The lattice Laplacian

For this lecture we put the particle systems away for now and work towards understanding two new systems called the massless and massive free fields on the lattice. For this we require some estimates on the lattice Laplacian and its resolvent. These are the topics of this lecture. In the next lecture we define the free fields.

The lattice Laplacian

Notation Think of $\mathbb{Z}^d$ as a graph with edges

$$E = E(\mathbb{Z}^d) = \{ \{x,y\} : x,y \in \mathbb{Z}^d, \ \|x-y\|_\infty = 1 \}$$

Use

$$xy = \{x,y\}$$

to denote an edge in E. For $\phi, \psi : \mathbb{Z}^d \to \mathbb{R}$,

$$(\phi, \psi) = \sum_{x \in \mathbb{Z}^d} \phi(x) \psi(x)$$

We will only need this for the case ϕ, ψ vanish outside a finite set.

Defn 1 For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, the lattice Laplacian with Dirichlet boundary conditions outside Λ is the unique* symmetric $\Lambda \times \Lambda$ matrix $\Delta = \Delta_\Lambda$ s.t

$$(\phi, -\Delta \phi) = \sum_{xy \in E} (\phi_x - \phi_y)^2$$

for all $\phi : \mathbb{Z}^d \to \mathbb{R}$, $\phi = 0$ outside Λ. $-\Delta$ is a linear operator $\mathbb{R}^\Lambda \to \mathbb{R}^\Lambda$.

The eigenvalues of $(-\Delta)$ are positive because $(\phi, -\Delta\phi) > 0$ for $\phi \neq 0$, therefore $(\epsilon - \Delta)^{-1}$ exists for $\epsilon \geq 0$.

$(\epsilon - \Delta = \epsilon I - \Delta)$

* Look up polarisation to see that $(\phi, -\Delta\phi)$ determines $(\phi, -\Delta\phi')$ for $\phi' \neq \phi$.

From this definition

$$-\Delta_{xy} = \begin{cases} 2d & \text{if } x = y \\ -1 & \text{if } xy \in E \\ 0 & \text{else.} \end{cases}$$

Let $W_{ab}(\Lambda)$ be all sequences in Λ of the form

$$\omega = (\omega_0, \omega_1, \cdots, \omega_n), \quad \omega_0 = a, \; \omega_n = b,$$
$$(\omega_i, \omega_{i+1}) \in E$$
$$n = n(\omega).$$

Proposition 1 For $\epsilon \geq 0$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$,

$$(\epsilon - \Delta)^{-1}_{ab} = \sum_{\omega \in W_{ab}(\Lambda)} \left(\frac{1}{2d+\epsilon}\right)^{n(\omega)+1}$$

Let

$$W_a(\Lambda) = \bigcup_b W_{ab}(\Lambda)$$

$$W_a^{(m)}(\Lambda) = \{\omega \in W_a(\Lambda) : n(\omega) = m\}$$

The idea is

$$\epsilon I - \Delta = D - O$$

D is diagonal, entries $2d+\epsilon$, O is off-diagonal,
$O_{xy} = 1$ iff $xy \in \text{edges}(\Lambda)$

Then the resolvent expansion

$$(D-O)^{-1} = \underbrace{D^{-1}}_{0\text{ step}} + \underbrace{D^{-1}OD^{-1}}_{1\text{ step}} + \underbrace{D^{-1}OD^{-1}OD^{-1}}_{2\text{ steps}} + \cdots$$

is the same as

$$(D-O)^{-1}_{ab} = \sum_{\omega \in W_{ab}(\Lambda)} (2d+\epsilon)^{-n(\omega)-1} \qquad \text{(resolvent)}$$

We make this into a

<u>Proof</u>

The right hand side of resolvent is absolutely convergent for $\epsilon > 0$ because

$$\sum_{\omega \in W_{ab}(\Lambda)} (2d+\epsilon)^{-n(\omega)-1} \leq \sum_{\omega \in W_a(\mathbb{Z}^d)} (2d+\epsilon)^{-n(\omega)-1}$$

$$= \sum_{n=0}^{\infty} (2d)^n (2d+\epsilon)^{-n-1} = \frac{1}{\epsilon}$$

and once we know $D^{-1} + D^{-1}OD^{-1} + \cdots$ is convergent, multiplying by $D-O$ shows it equals $(D-O)^{-1}$. By monotone convergence $\sum (2d)^{-n(\omega)-1} = \lim_{\epsilon \downarrow 0} \sum (2d+\epsilon)^{-n(\omega)-1}$
$= \lim (\epsilon - \Delta)^{-1}_{ab} = (-\Delta)^{-1}_{ab}$

∎

Define, for $k \in \mathbb{R}^d$,

$$\text{units} = \{ x \in \mathbb{Z}^d : \|x\| = 1 \}$$

$$f(k) = \sum_{x \in \text{units}} (e^{k \cdot x} - 1)$$

<u>Lemma 2</u> For $\epsilon > f(k)$, $k \in \mathbb{R}$,

$$e^{-k \cdot a} \sum_{\omega \in W_a(\mathbb{Z}^d)} (\epsilon + 2d)^{-n(\omega) - 1} e^{k \cdot \omega_{n(\omega)}}$$

$$= (\epsilon - f(k))^{-1}$$

<u>Proof</u>

$$e^{-k \cdot a} e^{k \cdot \omega_n} = e^{\sum_{i=0}^{n-1} k \cdot (\omega_{i+1} - \omega_i)}$$

$$= \prod_{i=0}^{n-1} e^{k \cdot (\omega_{i+1} - \omega_i)}$$

so LHS of Lemma becomes

$$\sum_{n \geq 0} \sum_{\omega \in W_a(\mathbb{Z}^d)} \prod_{i=0}^{n-1} \left(e^{k \cdot (\omega_{i+1} - \omega_i)} \frac{1}{\epsilon + 2d} \right) \frac{1}{\epsilon + 2d}$$

$$= \sum_{n \geq 0} \left(\frac{1}{\epsilon + 2d} \sum_{x \in \text{units}} e^{k \cdot x} \right)^n \frac{1}{\epsilon + 2d} = (\epsilon - f(k))^{-1}$$

<u>Corollary 3</u> For $\epsilon > 0$

$$\sum_{b \in \Lambda} (\epsilon - \Delta_\Lambda)^{-1}_{ab} \leq 1/\epsilon$$

and the LHS increases to $1/\epsilon$ as Λ increases to $\mathbb{Z}^d$.

<u>Proof</u> set $k = 0$ in Lemma 2 and use dominated convergence for controlling the limit $\Lambda \nearrow \mathbb{Z}^d$.

<u>Corollary 4</u> For (ϵ, λ) s.t. $\epsilon > \sup \{ f(k) : \|k\| = \lambda \}$

$$(\epsilon - \Delta)^{-1}_{ab} \leq \left(\epsilon - \sup \{ f(k) : \|k\| = \lambda \} \right)^{-1} \cdot e^{-\lambda \| b - a \|}$$

<u>Proof</u> By Lemma 2, $(\epsilon - \Delta)^{-1}_{ab} \leq \frac{1}{\epsilon - \sup(\cdots)} e^{-k \cdot (b-a)}$
Choose $k \,/\!/\, a - b$.

<u>Corollary 5</u> $\forall \epsilon > 0$, $a \in \mathbb{Z}^d$,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} (\epsilon - \Delta_\Lambda)^{-1}_{aa} = (2\pi)^{-d} \int_{[\pi, \pi]^d} (\epsilon - f(ik))^{-1} dk$$

The RHS is bounded uniformly in $\epsilon > 0$ if $d \geq 3$, else it diverges as $\epsilon \downarrow 0$.

Proof

$$(\epsilon-\Delta_\Lambda)^{-1}_{aa} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\Big(\sum_{b\in\Lambda}(\epsilon-\Delta)^{-1}_{ab}\, e^{ik(b-a)}\Big)\,dk$$

$$\xrightarrow[\substack{\Lambda\uparrow \\ \text{dominated} \\ \text{convergence,} \\ \text{Lemma 2}}]{} \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\big(\epsilon - f(ik)\big)^{-1}\,dk$$

This proves the Corollary apart from the claim about $\epsilon \downarrow 0$. For $\epsilon \downarrow 0$ note

$$f(ik) = \sum_{x\in\text{units}}\left(e^{ik\cdot x}-1\right)$$

$$= \sum_{x\in\text{units}}(\cos k\cdot x - 1) \qquad \text{real, } \leq 0$$

$$= 0 \text{ in } [-\pi,\pi]^d \text{ iff } k=0$$

Near $k=0$

$$\big(\epsilon - f(ik)\big)^{-1} = \frac{1}{\epsilon + \|k\|^2 + o(\|k\|^2)}$$

is integrable iff $d\geq 3$. The claim follows from monotone convergence. $\boxtimes$