Lecture 11

At the $O(V)$ level of the last lecture the action of RG is to replace

$$V = g:\phi^4: + a:\phi^2:$$

by

$$\tilde{V} = \tilde{g}:\phi^4: + \tilde{a}:\phi^2:$$

with

$$\tilde{g} = g L^{d-4[\phi]}, \qquad \tilde{a} = L^{d-2[\phi]} a .$$

To include all $O(V^2)$ corrections we today introduce an error term K such that under the action of RG

$$e^{-V} + K \longrightarrow e^{-V'} + K' .$$

Today we introduce some of the main tools for controlling this K: what space K is in and how to measure its size

The model

$$Z = \mathbb{E}\left(e^{-V} + K\right)^{\Lambda}$$

where

$$\left(e^{-V}+K\right)^{\Lambda} = \prod_{x\in\Lambda}\left(e^{-V_x} + K_x\right)$$

$$V_x = g:\phi_x^4: + a:\phi_x^2: + b, \qquad |a| \leq \sqrt{g}$$

forgot this in my lecture

$$K_x = K(\phi_x)$$

is even and

$$K(\phi_x) = O(\phi_x^6) \quad \text{as } \phi_x \to 0$$

(Initially one could choose $K = 0$)

E is expectation for the hierarchical field

$$\phi_x \overset{\mathcal{L}}{=} L^{-[\phi]} \phi_{L^{-1}x} + \zeta$$

$$\mathrm{Var}(\zeta_x) = \cancel{\log_3 L} \sum_{j=1}^{\log_3 L} 3^{-2[\phi]} \qquad (*)$$

<u>Remark</u>. if these equations hold for $L = 3$ then they hold for $L \in \{3, 3^2, 3^3, \dots\}$ (with different ζ's).

<u>Assumption 2</u>

$$d - 4[\phi] < 0$$

As in last lecture this means $g \to 0$ within $O(V)$ calculations. When coupling constants are contracted according to $O(V)$ calculation we say they are <u>irrelevant</u>.

* Correction to wrong equation in my lecture.

The T_Φ norm

For F a function of finitely many $\{\phi_x, x \in \Lambda_\infty\}$

$$\|F\|_{T_\Phi} = \|F\|_{T_\Phi, h}$$

$$= \sum_{x \in \Lambda_\infty^*} \frac{h^n}{n!} \left| \frac{\partial^n F(\phi)}{\partial \phi_{x_1} \cdots \partial \phi_{x_n}} \right|$$

$n = n(x)$ length of sequence x.

$h > 0$

Example 3

If $F = F(\phi_x)$

$$\|F\|_{T_\Phi} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \left| \frac{\partial^n F(\phi)}{\partial \phi_x^n} \right|$$

which is finite if F is analytic on $\mathbb{R}$, in fact in a strip of width h.

Lemma 4

(i) $\| F^{X} \|_{T_\phi} \leq \| F \|^{X}_{T_\phi}$

$\| F_1 F_2 \|_{T_\phi} \leq \| F_1 \|_{T_\phi} \| F_2 \|_{T_\phi}$ any F_1, F_2, even if they depend on the same variables.

(ii) $$\sum_{x \in \Lambda_\infty^*} \frac{h^n}{n!} \left\| \frac{\partial^n F}{\partial \phi_{x_1} \cdots \partial \phi_{x_n}} \right\|_{T_\phi, h}$$

$$\leq \| F \|_{T_\phi\, 2h}$$

Proof is omitted (see problems).

(ii) says that $\left\| \frac{\partial^n F}{\partial \phi \cdots \partial \phi} \right\|_{T_\phi, h}$

$$\leq \frac{n!}{h^n} \| F \|_{T_\phi, 2h}$$

which is a Cauchy bound. The point we will use is that derivatives are very small for h large.

Example 5

Suppose $F = F(\phi_0)$

$$(\mathbb{E}_1 F)(\phi_0') = \int d\mu_C(\zeta)\ F(\phi_0' + \zeta)$$

$$RG(F)(\phi_0) = \int d\mu_C(\zeta)\ F(L^{-[\phi]}\phi_0 + \zeta)$$

$$\frac{\partial}{\partial\phi_0} RG(F)(\phi_0) = L^{-[\phi]} \int d\mu_C(\zeta)\ F'(L^{-[\phi]}\phi_0 + \zeta)$$

$$\| RG(F) \|_{T_{\phi}, h} \leq \int d\mu_C(\zeta) \sup_{\zeta} \| F \|_{L^{-[\phi]}\phi_0 + \zeta,\ L^{-[\phi]}h}$$

$$\leq \underbrace{\sup_{\phi} \| F \|_{T_{\phi},\ L^{-[\phi]}h}}$$

Call this $\| F \|_{L^{-[\phi]}h}$

Lemma 6 For $h \le g^{-1/4}$, $\exists\, C$

$$\| e^{-g\phi_x^4} \|_{T_\phi, h} \le e^{O(gh^4) - \frac{1}{2} g \phi_x^4} \le C e^{-\frac{1}{2} g \phi_x^4}$$

Proof

$$\| (1 - g/N\, \phi_x^4)^N \|_{T_\phi, h}$$

$$\le \| 1 - g/N\, \phi_x^4 \|_{T_\phi, h}^N$$

$$= \left| \left(1 - \tfrac{g}{N} \phi_x^4\right) + \tfrac{g}{N} 4 |\phi_x|^3 h + \tfrac{g}{N} 6 |\phi_x|^2 h^2 \cdots + \tfrac{g}{N} h^4 \right|^N$$

$$\le \left| 1 + \tfrac{g}{N} h^4 \left(-t^4 + 4t^3 + 6t^2 + 4t + 1\right) \right|^N, \qquad t = \frac{|\phi_x|}{h}$$

$$\le \left| 1 + \tfrac{g}{N} h^4 \left(-\tfrac{1}{2} t^4 + c\right) \right|^N \xrightarrow[N \to \infty]{} e^{-\frac{1}{2} g \phi^4} e^{c g h^4}$$

∎

It is easy to improve this Lemma to bound $e^{-g :\phi^4: - a :\phi^2:}$ for $|a| \le \sqrt{g}$.

<u>Notation</u>

$$h = g^{-1/4}, \qquad \|F\|_h = \sup_\phi \|F\|_{T_\phi, h}$$

$$\tilde{h} = 2\left(L^{d-4[\phi]} g\right)^{-1/4}, \qquad \|F\|_{\tilde{h}} = \sup_\phi \|F\|_{T_\phi, \tilde{h}}$$

We choose $h = h$ in the T_ϕ norm.

Given (V, K), define $(\tilde{V}, \tilde{K})$ by

$$RG\left(e^{-V} + K\right)^B = e^{-\tilde{V}_x} + \tilde{K}_x$$

This is an equation for $\tilde{K}$ because $\tilde{V}$ is determined by $V(B)$.

where

$$x = L^{-1} B$$

$$\tilde{V}_x = RG(V(B))$$

Define

$$K_{main, x} = RG\left(e^{-V(B)}\right) - e^{-RG(V(B))}$$

Proposition 7

$\exists$ $C(L)$ such that if

$$g \leq C(L),$$

$$\|K\|_h \leq C(L),$$

THEN

$$\lim_{L \to \infty} \frac{\|\tilde{K} - K_{main}\|_{\tilde{h}}}{\|K\|_h} = 0$$

In this bound $\tilde{K} = \tilde{K}_x$, $K_{main} = K_{main,x}$ $K = K_y$ but it does not matter how you choose x, y because $K_x = K(\phi_x)$ is the same function K at every $x \in \Lambda_\infty$.

$C(L)$ is determined in the proof s.t. $\lim_{L \to \infty} C(L) = 0$ so we will assume that $\|K\|_h \leq 1$ etc at very places in the proof.

Part of Proof

$$\tilde{K}_x - K_{main,x}$$

$$= \sum_{y \in B} \hat{L}^{-1} e^{-V(B \setminus \{y\})} K_y \qquad \mathrm{I}$$

$$+ \sum_{y \in B} \hat{L}^{-1} (\mathbb{E}_1 - \mathrm{Id}) \left(e^{-V(B \setminus \{x\})} K_y \right) \qquad \mathrm{II}$$

$$+ \sum_{\substack{Y \subset B \\ |Y| \geq 2}} \hat{L}^{-1} \mathbb{E}_1 \left(e^{-V(B \setminus Y)} K(Y) \right) \qquad \mathrm{III}$$

Note that

$$L^{-[\phi]} \tilde{h} = L^{-[\phi]} \, 2 \left(L^{d-4[\phi]} g \right)^{-1/4}$$

$$= 2 L^{-d/4} g^{-1/4} \leq g^{-1/4} = h$$

We prove that term III is bounded as claimed in Prop 7 as follows

By Example 5 and Lemma 6

$$\| \mathrm{III} \|_{T_\phi, \tilde{h}}$$

$$\leq \sum_{\substack{Y \subset B \\ |Y| \geq 2}} \| e^{-V(B \setminus Y)} K^Y \|_{\underbrace{L^{-[\phi]}\tilde{h}}_{\leq h}}$$

$$= \Big(\sum_{\substack{Y \subset B \\ \cancel{|Y| \geq 2}}} \| e^{-V} \|_h^{B \setminus Y} \, 1 \Big) \, \| K \|_h^2$$

$$\leq \left(\| e^{-V} \|_h + 1 \right)^{|B|} \| K \|_h^2$$

$$\leq \underbrace{\left(c^{L^d} \| K \|_h \right)}_{(*)} \| K \|_h$$

By choice of $C(L)$ we make $(*) \to 0$ as $L \to \infty$