MATH 110 : Learning Goals

Course Objectives

In this course students will learn the basic ideas, tools and techniques of Differential Calculus and will use them to solve problems derived from simple, real-life applications. Specifically, students will learn to

- analyze the behaviour of basic mathematical functions (polynomial, radical, rational, trigonometric, exponential, logarithmic, absolute-valued, composite, and piecewise functions) both graphically and analytically;
- perform differentiation operations and other basic algebraic operations on the above functions fluently;
- recognize when and explain why differentiation operations are required, and use them to solve simple abstract and applied problems.
- use basic calculus concepts to find approximations of functions.

Students will apply the above knowledge and skills to translate a problem involving real-life applications into a mathematical problem, and solve it by means of Calculus. The applications include science and engineering problems involving velocity/acceleration of moving objects and other rates of change of physical quantities, exponential growth/decay models of populations, optimization problems, curve sketching, approximation techniques.

In general, when solving a problem, students should be able to:

- after reading a problem, correctly state what the problem is asking in mathematical terms, and what information is given that is needed in order to solve the problem;
- after restating the problem, identify which mathematical concepts and techniques are needed to find the solution;
- apply those concepts and techniques and correctly perform the necessary algebraic steps to obtain a solution;
- interpret results within the problem context and determine if they are reasonable.

Students will also learn how to construct simple proofs. They will learn to show that a given mathematical statement is either true or false by constructing a logical explanation (proof) using appropriate Calculus theorems and properties of functions studied in the course. In particular, when applying a theorem, students will recognize the importance of satisfying its hypotheses and drawing logical conclusions.

Prerequisites

As a prerequisite to this course, students are required to have a reasonable mastery of pre-calculus mathematics (e.g., B.C. Principles of Mathematics 10--12), as this material is required throughout the course. In particular, students are expected to be able to perform fluently the tasks listed below.

Note: The skills labels by (R) are reviewed explicitly in the course.

Prerequisites Topics:

1. Algebra

1.1: When working with expressions containing the following basic mathematical functions: polynomial, radical, rational, trigonometric, exponential, logarithmic, absolute-valued, and compositions of the above, students should be able to perform the following algebraic manipulations:

a) factor a GCF, difference of two squares, difference of two cubes, trinomials, and expressions with exponential and logarithmic functions,
b) apply the laws of exponents to simplify expressions (R),
c) apply the laws of logarithms to simplify expressions (R),
d) add and subtract and/or simplify complicated rational expressions,
e) rationalize a denominator,
f) convert an expression with a negative exponent to a reciprocal, and vice versa.

1.2: When solving equations and systems of equations, students should be able to

a) manipulate and solve linear equations,
b) manipulate and solve quadratic equations using factoring or the quadratic formula,
c) determine under what conditions a quadratic equation has one, two, or no solutions,
d) manipulate and solve rational equations and simple equations containing radical, absolute-valued, trigonometric (R), exponential (R), logarithmic (R) expressions,
e) solve systems of linear and quadratic equations,
f) interpret the solution(s) of an equation graphically as the x-intercept(s) of a curve (R), similarly interpret the solution(s) of a system of equations as the points of intersection of two or more curves.

1.3: When working with inequalities, students should be able to manipulate and solve linear and quadratic inequalities (R).

2. Geometry

2.1: Euclidean geometry. Students should be able to

a) apply Pythagoras' theorem,
b) determine trigonometric relationships involving the sides and angles of a right triangle (R),
c) express proportional relations between similar triangles,
d) apply properties of isosceles and equilateral triangles,
e) compute perimeter, area of basic shapes (e.g. triangle, square, rectangle, trapezoid, etc.) and volume of basic solids (e.g. rectangular box, sphere, circular cylinder, cone, etc.);

2.2: Analytic geometry: points on the plane. Students should be able to:

a) represent and identify points on the plane using the Cartesian coordinate system (R),
b) compute the distance between two points (R),
c) find the midpoint of a line segment (R),
d) compute the slope between two points (R).

2.3: Analytic geometry: lines and other curves. Students should be able to

a) find the slope of a line from a graph or given an equation (R),
b) given an equation of a line in any form, sketch the graph (R),
c) using either the point-slope or the point-intercept form of a line, find the equation of:

c.1. a line of known slope that goes through a given point (R),
c.2. a line that goes through two given points (R),
c.3. a line given its graph (R),

d) find the equation of a vertical/horizontal line that goes through a given point,
e) determine if two lines are parallel or perpendicular, relate the slopes of parallel lines, relate the slopes of perpendicular lines (R),
f) determine whether a point of known coordinates lies on a line/curve of given equation (R),
g) find the x- and y-intercepts of a line/curve, find the intersection point(s) of two curves/lines (R),
h) given a quadratic equation, sketch its graph (parabola) and vice versa, relate the graph of a parabola to its quadratic equation (R),
i) using the above skills, solve simple geometrical problems.

3. Functions (R)

3.1 Basic concepts: Students should be able to

a) give a definition of function (in your own words),
b) identify independent and dependent variables in a function,
c) give examples of functions in any of the three basic representations (graph, table of values, equation),
d) given a function (graph, table of values, equation), find function values
e) given one of the three basic representations of a function (graph, table of values, equation), construct a different representation,
f) given a graph/curve, determine if it represents a function, give examples of graphs that do not represent a function.

3.2. Polynomials, rational functions, ratios p(x)/q(x), where p and q are compositions of polynomials and roots, and piecewise functions defined by polynomials, rational functions, ratios p(x)/q(x) : Students should be able to

a) find the domain
b) find the range (in simple cases)
c) give examples of functions of given domain and range
d) given an equation, sketch the graph of: linear and quadratic functions, x^3, 1/x, piecewise functions composed by linear, quadratic functions, x^3, and 1/x,
e) find zeros and y-intercepts (or determine they don't exist).

3.3 Absolute-valued functions: Students should be able to,

a) find the domain
b) find the range (in simple cases)
c) write down the equation of an absolute-valued function as a piecewise defined function,
d) given an equation, sketch the graph
e) find zeros and y-intercepts (if they exist).

3.4 Exponential functions: Students should be able to

a) find the domain
b) find the range (in simple cases)
c) sketch the graph of a^x for $0 < a < 1$ and $a > 1$, and in particular the case $a = e$.
d) find zeros and y-intercepts (or determine they don't exist),
e) explain the different between a^x and x^a for any $a > 0$.

3.5 Inverse functions: Students should be able to

a) give a definition of a one-to-one function (in your own words), and recognize whether a given function is one-to-one,
b) give a definition of inverse function (in your own words),
c) given a function, determine if it is invertible, and explain why,
d) given the graph of a one-to-one function, find domain and range and sketch the graph of its inverse,
e) find domain and range and sketch the graph of $\sqrt[2]{x}$ and $\sqrt[3]{x}$.

3.7 Logarithmic functions: Students should be able to

a) find the domain,
b) find the range (in simple cases),
c) sketch the graph of $\log x$ (the logarithm to base e),
d) find zeros and y-intercepts (or determine they don't exist),
e) state the relationship between $\log x$ and e^x.

3.8 Function composition: Students should be able to

a) given any of the above functions, use function composition to build new functions
b) given a composite function, identify the individual functions that make up the composite function.

Calculus Topics

4. Limits and Continuity

4.1 The limit of a function: Students should be able to

a) explain in your own words and using a picture the meaning of $\lim_{x\to a} f(x) = L$, $\lim_{x\to a+} f(x) = L$, and $\lim_{x\to a-} f(x) = L$, where a is a finite number and L is either a finite number or infinity,
b) given an equation or graph of any of the functions listed in Section 3, evaluate the above limits either algebraically or graphically, or determine the limit does not exist, explain why; if the limit is of the form 0/0, use algebraic manipulations (e.g. factoring) to evaluate it.
c) give examples of rational functions whose limit at a finite number of points does not exist.
d) evaluate limits that contain a parameter.
e) evaluate limits that arise from the definition of derivative (see Section 3).
f) given $\lim_{x\to a} f(x) = L$ or the analogous one-sided limits (where L is either a finite number or infinity), sketch the graph of a function near the point $x = a$.
g) recognize when L'Hopital's rule applies and apply it to resolve undeterminate limits.

4.2 Limits at infinity: Students should be able to

a) explain in your own words and using a picture the meaning of $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$, where L is either a finite number or (positive or negative) infinity.
b) given an equation or graph of any of the functions listed in Section 3, evaluate the above limits either algebraically or graphically, or determine the limit does not exist and explain why; if the limit is of the form ∞/∞, use algebraic manipulations (e.g. factoring) to evaluate it.

4.3 Continuity: Students should be able to

a) explain in your words and using a graph what it means for a function to be continuous at a point,

b) using the limit notation, give a definition of continuous function at a point,
c) given an equation or graph of any of the functions listed in Section 3, determine whether the function is continuous on its domain,
d) give examples of real-life quantities that are continuous/discontinuous,
e) given a piecewise function containing parameters, determine the parameter value(s) that make the function continuous on an interval,
f) state the Intermediate Value Theorem, explain in your own words the meaning of the theorem, recognize when the theorem is applicable,
g) give examples of functions that do not satisfy the Intermediate Value Theorem by virtue of their discontinuity either in the interior of an interval or at its endpoints,
h) given an algebraic equation, apply the Intermediate Value Theorem to prove the existence of roots; apply the same idea to estimate zeros of functions (for the functions listed in Section 3).

5. Estimating instantaneous velocities and slopes of tangent lines

5.1 The difference quotient: Students should be able to
a) given an equation or graph of a function, compute the change in function values for a change in the independent variable, use a difference quotient to compute the average rate of change of the function and the average slope of the graph between two points (i.e. the slope of the secant line between two points), relate the average slope of the curve to the average rate of change of the function, and vice versa,
b) given an equation or graph representing the position of a moving object as a function of time (position function), use the skills in a) to compute the object's change in position and average velocity over a time interval, and relate the object's average velocity to the average slope of the position curve over a time interval.

5.2. Velocity and the idea of limit: Students should be able to
a) using difference quotients, explain how to estimate the instantaneous velocity of an object and argue why the concept of instantaneous velocity requires the notion of limit of a function,
b) intepret veloctiy as a rate of change and apply the skills in a) to any function in order to estimate the rate of change of the function at a point.

5.3 The tangent line and the idea of limit: Students should be able to
a) given a graph and using secant lines and difference quotients, explain how to define and estimate the slope of a tangent line to the graph at point and argue why the concept of slope of a tangent line requires the notion of limit,
b) given a graph, draw the tangent line to the curve at a given point.

6. The Derivative

6.1 The definition of derivative: Students should be able to
a) given a function $f(x)$ (equation or graph), interpret the quantity $[f(x+h)-f(x)]/h$ for some $h \neq 0$ as a difference quotient and rewrite it in simplified form,
b) using difference quotients and limits, give a rigorous definition (limit defiinition) of derivative of a function at a point; intepret the derivative as the rate of change of the function, relate the rate of change to the slope of a tangent line to the graph of the function,
c) using the limit definition of derivative, calculate derivatives (at a point or as a function) of polynomial functions of degree 3 or less and rational functions $p(\mathrm{x})/q(\mathrm{x})$ where p and q are linear or

quadratic functions; apply the limit definition to express the derivative of (simple) functions in general form (e.g. some unknown function $cf(x)$ where c is a constant),

d) use different notations (f 'and df/dx) to denote the derivative of a function f,
e) given the graph of a function, sketch a rough graph of its derivative, and vice versa.

6.2 Differentiable functions: students should be able to

a) explain what it means for a function to be differentiable at a point,
b) explain why differentiable functions are continuous,
c) demonstrate, using an example, that continuous functions need not be differentiable,
d) using the limit definition of derivative, compute the derivative of |x|,
e) given a piecewise function containing parameters, determine the parameter value(s) of that make the function differentiable on a given interval.

6.3 Derivatives of elementary functions: Students should be able to

a) state the derivative of basic trigonometric, exponential, logarithmic functions,
b) recognize when each one of the following differentiation rules applies: the constant-multiple rule, the sum/difference rule, the power rule, the product rule, the quotient rule, the chain rule,
c) given an equation of any function listed in section 3, apply any of the above rules (separately or in combination with other rules) to compute the derivative of the function (at a point or as a function).

6.4. Derivatives and tangent lines: Students should be able to

a) given an equation of a curve and using the relationships identified in 6.1.b and derivatives, compute the slope of and find the equation of a line tangent to the curve at a point,
b) using derivatives, solve geometrical problems involving tangent lines.

6.5 Derivatives and velocity: Students should be able to

a) given an equation of a position function and using the relationships identified in 6.1.b, compute instantaneous velocities,
b) sketch the graphs of velocity and acceleration of a particle given a graph of its position.

6.6 Derivatives and rates of change: Students should be able to

a) given a mathematical model (equation) for a physical quantity or population size and using the relationships identified in 6.1.b, compute rates of change and determine its units,
b) given a mathematical model for exponential growth/decay, use the properties of exponentials and logarithms to find any unknown constant in the model and make predictions,

6.7 Higher order derivatives: Students should be able to

a) give a definition of the second (and higher order) derivative of a function, and interpret it in terms of rate of change and slope,
b) given an equation of a function and using differentiation rules, compute the second derivative of the function (at a point or as a function),
c) given an equation of a position function, relate acceleration to the second derivative of position,
d) given an equation of a velocity function, relate accelaration to the first derivative of velocity,
e) solve simple velocity and acceleration problems using derivatives.

6.8 Implicit Differentiation. Students should be able to,

a) given an implicit relationship between x and y, find dy/dx by applying the technique of implicit differentiation,

b) justify the technique of implicit differentiation using the chain rule,
c) given the equation of a curve that does not represent a function (and has equation of the form $f(x,y) = g(x,y)$ where f and g are composed of elementary functions), find the equation of lines tangent to the curve.

7. Related Rates Problems: Sudents should be able to solve problems where two (or more) quantities are related to each other and one seeks the instantaneous rate of change of one quantity given the rate of change of the other quantity. If the relationships between variables are not given, students should be able to derive them from geometrical arguments.

8. Shape of a function

8.1 Mean Value Theorem: Studens should be able to

a) state the theorem (clearly listing its assumptions and conclusion), explain its conclusion using a graph.
b) explain how the theorem connects the shape of the graph of a function on an interval to the value of the derivative at an interior point,
c) apply the theorem to construct simple proofs.

6.7 Increasing/decreasing functions and concavity: Students should be able to, given an equation/graph of a function

a) find critical numbers of the function,
b) determine the intervals of increase/decrease of the function,
c) state and apply the first and second derivative tests,
d) find local/relative extreme values of the function,
e) define the terms “concave up”, “concave down” and “inflection point” in the context of functions, explain the connection between the concavity of a function and the sign of its second derivative,
f) determine intervals of concavity and find inflection points (if they exist).

8.3 Asymptotes: students should be able to

a) give a rigorous definition of vertical and horizontal asymptote using limits,
b) given an equation of a function, find vertical and horizontal asymptotes (if they exist).

8.4 Curve Sketching: Students should be able to sketch the graph of a function, indicating its domain, intercepts, asymptotes, intervals of increase/decrease and extreme values, intervals of concavity and inflection points.

9. Optimization: Students should be able to

a) find the absolute/global maximum and minimum of a function,
b) solve applied optimization problems.

10. Approximations

10.1 Linear Approximation: Students should be able to

a) using a graph, explain how to obtain the linear approximation of a function at a point,

b) compute the linear approximation of a given function at a point (“center”),
c) use appropriate linear approximations to estimate a function value.

10.2 Taylor Polynomials: Students should be able to

a) find the Taylor polynomial (of a given degree) of a function at a point (“center”),
b) explain of the derivatives of the function relate to the coefficients of the polynomial,
c) explain how linear approximation relates to Taylor polynomials,
d) use Taylor polynomials to approximate function values.