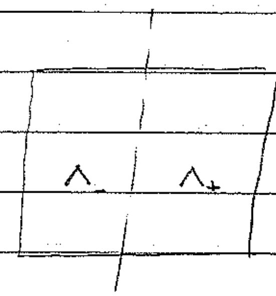
Change variables $\phi_x = \phi'_x + h_x$, choose h to eliminate terms linear in ϕ'

$$= \underbrace{\frac{Z(-h)}{Z}}_{\le 1 \text{ by Prop 3}}\ e^{\frac{1}{2}(f, (-\Delta)^{-1} f)}$$

$\blacksquare$

Reflection Positivity

$\Theta : \Lambda \to \Lambda$

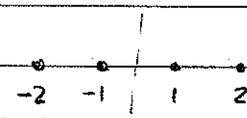


$\Theta \Lambda_+ = \Lambda_- \qquad \Theta \Lambda_- = \Lambda_+$

Let $F = F\left((\phi_x)_{x \in X}\right)$

Define ΘF by replacing ϕ_x by $\phi_{\Theta x}$, $x \in X$.

Example 5



$$\Theta\, e^{\phi_1 + \phi_2} = e^{\phi_{-1} + \phi_{-2}}$$

Defn $\langle\ \rangle$ satisfies OS (Osterwalder – Schrader positivity) if

$$\langle \Theta F \; F \rangle \geq 0 \qquad \forall\, F \in \mathcal{F}_{\Lambda_+}$$

Theorem 6 Nearest neighbour ferromagnetic models are OS.

This is proved in the Froehlich – Simon – Spencer 1976 paper.

Proof (Prop 3) $\quad Z(h) \leq Z(c)$

OS $\Rightarrow$

$$\langle \Theta(F)\, G \rangle \leq \langle \Theta(F)\, F \rangle^{1/2} \langle \Theta(G)\, G \rangle^{1/2}$$

because $(F, G) \stackrel{\text{def}}{=} \langle \Theta(F)\, G \rangle$ is an inner product. Use

$$\begin{array}{|c|c|}\hline \square & \triangle \\ \hline \bigcirc & \times \\ \hline \end{array} \leq \begin{array}{|c|c|}\hline \square & \square \\ \hline \bigcirc & \bigcirc \\ \hline \end{array}^{1/2} \begin{array}{|c|c|}\hline \triangle & \triangle \\ \hline \times & \times \\ \hline \end{array}^{1/2}$$

$$\leq \begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}^{1/2} \begin{array}{|c|c|}\hline \bigcirc & \bigcirc \\ \hline \bigcirc & \bigcirc \\ \hline \end{array}^{1/2} \begin{array}{|c|c|}\hline \triangle & \triangle \\ \hline \triangle & \triangle \\ \hline \end{array}^{1/2} \begin{array}{|c|c|}\hline \times & \times \\ \hline \times & \times \\ \hline \end{array}^{1/2}$$

$$Z(\text{const}) = Z(0)$$

by undoing the translation $\phi = \phi' - \text{const}$.
noting

$$e^{-\frac{1}{2}(\phi, -\Delta\phi)} = e^{-\frac{1}{2}(\phi', -\Delta\phi')}$$

Discussion

This is a very unstable method of proof

(1) Add next to nearest neighbour ferromagnetic interactions: ruins OS but must strengthen trend towards order?

(2) The Fermions $d\phi$, $d\bar{\phi}$ ruin OS so we cannot prove existence of collapsed phases of self interacting walk by OS.

Proving existence of phase transitions in systems with $O(N-1)$ symmetry, $N>1$, is almost unimaginably hard by cluster expansions. OS is essentially the only reasonable technique we have. (There are duality transformations for $N=2$)

Outstanding problem: The quantum antiferromagnet is OS so we can prove $\exists$ phase transition. The quantum ferromagnet is not OS. We can't prove $\exists$ phase transition!

Problems

(1) Do (homework problem 1)

(2) Fill in the details of proof of Prop 3