Lecture 13

In this and the remaining lectures we will see how the hierarchical model techniques can be lifted to the Euclidean $\mathbb{Z}^d$ case. First we will discuss the scheme in an abstract way and then describe how it is applied to the anharmonic lattice,

$$Z = \int_{\mathbb{R}^\Lambda} \prod_{xy \in \text{edges}(\Lambda)} e^{-f(\phi_x - \phi_y)} \, d^\Lambda \phi$$

where f is "nearly" Gaussian.

$$f(\phi_x - \phi_y) \simeq \frac{1}{2}(\phi_x - \phi_y)^2$$

Proposition 1 (Brydges Guadagni Mitter 2003)

Let Φ be the $\mathbb{Z}^d$ massless free field.
Let $L \in \mathbb{N}$, $L \geq 2$. Let $d \geq 3$.

$\exists$ independent $\{\zeta_j, j \geq 1\}$, where
$\zeta_j = \{\zeta_j(x), j \geq 1\}$, s.t.

(1) ζ_j is Gaussian, $\mathbb{Z}^d$ invariant law,

(2) $\mathrm{Cov}(\zeta_j(x)\ \zeta_j(y)) = 0$ if $|x-y| \geq L^j/2$,

(3) $\Phi = \sum_{j \geq 1} \zeta_j$.

Furthermore the same is true for the massive $\mathbb{Z}^d$ free field for $d \geq 1$.

These "increments" ζ_j cannot be scalings of $\zeta = \zeta_1$ because a scaling would live on $L^{-j}\mathbb{Z}^d \neq \mathbb{Z}^d$.

However $C_j(x,y) \overset{\text{def}}{=} \mathrm{Cov}(\zeta_j(x), \zeta_j(y))$ obeys

<u>Scaling Estimates</u> (c.f. problem 1)

$$\left|\left(\nabla_x^{\alpha} \nabla_y^{\alpha} C_j\right)(x,y)\right|_{x=y}$$
$$\leq C(\alpha)\ L^{-2(j-1)([\varphi]+|\alpha|)}$$

where,

$$\nabla_e f(x) = f(x+e) - f(x)$$

$$\alpha = (e_1, e_2, \cdots, e_n) \in (\text{unitvectors})^*$$

Since $C(x,y) = C(x-y)$, $\nabla_x^{\alpha} \nabla_y^{\alpha} C(x,y)\big|_{x=y}$
$= \nabla_x^{2\alpha} C(0,0)$.

<u>Remarks</u>

In the massive case C_j does more: it becomes essentially zero for $j \geq \log_L(\text{mass})^{-1}$.

Massless $d = 1,2$: $\exists\ \zeta_j$ s.t. $\nabla\Phi = \sum \nabla \zeta_j$.

The coordinates

Hierarchical:

$$(e^{-V} + K)^{\Lambda}$$

$$= \sum_{X \subset \Lambda} e^{-V(\Lambda \setminus X)} K^X$$

$$= \sum_{X \in \mathcal{P}_0(\Lambda)} e^{-V(\Lambda \setminus X)} K^X \qquad \text{(hierarchical-rep)}$$

Recalling that $\mathcal{P}_0$ is all unions of L^0 blocks, i.e. points in Λ.

K^X factors: $K^X = \prod_{x \in X} K^x$

Euclidean

Replace (hierarchical-rep) by (at scale j) by

$$\sum_{X \in \mathcal{P}_j(\Lambda)} e^{-V_j(\Lambda \setminus X)} K_j(X)$$

where

$$K_j = \{ K_j(X) : X \in \mathcal{P}_j \}$$

(1) $K_j(X)$ depends on

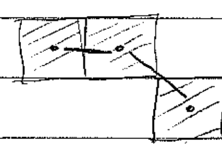
$$\{\phi_x : x \in X^*\}$$

where X^* is a neighbourhood of X defined later.

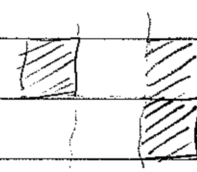
(2) $$K_j(X) = \prod_{Y \in cc(X)} K_j(Y) \qquad \text{(Factorisation)}$$

where $cc(X) = \{$ connected components of $X\}$

$Y \in \mathcal{P}$ is connected if any pair of points $a, b \in Y$ are such that there is a sequence $(a = x_1, x_2, \cdots, x_n = b)$ with
$\|x_i - x_{i-1}\|_\infty = 1$ for $i = 2, 3, \cdots, n$
and $x_i \in Y$, $i = 1, \cdots, n$.



is connected.



is not connected

Defn 2 For $X \in \mathcal{P}_j$, F, G functions on $\mathcal{P}_j$,

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}_j(X)} F(Y)\, G(X \setminus Y)$$

With this definition the "coordinates" (V_j, K_j) represent a random variable that depends on all the fields $\{\phi_x : x \in \Lambda\}$ by

$$(V_j, K_j) \longrightarrow (e^{-V_j} \circ K_j)(\Lambda)$$

The Euclidean RG

is a method to compute

$$\mathbb{E}\, e^{-V_0(\Lambda)}, \qquad V_0(\Lambda) = \sum_{x\in\Lambda} V_{0,x}$$

via

$$= \lim_{N\to\infty} \mathbb{E}_N \cdots \mathbb{E}_1\, e^{-V_0(\Lambda)}$$

(No rescaling: RG step is $F_j \mapsto \mathbb{E}_{j+1} F_j$)

$j=0$

$$e^{-V(\Lambda)} = \sum_{X\in\mathcal{P}_0(\Lambda)} e^{-V_0(\Lambda\setminus X)}\, K_0(X)$$

$$K_0(X) = \begin{cases} 1 & \text{if } X=\emptyset \\ 0 & \text{else} \end{cases}$$

RG step: find (V_{j+1}, K_{j+1})

$$(V_j\,, K_j) \xrightarrow{\mathbb{E}_{j+1}} (V_{j+1}\, K_{j+1})$$

s.t.

$$\mathbb{E}_{j+1}\left(e^{-V_j}\circ K_j\right)(\Lambda) = \left(e^{-V_{j+1}}\circ K_{j+1}\right)(\Lambda)$$

$$= \sum_{X\in\mathcal{P}_{j+1}(\Lambda)} K_{j+1}(X)\, e^{-V_{j+1}(\Lambda\setminus X)}$$

To define the maps

$$(V_j, K_j) \xrightarrow{\mathbb{E}_{j+1}} (V_{j+1}, K_{j+1}), \qquad j = 0, 1, 2, \ldots$$

is a 2½ step process beginning as in hierarchical case with a "linear guess" that $V_{j+1} \simeq \tilde{V}$ where

$$\tilde{V}(X) = \mathbb{E}_{j+1} V_j(X)$$

Our first objective is a formula for $\tilde{K}$ s.t. $(V_j, K_j) \xrightarrow{\mathbb{E}_{j+1}} (\tilde{V}, \tilde{K})$

Defn 3

For $X \in \mathcal{P}_j$, $\bar{X}$ is the smallest set in $\mathcal{P}_{j+1}$ that contains X.

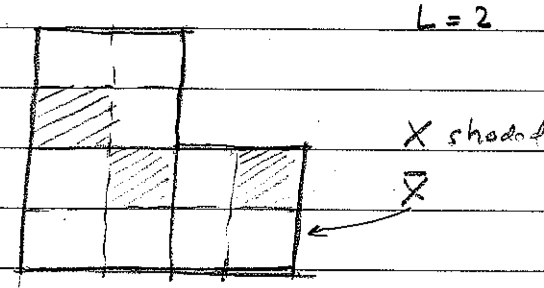
Defn 4

For $U \in \mathcal{P}_{j+1}$, we say

$$X \in \bar{\mathcal{P}}_j(U)$$

if

$$\bar{X} = U$$



Example 5

Let

$$I_x = e^{-V_x}$$
$$\tilde{I}_x = e^{-\tilde{V}_x}$$
$$\delta I_x = I_x - \tilde{I}_x$$

Then

$$I^X = (\tilde{I} + \delta I)^X$$

$$= \sum_{Y \subset X} \delta I^Y \, \tilde{I}^{X \setminus Y}$$

$$= \left((\delta I) \circ \tilde{I} \right)(X)$$

$\delta I(Y) = \delta I^Y$
$\tilde{I}(Y) = \tilde{I}^Y$

Properties of $\circ$

$$A \circ B = B \circ A$$
$$A \circ (B \circ C) = (A \circ B) \circ C$$

Lemma 6 $(V_j, K_j) \xrightarrow{\mathbb{E}_{j+1}} (\tilde{V}, \tilde{K})$ where

For $U \in \mathcal{P}_{j+1}$

$$\tilde{K}(U) = \sum_{X \in \bar{\mathcal{P}}_j(U)} \tilde{I}^{U \setminus X} \, \mathbb{E}_{j+1}(K_j \circ \delta I)(X)$$

and $\tilde{K}$ satisfies (factorisation), as a function on $\mathcal{P}_j$,

$$\tilde{K}(U) = \prod_{X \in c.c.(U)} \tilde{K}(X) \qquad \text{(factorisation 2)}$$

where $X \in c.c(U)$ means that $X \in \mathcal{P}_{j+1}$ is a connected component of U as a set in $\mathcal{P}_{j+1}$.

Remark 7

This Lemma does not depend on the choice $\tilde{V} = \mathbb{E}V$. It holds for any $\tilde{V}$ which is not a function of ζ_{j+1} so that

$$\mathbb{E}_{j+1}(\tilde{I}^X (\cdot)) = \tilde{I}^X \, \mathbb{E}_{j+1}(\cdot)$$

Proof

$$\mathbb{E}_{j+1}\,(I\circ K)(\Lambda)$$

$$= \mathbb{E}_{j+1}\Big(\big(\tilde{I}\circ \delta I\big)\circ K\Big)(\Lambda)$$

$$= \mathbb{E}_{j+1}\Big(\tilde{I}\circ\big(\delta I\circ K\big)\Big)(\Lambda)$$

$$= \sum_{X\in\mathcal{P}_j(\Lambda)} \mathbb{E}_{j+1}\ \tilde{I}(\Lambda\setminus X)\ \ (\delta I\circ K)(X)$$

$$= \sum_{X\in\mathcal{P}_j(\Lambda)} \tilde{I}^{\Lambda\setminus X}\ \ \mathbb{E}_{j+1}\,(\delta I\circ K)(X)$$

$$= \sum_{U\in\mathcal{P}_{j+1}(\Lambda)}\ \sum_{X\in\overline{\mathcal{P}}_j(U)} \tilde{I}^{\Lambda\setminus\overset{=U}{\overline{X}}}\ \tilde{I}^{\overset{=U}{\overline{X}}\setminus X}\ \mathbb{E}_{j+1}(\delta I\circ K)(X)$$

$$= \sum_{U\in\mathcal{P}_{j+1}(\Lambda)} \tilde{I}^{\Lambda\setminus U}\ \ \tilde{K}(U)$$

$$= \big(\tilde{I}\circ\tilde{K}\big)(\Lambda)$$

Factorisation depends on the finite range property of I_{j+1} and (factorisation). See problem 2.

∎

Problems

1. Prove that if $u \in C_0^\infty(\mathbb{R}^d)$ then

$$\int_{L^{j-1}}^{L^j} \frac{dl}{l} \, l^{-2[\phi]} \, u\left(\frac{x-y}{l}\right)$$

obeys scaling estimates.

2. Prove (factorisation 2).

3. Show that Lemma 6 returns the hierarchical formula for $\tilde{K}$ when connectedness is defined with the hierarchical metric and Λ_∞ replaces $\mathbb{Z}^d$.