

# Learning goals: Math 257/316

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## Overarching Goals

After completing this course, students should be able to:

- recognize the three basic 2nd order linear Partial Differential Equations: Heat (parabolic), wave (hyperbolic), and Laplace (Elliptic) equation, and associate these model equations to their characteristic physical processes;
- exploit the linearity of the equation in order to come up with series solutions by superposition;
- apply specific solution techniques (separation of variables, eigenfunctions expansions, Fourier series, using linearity to solve problems with inhomogeneous boundary conditions);
- write down an appropriate finite difference scheme for one of the three fundamental equations and approximate boundary conditions with a prescribed truncation error and implement it in Excel Spreadsheets.

## Topic-Level Learning Goals

### 1. Introduction to Partial differential equations

- Recognize the classic types (parabolic, elliptic, hyperbolic equations) of second order linear PDEs.
- Identify the form of the heat equation, the wave equation, and Laplace's equation, and recognize their occurrence in physical examples, such as transient heat conduction, vibrating strings, steady heat conduction etc.

### 2. Preliminaries: This is a review section at the end of which you should be able to

- Derive Taylor series of common functions.
- Find solutions to first order separable and linear ODEs.
- Find solutions to second order homogeneous constant coefficient linear ODEs.
- Find solutions to homogeneous Euler/equidimensional equations.
- Solve inhomogeneous second order equations using undetermined coefficients.

### 3. Series Solutions of Differential Equations

- For a given ODE classify a given point in the domain as ordinary or singular and be able to distinguish regular from irregular singular points.
- Find series solutions to ODEs about a specified point: Identify the appropriate series expansion to use about the point, develop Frobenius or Taylor series expansions depending on the classification of the expansion point.
- Calculate as many non-zero terms in the series solution as is necessary.
- Find the radius of convergence of a given series solution.

### 4. Fourier Series and Separation of Variables

- Expand a given function as a full range or half range Fourier series.
- Identify the relationship between the smoothness of a given function and the rate of convergence of its Fourier series. Identify the non-uniform convergence of the Fourier series in the neighbourhood of points of discontinuity and recognize the Gibbs phenomenon.

### 5. Boundary Value Problems and separation of variables

- Categorize initial boundary value problems based on the type of boundary conditions: Dirichlet or Neumann or mixed boundary conditions; homogeneous or inhomogeneous boundary conditions; constant or time dependent boundary conditions.
- Apply the method of separation of variables to solve initial-boundary value problems for the heat equation with homogeneous boundary conditions. For a given boundary value heat conduction problem with homogeneous boundary conditions identify the appropriate eigenvalues and eigenfunctions, and find the general solution as a superposition of these eigenfunctions in the form of an infinite series.
- Solve initial-boundary value heat conduction problems with inhomogeneous boundary conditions and/or forcing terms by reducing the problem to one with homogeneous boundary conditions where separation of variables applies; by exploiting the linearity of the problem decompose the solution into a linear combination of a convenient function that satisfies the inhomogeneous boundary conditions (typically the steady state solution or some other particular solution) and a solution to the PDE that satisfies the homogeneous boundary conditions; treat forcing by means of eigenfunction expansions.
- Determine D'Alembert's solution to the second order wave equation for an infinite domain. Use the method of separation of variables to solve the wave equation on a finite domain with homogeneous boundary conditions. In particular, use the series solution to identify the appropriate periodic extension of the initial displacement and velocity fields to be able to express the solution to the wave equation on a finite domain in the form of D'Alembert's solution. As with the heat conduction problem, exploit the linearity of the problem to solve problems with inhomogeneous boundary conditions and forcing terms.
- Find steady state solutions of the 2D heat or wave equations that do not vary with time by solving the 2D Laplace equation. In particular, by separation of variables find a solution to Laplace's equation on various domains (rectangular, circular, wedge shaped, annular, interior and exterior circular openings). When appropriate, perform a change of coordinates (polar coordinates) and apply separation of variables on the transformed equation.

## 6. Numerical solutions of PDEs using Finite Difference Methods

- Identify the appropriate first and second order finite difference approximations to the first and second derivatives of a function using Taylor's Theorem.
- Using a first order approximation to the time derivative and a second order approximation to the spatial derivative, derive an explicit finite difference approximation to the heat equation. Implement the finite difference scheme in a spread sheet to solve the heat equation with various boundary conditions and compare the numerical solution to a Fourier series solution obtained by separation of variables. Introduce ghost mesh points in order to approximate derivative boundary conditions. Use Fourier analysis to derive a restriction on the time step to guarantee stability for a given spatial discretization.
- Using second order approximations to the space and time derivatives derive a finite difference approximation to the wave equation. Use an array of ghost mesh points in order to implement the time derivative initial conditions. Use Fourier analysis to derive a restriction on the time step to guarantee stability for a given spatial discretization.
- Use second order finite difference approximations to the partial derivatives in Laplace's equation to obtain a system of finite difference equations. Use the iteration feature in EXCEL to relax the solution to the steady state. Use rows and columns of ghost points to implement derivative boundary conditions.

## 7. Sturm-Liouville Theory

- Identify eigenvalues and eigenfunctions for variable coefficient problems in the Sturm-Liouville form.
- Reduce non constant coefficient second ordinary linear equations to Sturm-Liouville form.
- Determine the weight function for orthogonality.
- Separate variables for variable coefficient PDEs.