# Lecture 2

The next simplest case after the ideal gas is called mean field theory. It is important because it exhibits the phenomenon of a "phase transition". In fact it is a reasonable model for the transition in which liquid water becomes steam. As you know from every day experience there is a very well defined temperature (100°C), at which the density of water has a jump: liquid water is much denser than steam.

Mean field theory should be formulated for the continuum models of last lecture, but in order to avoid a problem with (1. stability) we will consider lattice systems instead. The topics of this lecture are (1) how lattice systems are a special case of the continuum systems of lecture 1 (2) the limit as $\Lambda \nearrow \mathbb{R}^d$ (3) mean field theory for lattice systems. (4) phase transitions.

Notation

Paving $\mathbb{R}^d$ by blocks:

Let $L \in \mathbb{N}$. For $x \in \mathbb{Z}^d$,

$$B(x) = \{y \in \mathbb{R}^d : \|y - Lx\|_\infty < L/2\}$$

$$\|y\|_\infty = \max_{i=1,\dots,d} |y_i| \qquad \text{for } y \in \mathbb{R}^d$$

$B(x)$ is called a block. The set of all blocks is

$$\mathcal{B} = \mathcal{B}_L = \{B(x) : x \in \mathbb{Z}^d\}$$

$$\mathcal{P}(\Lambda) = \{\text{all finite unions of } \overline{B},\ B \in \mathcal{B}(\Lambda)\}$$

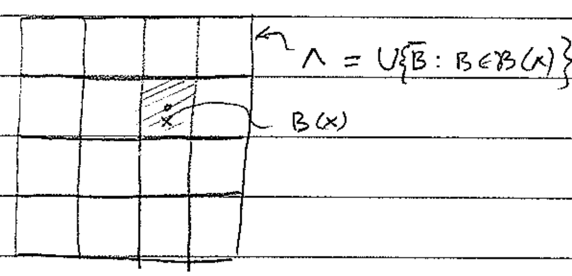
Note $\emptyset \in \mathcal{P}(\Lambda)$

Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$.

A set $X \in \mathcal{P}$ is called a polymer. For $X \in \mathcal{P}$

$$|X|_1 = |\mathcal{B}(X)|$$

is the number of $L = L^1$ blocks in $X$.




Choose $L = 1$ for this and next lecture.

The random variables

After passing by blocks we are only interested in

$N(B)$ = # particles in $B$

$$N = \sum_{B \in \mathcal{B}(\Lambda)} N(B)$$

$$\underline{N}_X = \left( N(B) : B \in \mathcal{B}(X) \right), \qquad X \in \mathcal{P}$$

$\mathcal{F}_X$ is the $\sigma$ algebra generated by $\underline{N}_X$

$F \in m\mathcal{F}_X$ means $F$ is measurable wrt $\mathcal{F}_X$.

## The infinite volume limit

This refers to studying the joint distributions of $\underline{N}_X$, $X \in \mathcal{P}$, in a limit

$$\Lambda_1 \subset \Lambda_2 \subset \cdots, \qquad \Lambda_i \in \mathcal{P}, \qquad \cup \Lambda_i = \mathbb{R}^d.$$

An infinite volume limit is a probability space $(\Omega_\infty, \mathbb{P}_\infty)$ carrying random variables $(N(B), B \in \mathcal{B})$ such that for some sequence $\Lambda_i$, for every $X \in \mathcal{P}$,

$$\lim_{i \to \infty} \mathbb{P}_{\Lambda_i}\{\underline{N}_X = \underline{n}\} = \mathbb{P}_\infty\{\underline{N}_X = \underline{n}\}. \qquad \text{(def-infinite-vol)}$$

Later in these lectures, when we encounter random variables which are not discrete (values in $\mathbb{N}_0$), we will use the notion of weak convergence, which is equivalent to demanding that expectations of all bounded continuous functions of $\underline{N}_X$ converge to $\infty$ volume expectation of the same.

## Mean Field Theory

Define by, $\beta > 0$,

$$V = \begin{cases} \infty & \text{if } N(B) > 1 \text{ for some } B \in \mathcal{B}(\Lambda) \\ -\frac{\beta}{|\Lambda|_1} N^2/2 & \end{cases}$$

Let

$$\Omega = \{0,1\}^{\mathcal{B}(\Lambda)}$$

0 or 1 particle in each B

$\Omega = \{\text{values of } N_\Lambda\}$

For $\underline{n} \in \Omega$,

$$z^{\underline{n}} = \prod_{B \in \mathcal{B}(\Lambda)} z^{n(B)}$$

$$H(\underline{n}) = -\frac{\beta}{|\Lambda|_1} \left( \sum_{B \in \mathcal{B}(\Lambda)} n(B) \right)^2$$

Then, under grand canonical ensemble,

$$\mathbb{P}\{N_\Lambda = \underline{n}\} = \begin{cases} \frac{1}{Z} z^{\underline{n}} e^{-H(\underline{n})} & \underline{n} \in \Omega \\ 0 & \underline{n} \notin \Omega \end{cases}$$

(lattice)

$$Z = \sum_{\underline{n} \in \Omega} z^{\underline{n}} e^{-H(\underline{n})}$$

Proof  Since $V \in m\mathcal{F}_\Lambda$

$$\int_{\{\underline{N}_\Lambda = \underline{n}\}} e^{-V} \, d\mathbb{P}_{V=0}$$

$$= \begin{cases} e^{-H(\underline{n})} \, \mathbb{P}_{V=0}\{\underline{N}_\Lambda = \underline{n}\} & \underline{n} \in \Omega \\ 0 \quad \text{if } \underline{n} \notin \Omega \end{cases}$$

$$\underset{\text{Lemma 1.1}}{=} e^{-H(n)} \prod_{B \in \mathcal{B}(\Lambda)} \left( \frac{z^{n(B)}}{n(B)!} e^{-z} \right) \qquad n \in \Omega$$

$$= e^{-H(\underline{n})} \, z^{\underline{n}} \, e^{-z|\Lambda|}$$

Then divide by normalisation and use

$$\mathbb{P}(E) = \frac{1}{Z} \int_E e^{-V} \, d\mathbb{P}_{V=0} \qquad \blacksquare$$

This argument never used the specific form of $V$ beyond $V \in m\mathcal{F}_\Lambda$, so by the same argument, a lattice model arises whenever, for the continuum model, $V \in m\mathcal{F}_\Lambda$, and this is equivalent to

$$v(x,y) = v([x],[y]) \quad \text{a.e. Lebesgue} \qquad (v \text{ lattice})$$

in Example 1.2. $[x]$ is the point in $\mathbb{Z}^d$ closest to $x \in \mathbb{R}^d$, in the sense $x \in B \Leftrightarrow B = B([x])$. $[x]$ is well defined a.e. in $x \in \mathbb{R}^d$.

Proposition 2 In the infinite volume limit, for every $X \in \mathcal{P}$ the probability law for $\underline{N}_X$ is a convex combination of Bernoulli $(1 : ze^{\phi})$ where $\phi$ is a constant in the set $M_0$ of global minima to

$$S(\phi) = \frac{\phi^2}{2\beta} - \log(1+ze^{\phi})$$

If $(\beta, z) \notin \{ze^{\beta/2}=1\}$ or if $\beta \leq 4$ there is a unique global minimum $\phi$ and

$$\mathbb{P}(\underline{N}_X = \underline{n}) = \left(\frac{ze^{\phi}}{1+ze^{\phi}}\right)^{\underline{n}}, \quad \underline{n} \in \Omega(X) \qquad \text{(1 phase)}$$

Otherwise $|M_0| = 2$ and

$$\mathbb{P}(\underline{N}_X = \underline{n}) = \frac{1}{2} \sum_{\phi \in M_0} \left(\frac{ze^{\phi}}{1+ze^{\phi}}\right)^{\underline{n}} \qquad \text{(2 phase)}$$

$Y \sim$ Bernoulli $(1 : t)$ means $Y = \begin{cases} 1 & \text{with prob } \frac{t}{1+t} \\ 0 & \text{with prob } \frac{1}{1+t} \end{cases}$

## Discussion

~~Let $p \in (0,1]$.~~

There exists a probability space $(\Omega_\infty^{(p)}, \mathbb{P}_\infty^{(p)})$ on which are defined random variables

$$\left( N(B),\quad B \in \mathcal{B}(\mathbb{R}^d) \right), \qquad N(B) : \Omega_\infty^{(p)} \to \mathbb{N}$$

and under the law $\mathbb{P}_\infty^{(p)}$ all those random variables are independent Bernoulli$(p)$. By taking two copies, each carrying $\frac{1}{2}$ probability, we define a new probability space

$$(\Omega_\infty, \mathbb{P}_\infty), \quad \Omega_\infty = \Omega_\infty^{(p_1)} \cup \Omega_\infty^{(p_2)}$$

$$\mathbb{P}_\infty \big|_{\Omega_\infty^{(p_i)}} = \tfrac{1}{2}\, \mathbb{P}_\infty^{(p_i)}, \quad i=1,2,$$

with an additional random variable

$$\rho = \begin{cases} p_1 & \text{on } \Omega_\infty^{(1)} \\ p_2 & \text{on } \Omega_\infty^{(2)} \end{cases}$$

Choose $p_i = \frac{z e^{\phi_i}}{1+ z e^{\phi_i}}$, $\phi_i \in M_0$, $i=1,2$ as in (2phase). Then $(\Omega_\infty, \mathbb{P}_\infty)$ is the infinite volume limit of MFT in case (2phase): For $X \in \mathcal{P}$, Prop. 2 says

$$\lim_{i\to\infty} \mathbb{P}_{\Lambda_i}(\underline{N}_X = \underline{n}) = \mathbb{P}_\infty(\underline{N}_X = \underline{n}) \qquad \text{(def } \infty \text{vol)}$$

However $\rho$ is not as new as it looks because we can create it from the random variables $(N(B),\ B \in \mathcal{B}(\mathbb{R}^d))$ by the construction

$$\rho = \lim_{X \nearrow} \frac{1}{|X|_1} \sum_{B \in \mathcal{B}(X)} N(B) \quad \text{a.s. } \mathbb{P}_\infty \qquad (\rho)$$

<u>Proof</u> under $\mathbb{P}_\infty(\cdot \mid \rho)$ $N(B)$ are independent with expectation $\rho$ so by the strong law of large numbers

$$\frac{1}{|X|_1} \sum_{B \in \mathcal{B}(X)} N(B) \longrightarrow \mathbb{E}(N(B) \mid \rho)$$

$$= \rho$$

a.s. $\mathbb{P}_\infty(\cdot \mid \rho)$ convergence implies a.s. $\mathbb{P}_\infty$ convergence. ▨

If we define $\tilde{\mathcal{F}}_X$ to be the $\sigma$-algebra generated by $\underline{N}_X$ then $(\rho)$ implies $\rho$ is $\tilde{\mathcal{F}}_{X^c}$ measurable for all $X$. In down to earth language, $\rho$ does not depend on $\underline{N}_X$ because the $|X|_1 \to \infty$ limit in $(\rho)$ washes out the contribution from $\underline{N}_X$. Thus $\rho$ is $\mathcal{T}$ measurable where

$$\mathcal{T} = \bigcap_{X \subset \mathbb{R}^d} \tilde{\mathcal{F}}_{X^c} \qquad (\mathcal{T})$$

$\mathcal{T}$ is called the <u>Tail</u> <u>$\sigma$-algebra</u> or the <u>algebra at $\infty$</u>. We say it is <u>non-trivial</u> because it contains sets which have prob $\neq 0$ or 1; equivalently, there are non constant $\mathcal{T}$ measurable functions such as $\rho$.

In case (1 phase) the infinite volume limit is $(\Omega_\infty^{(p)}, \mathbb{P}_\infty^{(p)})$, $p = ze^{\phi}/_{1+ze^{\phi}}$, $\phi \in M_0$ is unique. In this case the only $\mathcal{T}$ measurable functions are constants, by the Hewitt–Savage 0-1 law.

Physically $\rho$ is the density. In case (2 phase) the system has two co-existing "phases", one has a higher density than the other, much like liquid water and gaseous water. The ½ ½ mixture of the two is caused by me trying to keep it simple.

By only allowing a simplified form of $V$ for MFT I have only revealed the convex combination with coefficients ½, ½. The infinite volume limit is normally set up in a more general way which includes in $V$ an external field term that represents the interaction of particles inside $\Lambda$ with a fixed configuration of particles outside $\Lambda$. By taking these more general $\infty$ vol. limits one can achieve other convex combinations.

## Problems

1. For $v(x,y)$ as in (vlattice) find $H(\underline{n})$ so that (lattice) holds. In other words express $\sum_{1 \le i < j \le N(\Lambda)} v(x_i, x_j)$ as an explicit function of the random variables $(N(B), B \in \mathcal{B})$.

2. Ising Models are usually expressed in terms of

$$\Omega_{Ising} = \{-1, 1\}^{\Lambda \cap \mathbb{Z}^d}$$

e.g.

$$Z_{Ising} = \sum_{\sigma \in \Omega_{Ising}} e^{\beta \sum_{xy \in \Lambda \cap \mathbb{Z}^d} \sigma_x \sigma_y}$$

What Ising model is "the same as" our mean field theory in case (2phase)? $(n = 0, 1 \longleftrightarrow \sigma = -1, 1)$

3. Look up the de Finetti theorem in Durrett, Probability, Theory and Examples, or any other good textbook, and explain what it has to do with MFT.