MATH 305 – Applied Complex Analysis

Learning Objectives

Session 2011W Term 1 (Sept–Dec 2011)

Course pre-requisites:

As a pre-requisite to this course students are required to have a reasonable mastery of multivariable calculus and differential equations. This includes being able to

- extensively understand algebraic and transcendental functions;
- describe and parameterize curves and regions in two-dimensional space;
- understand and evaluate partial derivatives and integrals of multivariable functions;
- understand and find Taylor series and determine their intervals of convergence;
- solve boundary value problems.

Course-level learning goals:

In this course students will learn the algebra and geometry of complex numbers, mappings in the complex plane, the theory of multi-valued functions, the calculus of functions of single complex variable and the Fourier transform. In particular, students after completing this course are expected to be able to

- perform basic mathematical operations (arithmetics, powers, roots) with complex numbers in Cartesian and polar forms;
- determine continuity/differentiability/analyticity of a function and find the derivative of a function;
- work with functions (polynomials, reciprocals, exponential, trigonometric, hyperbolic, etc) of single complex variable and describe mappings in the complex plane;
- work with multi-valued functions (logarithmic, complex power) and determine branches of these functions;
- evaluate a contour integral using parametrization, fundamental theorem of calculus and Cauchy's integral formula;
- find the Taylor series of a function and determine its circle or annulus of convergence;
- compute the residue of a function and use the residue theory to evaluate a contour integral or an integral over the real line;
- find the Fourier transform and the inverse Fourier transform of a function;
- determine the number of zeros of a polynomial in the unit disk and in the right half plane;
- explain the concepts, state and prove theorems and properties involving the above topics.

Topic-level learning goals:

Here are the major learning objectives of the course. Not all of these outcomes are of equal importance.

Fundamentals of complex numbers:

1. Write a complex number in *Cartesian form* (real and imaginary parts). Perform basic mathematical operations and prove basic properties of complex numbers in Cartesian form using *complex arithmetic*, *complex conjugates* and *moduli*. Represent complex numbers and their mathematical operations geometrically.

2. Write a complex number in *polar form* (modulus and arguments) using the *Euler's Equation*. Perform basic mathematical operations and prove basic properties of complex numbers in polar forms. Distinguish between a general *argument* and *the principal value of the argument*.

3. Find the *powers* and the *roots* of a complex number. Compute a *complex exponential*. Be able to recognize and apply the *De Moivre's formula*.

4. Algebraically or geometrically represent a *planar set* defined by either equations or inequalities. Be able to recognize *open/closed*, *connected/disconnected* and *bounded/unbounded* sets.

5. Recognize *functions of a complex variable*. Find the *domain* and *range* of a function. Find the *image* of a set under a function or a composition of functions.

Analyticity:

1. Be able to define *continuity* of a function using limits. Determine where a function is *continuous/discontinuous*.

2. Be able to define *differentiability* of a function using limits. Determine where a function is *differentiable/non-differentiable*.

3. Be able to define *analyticity* of a function. Determine whether a function is *analytic/not analytic* or *entire/not entire*.

4. Use the *Cauchy-Riemann Equations* to determine whether/where a function is differentiable and find the derivative of a function.

5. Use the *two-dimensional Laplace's equation in Cartesian or polar coordinates* to determine whether a real-valued function is *harmonic* or not. Find the *harmonic conjugate* of a harmonic function. Find a harmonic function satisfying given boundary conditions.

Elementary functions:

1. Evaluate *exponential*, *trigonometric* and *hyperbolic* functions of a complex number. Be able to prove and apply properties involving these functions.

2. Solve an equation involving exponential, trigonometric and hyperbolic functions.

3. Find the image of a set under exponential, trigonometric and hyperbolic functions.

<u>Multi-valued functions:</u>

1. Evaluate *logarithms* and *complex powers* of a complex number. Be able to prove and apply properties involving logarithms and complex powers.

2. Identify the *principal value* of a logarithm or a complex power. Locate *branch points/cuts* and determine *branches* of a logarithmic or a power function.

3. Solve an equation involving logarithms and complex powers.

<u>Complex integration:</u>

1. Be able to define a *smooth arc* or a *smooth closed curve*. Parameterize a smooth arc or a closed curve with a specific direction.

2. Parameterize a *contour* in the complex plane as a union of smooth arcs. Be able to recognize *simple closed contours* and distinguish between the *interior domain* and the *exterior domain* which are separated by the simple closed contour. Distinguish between a *positively oriented* and a *negatively oriented* simple closed contour.

3. Evaluate a *contour integral* directly using the parametrization of the contour. Be able to prove and apply properties of contour integrals. Be able to recognize the *fundamental theorem of calculus* and the criteria for the *independence of path* in a contour integral. Estimate the *upper bound of the modulus of a contour integral*.

4. Be able to recognize *simply/multiply connected domains* and the admissibility of *continuous deformation* between two contours. Know the *Cauchy's integral theorem*.

5. Evaluate a contour integral with an integrand which have *singularities* lying inside or outside the simple closed contour.

6. Be able to recognize and apply the *Cauchy's integral formula* and the *generalized Cauchy's integral formula* (relationship between the derivative and the contour integral of a function).

7. Be able to recognize and apply the *Liouville's theorem*, the *mean-value property* of a function and the *maximum modulus principle*.

8. Know the *fundamental theorem of algebra*.

<u>Series:</u>

1. Determine whether a series is *convergent* or *divergent* by using the *ratio test*.

2. Determine the *circle of convergence* of a *power series*. Find the derivative of a convergent power series by termwise differentiation. Find the *Taylor series* of a given function and find its circle of convergence by either the ratio test or by locating singularities of the function.

3. Classify *zeros* and *singularities* of an analytic function. Find the *Laurent series* of a rational function. Determine the *annulus of convergence* of a Laurent series.

4. Find the *residues* of a function at given points or singularities. Use the *residue theorem* to evaluate a contour integral.

Further topics in integration:

1. Rewrite a *trigonometric integral* over $[0, 2\pi]$ as a contour integral and evaluate using the residue theorem.

2. Define the *Cauchy principal value* of an *improper integral* over $(-\infty, \infty)$ or $[0, \infty)$ and distinguish from a general improper integral.

3. Rewrite an *improper integral* involving possibly a trigonometric function over $(-\infty, \infty)$ or $[0, \infty)$ as a contour integral and evaluate its Cauchy principal value using the residue theorem.

4. Use the residue theorem to evaluate an integral involving *multi-valued function* along a contour defined in a branch. Rewrite an improper integral involving multi-valued functions over $[0, \infty)$ as an integral along a contour defined in a branch to be determined and evaluate.

Fourier transforms:

1. Recognize the relationship between the *Fourier series expansion* and the *Fourier transform*. Find the Fourier transform or the inverse Fourier transform of a function.

2. Be able to recognize and apply properties of Fourier transform and inverse Fourier transform.

Nyquist criterion:

1. determine the number of zeros of a polynomial in the unit disk and in the right half plane.