Analysis on arithmetic quotients

This talk will be found at

http://www.math.ubc.ca/~cass/sydney/talk2.pdf

Let G be the group of real points on a reductive group defined over $\mathbb{Q}$, Γ a group of finite index in $G(\mathbb{Z})$. The spectrum of $L^2(\Gamma\backslash G)$ is of great number-theoretical interest.

In this talk I'll try to give some idea of some of the difficulties involved in the subject, by explaining something of what happens for SL_2. This talk will be largely a talk in analysis, since it is the analysis involved that most people find intimidating. I'll say almost nothing of the connections with number theory.

Geometry

$$G = \mathrm{SL}_2(\mathbb{R})$$

$$\Gamma = \mathrm{SL}_2(\mathbb{Z})$$

$$\Gamma(N) = \{\gamma \equiv I \textbf{ mod } N\}$$

$$K = \mathrm{SO}_2$$

$$\mathcal{H} = \{z = x + iy \,|\, y > 0\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az + b \\ cz + d \end{bmatrix}$$

$$\sim \begin{bmatrix} (az + b)/(cz + d) \\ 1 \end{bmatrix}$$

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \longrightarrow \frac{az + b}{cz + d}$$

$$\mathcal{H} \cong G/K$$

$$y(g(z)) = \frac{y}{|cz + d|^2}$$

$$P = \left\{ \begin{bmatrix} a & x \\ & a^{-1} \end{bmatrix} \right\}$$

$$A = \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \right\}$$

$$N = \left\{ \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \right\}$$

$$\Gamma \cap P = \left\{ \begin{bmatrix} \pm 1 & n \\ & \pm 1 \end{bmatrix} \right\}$$

The group P is the stabilizer of ∞.

For z in $\mathcal{H}$, z and 1 generate a lattice in $\mathbb{C}$. An element g in $\mathrm{SL}_2(\mathbb{Z})$ takes this to the lattice spanned by $az+b$ and $cz+d$, which span the same lattice. This is similar to the lattice spanned by $(az+b)/(cz+d)$ and 1.

In fact, points of the quotient $\Gamma\backslash\mathcal{H}$ parametrize similarity classes of lattices in $\mathbb{C}$.

The fundamental domain of Γ is the region

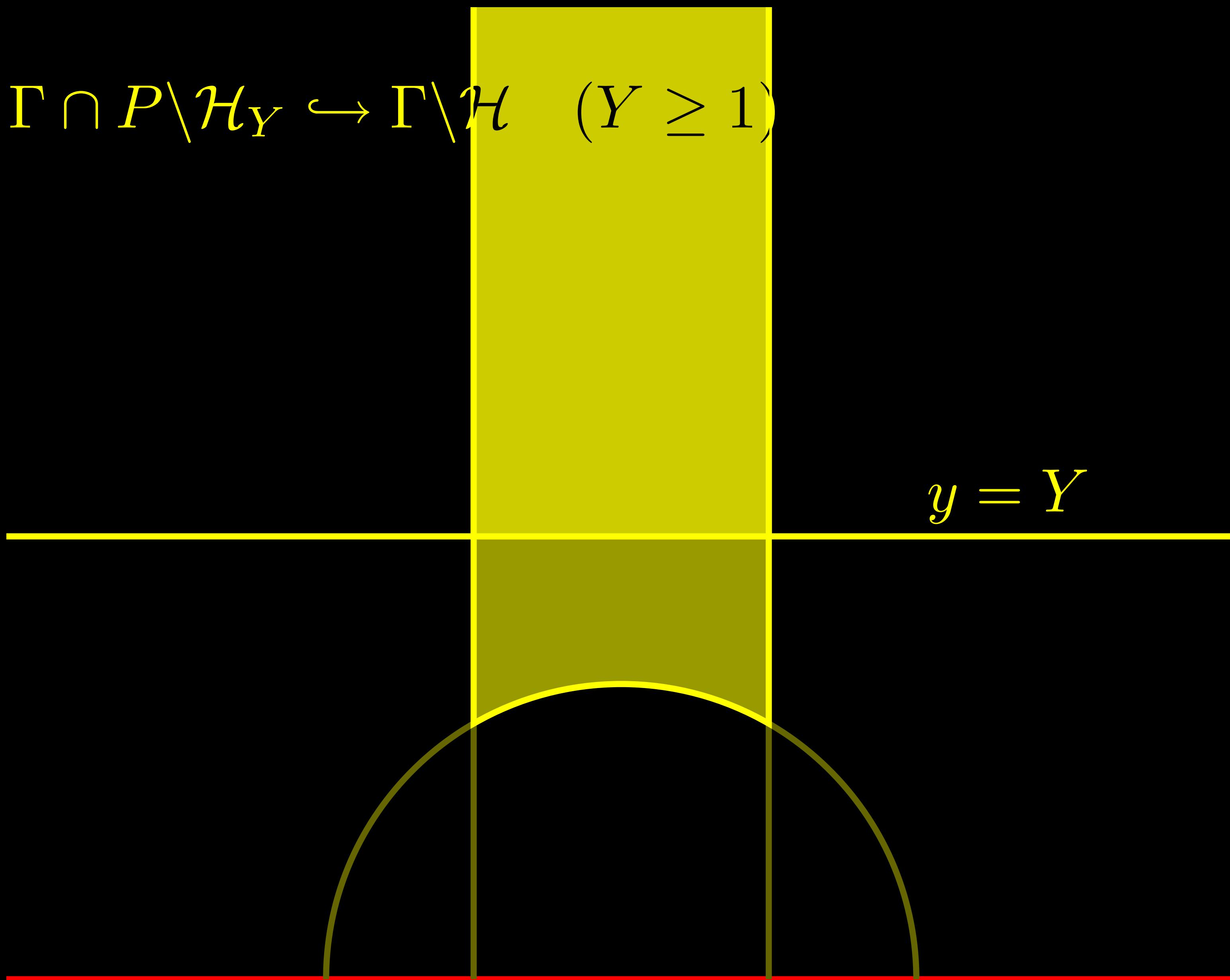
$$-1/2 \leq \Re(z) < 1/2, \quad |z| \geq 1 \; .$$

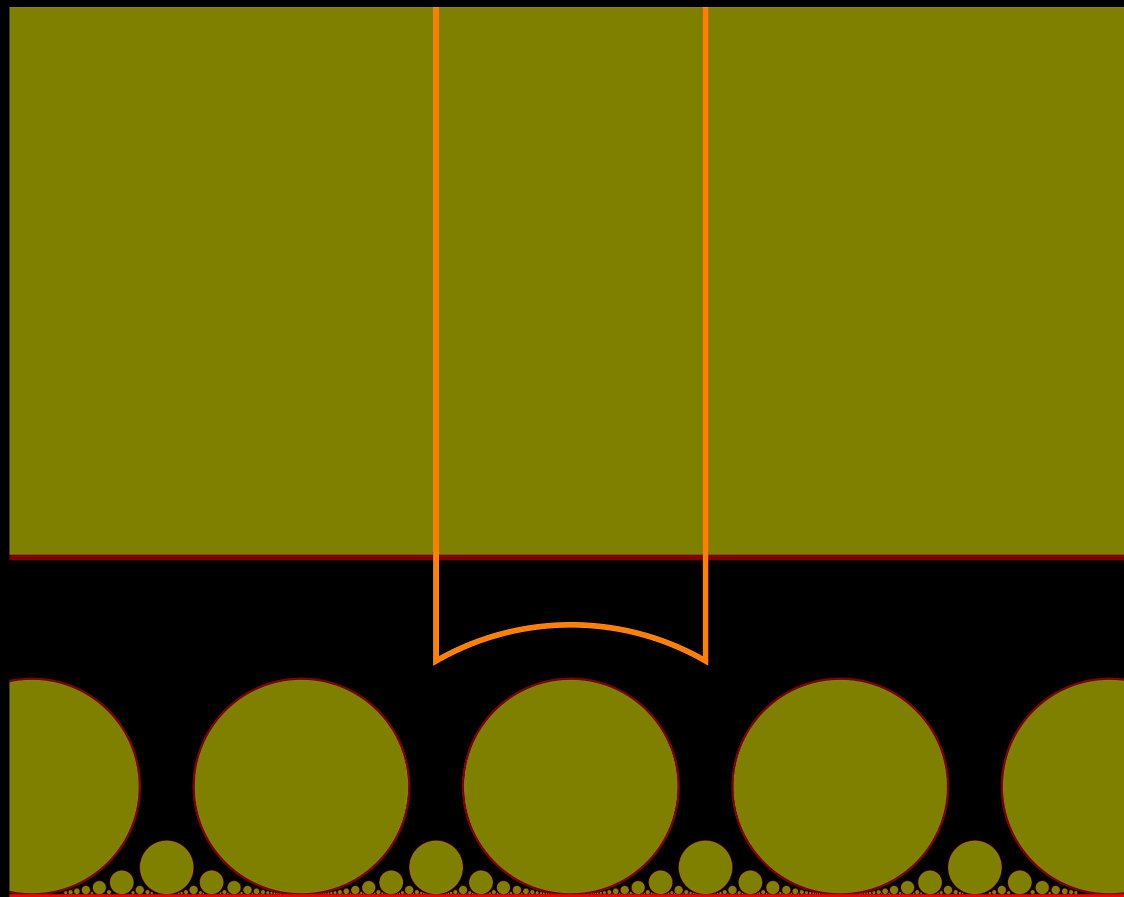
There is a simple reduction algorithm transforming an arbitrary z to one in this region, a succession of transformations

$$T\colon \; z \longmapsto z - n$$
$$S\colon \; z \longmapsto -1/z$$

The algorithm stops because in every cycle y increases. It is also related to the problem of finding a relatively orthogonal basis for lattices.

$$\Gamma \cap P \backslash \mathcal{H}_Y \hookrightarrow \Gamma \backslash \mathcal{H} \quad (Y \geq 1)$$





$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: **the image of** $\{y \geq Y\}$ **has height** $1/c^2Y$**.**

The problem

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

$$dA = \frac{dx\, dy}{y^2}$$

The differential operator Δ is essentially self-adjoint on $\mathrm{L}^2(\Gamma\backslash\mathcal{H})$.

What is its spectrum?

This is a special case of another question. Functions on $\Gamma\backslash\mathcal{H}$ may be identified with functions on $\Gamma\backslash G$ fixed on the right by K. The operator Δ is the restriction to $\mathcal{H}$ of the Casimir element $\mathfrak{C}$ of $U(\mathfrak{g})$, which lies in the centre $Z(\mathfrak{g})$.

What is the decomposition of $L^2(\Gamma\backslash G)$ into eigenspaces of $\mathfrak{C}$, or (equivalently) irreducible unitary representations of G?

The first step is to make a simple decomposition. Any reasonable Γ-invarant f on $\mathcal{H}$ is invariant under integral horizontal translations, so

$$f(x+iy) = \sum f_n(y) e^{2\pi i n x} \ .$$

The function f is called cuspidal if the constant term f_0 vanishes. With mild conditions on f, the difference $f - f_0$vanishes very rapidly at ∞. The subspace

$$L^2_{\mathrm{cuspidal}}(\Gamma\backslash\mathcal{H})$$

decomposes into finite-dimensional eigenspaces of Δ (looks like L^2 of a compact Riemannian manifold).

$$
\begin{aligned}
\mathrm{L}^2 &= \mathrm{L}^2_{\mathrm{continuous}} \oplus \mathrm{L}^2_{\mathrm{discrete}} \\
&= \mathrm{L}^2_{\mathrm{continuous}} \oplus \mathrm{L}^2_{\mathrm{cuspidal}} \oplus \ \mathbb{C} \cdot 1 \ .
\end{aligned}
$$

All three of these components are related to number theory.

The spectrum of $\mathrm{L}^2_{\mathrm{continuous}}$ is $(-\infty, -1/4]$, and we can describe the eigenfunctions contributing to it rather explicitly.

The decomposition of $\mathrm{L}^2_{\mathrm{cuspidal}}$ is a mystery. The few constituents explicitly known are related to representations of Galois groups.

The spaces $\mathrm{L}^2_{\mathrm{discrete}}$ and $\mathrm{L}^2_{\mathrm{continuous}}$ are linked together.

The continuous spectrum directly involves

$$c(s) = \frac{\xi(2s-1)}{\xi(2s)} \quad \left(\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)\right) .$$

The function $c(s)$ has a simple pole at $s = 1$ and the volume of $\Gamma\backslash\mathcal{H}$ is its residue there.

We have $|c(s)| = 1$ on $\Re(s) = 1/2$ and the distribution of the eigenvalues on $\mathrm{L}^2_{\mathrm{cuspidal}}$ is related to the winding of $c(s)$ on that line.

The principal tool is the trace formula, which still holds many secrets, even for $\mathrm{SL}_2(\mathbb{Z})$.

At any rate, understanding it requires understanding the continuous spectrum, which is spanned by Eisenstein series.

Eisenstein series

For s in $\mathbb{C}$ the function y^s is a function on $\mathcal{H}$ invariant with respect to N and indeed $\Gamma \cap P$.

$$\Delta y^s = s(s-1)y^s \ .$$

The Eisenstein series is

$$E_s = \sum_{\Gamma \cap P \backslash \Gamma} y(\gamma(z))^s, which$$

converges for $\Re(s) > 1$. Since $\Delta y^s = s(s-1)$

$$\Delta E_s = s(s-1)E_s \ .$$

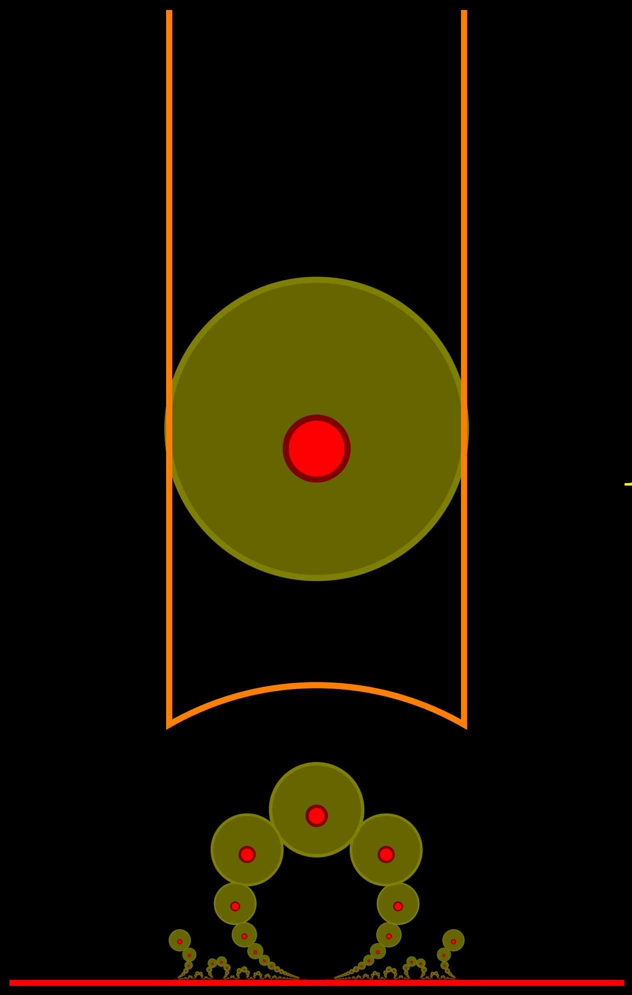
For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ since

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + nc & b + nd \\ c & d \end{bmatrix}$$

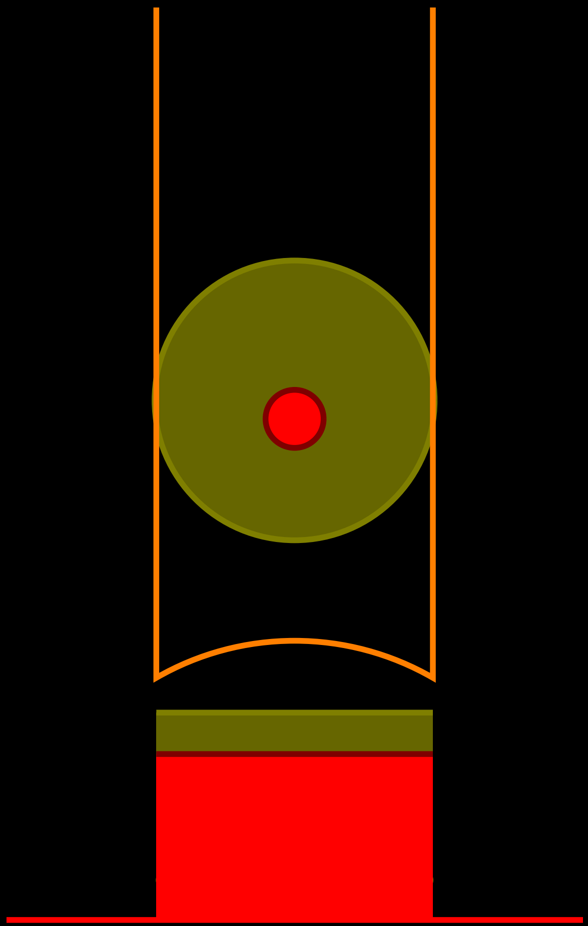
this is also

$$\sum_{c \geq 0, \mathrm{gcd}(c,d)=1} \frac{y^s}{|cz + d|^{2s}} \, ,$$

These series were first investigated by Maass, after some hints by Hecke. The hard theorems about them were first proven by Selberg.



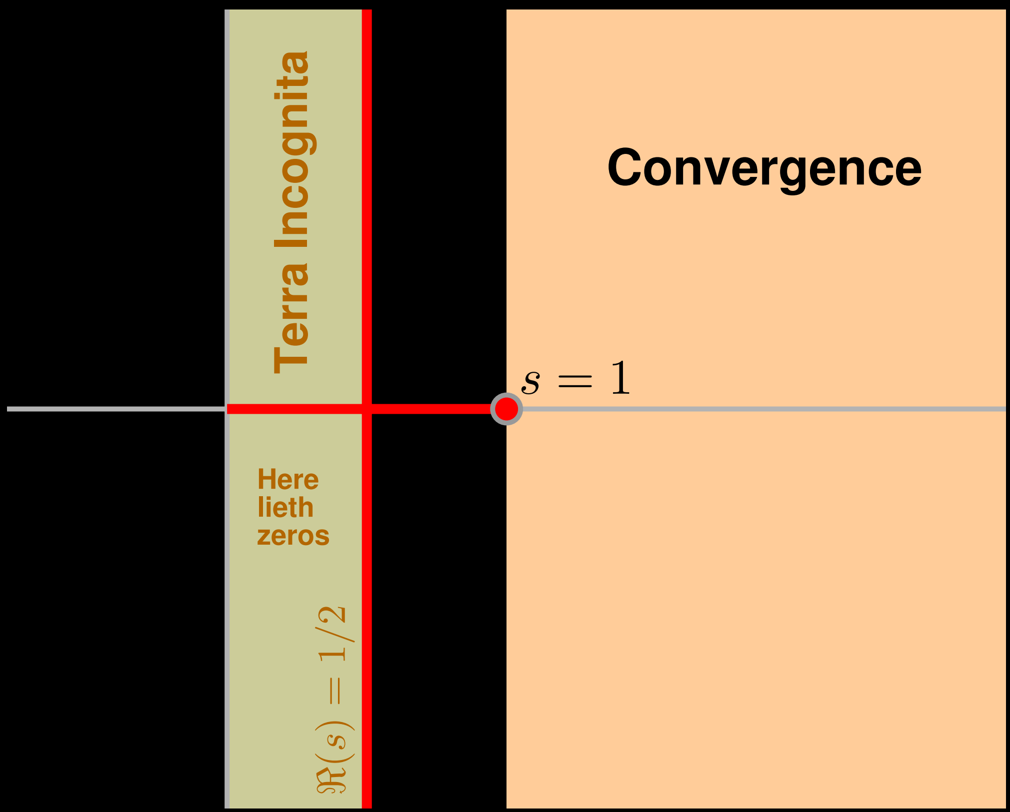
$$E_s = y^s + \sum_{\gcd(c,d)=1, c>0} \frac{y^s}{|cz+d|^{2s}}$$



$$|E_s - y^s| \leq \int_0^{1/y} v^\sigma \, \frac{dx\,dv}{v^2}$$

$$= y^{1-\sigma} \quad \big(\sigma = \Re(s) > 1\big)$$

The Laplacian is negative, which means that its spectrum is contained in $(\infty, 0]$. This corresponds to Eisenstein series E_s with $\Re(s) = 1/2$ as well as s on the line segment $[0, 1]$.



Theorem. The function E_s continues meromorphically into all of $\mathbb{C}$. It satisfies the functional equation

$$E_s = c(s)E_{1-s} \quad \left(c(s) = \frac{\xi(2s-1)}{\xi(2s)} \right)$$

Near $y = \infty$

$$E_s \sim y^s + c(s)y^{1-s}$$

In $\Re(s) \ge 1/2$ it has exactly one pole, which is a simple pole with residue the constant function $1/\xi(2)$.

A consequence is that

$$c(s)c(1-s) = 1$$

The Plancherel theorem

If $\varphi(s)$ is a continuous function of compact support on $1/2 + i\mathbb{R}$, define the integral

$$E_\varphi(z) = \frac{1}{2\pi i} \int_{\Re(s)=1/2} \varphi(s) E_s \, ds \ .$$

Then

$$\|E_\varphi\|^2 = 2\|\varphi\|^2 \ .$$

The map $\varphi \to E_\varphi$ induces an isometry (up to a constant factor)

$$\{\varphi \in \mathrm{L}^2(1/2 + i\mathbb{R}) \,|\, \varphi(1-s) = c(s)\varphi(s)\}$$

with $\mathrm{L}^2_{\mathrm{continuous}}(\Gamma\backslash\mathcal{H})$.

The principal theorem is the decomposition

$$L^2 = L^2_{\mathrm{continuous}} \oplus \mathrm{L}^2_{\mathrm{residual}} \oplus L^2_{\mathrm{cuspidal}} \ ,$$

along with variants.

$$
\begin{aligned}
E_\varphi &= \frac{1}{2\pi i}\int_{\Re(s)=1/2} \varphi(s)E_s\,ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=1/2} \varphi(s)c(s)E_{1-s}\,ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=1/2} \varphi(1-s)c(1-s)E_s\,ds \\
&= E_{\varphi_\#}
\end{aligned}
$$

where

$$
\varphi_\#(s) = \frac{\varphi(s) + c(1-s)\varphi(1-s)}{2}
$$

$$
c(s)\varphi_\#(s) = \varphi_\#(1-s)
$$

In the rest of this talk, I'll try to give some idea of how the proof of meromorphic continuation of E_s goes.

The constant term

Any function F on $\Gamma\backslash\mathcal{H}$ is invariant under $\Gamma \cap N \cong \mathbb{Z}$, and may therefore be expressed in a Fourier series

$$F(x+iy) = \sum_n F_n(y) e^{2\pi i n x}$$

$$F_n(y) = \int_0^1 F(x+iy) e^{-2\pi i n x}\, dx\ .$$

$F_0(y)$ **is called the constant term of** F

If F satisfies a certain mild growth condition on the fundamental domain as $y \longrightarrow \infty$, then $F - F_0$ is rapidly decreasing as $y \longrightarrow \infty$.

The constant term controls the asymptotic behaviour of F.

The cuspidal component of L^2 is that of functions whose constant term vanishes, and cuspidal functions are of rapid decrease at ∞. The cuspidal component looks like L^2 of a compact Riemannian surface.

Since $\Delta E_s = s(s-1)E_s$ and Δ commutes with N, if $\Phi(y)$ is the constant term of E_s then

$$y^2\Phi'' = s(s-1)\Phi \ .$$

This has solutions

$$y^s, \quad y^{1-s} \quad (s \neq 1/2)$$

Since we already know that y^s is the dominant term of E_s for $\Re(s) > 1$

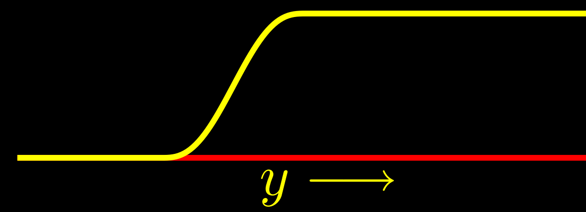
$$\Phi(y) = y^s + c(s)y^{1-s}$$

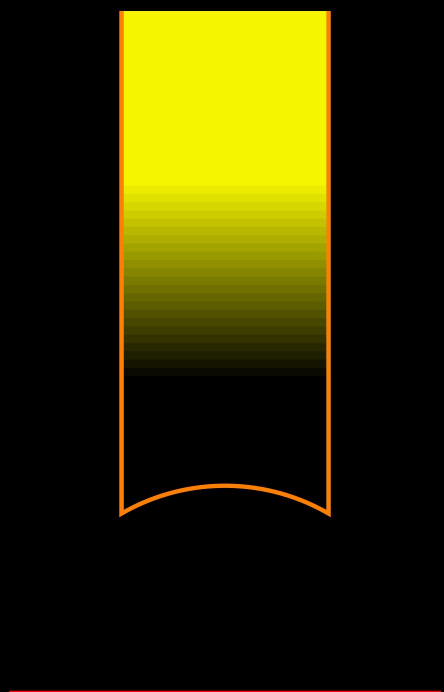
$$E_s = y^s + c(s)y^{1-s} + \text{ rapidly decreasing stuff}$$

$$E_s - y^s = \text{ is square integrable if } \Re(s) > 1/2$$

Analytic continuation
by spectral analysis of the Laplacian

Fix $Y > 1$. Let $\chi(y)$ be a smooth function identically 1 for large y and 0 for $y < Y$.





The function χ may be identified with one on $\Gamma\backslash\mathcal{H}$. If F is any function on $\Gamma\backslash\mathcal{H}$ satisfying a mild growth condition then $\Lambda_\chi F = F - \chi \cdot F_0$ will be rapidly decreasing at ∞.

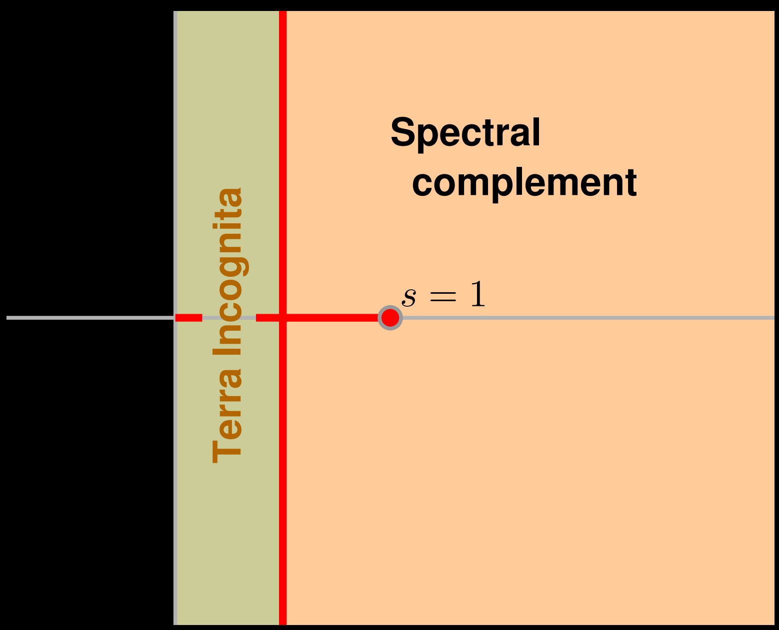
For s in the region of convergence, $E_s - \chi \cdot y^s$ will be in L^2. If $\lambda = s(s-1)$ then

$$
\begin{aligned}
(\Delta - \lambda)(E_s - \chi \cdot y^s) &= (\Delta - \lambda)E_s - (\Delta - \lambda)(\chi \cdot y^s) \\
&= -(\Delta - \lambda)(\chi \cdot y^s)
\end{aligned}
$$

a function of compact support on $\Gamma\backslash\mathcal{H}$, *defined for all* s. Since λ is not in the spectrum of the self adjoint operator Δ, $(\Delta - \lambda)^{-1}$ is a bounded operator on $\mathrm{L}^2(\Gamma\backslash\mathcal{H})$, and $E_s - \chi \cdot y^s$ will be the unique function in $\mathrm{L}^2(\Gamma\backslash\mathcal{H})$ satisfying this equation. This allows us to extend the definition to the whole region $\Re(s) > 1/2$ except the segment $(1/2, 1]$, since this region is in the complement of the spectrum.

Setting

$$E_s = (\Delta - \lambda)^{-1}\big((\Delta - \lambda)(E_s - \chi \cdot y^s)\big) + \chi \cdot y^s$$



will define E_s throughout this region compatibly with its definition by the series.

Meromorphic continuation
by Fredholm theory and Hecke operators

A compactly supported function f on $\mathcal{H}$ that's invariant under non-Euclidean rotation lifts to a function on G that's left and right invariant with respect to multiplication by K. If F is a function on $\Gamma\backslash\mathcal{H}$ then

$$R_f F = \int_G F(xg) f(g)\, dg$$

is again a function on $\Gamma\backslash\mathcal{H}$. Convolution makes the set of all such functions into a ring $\mathfrak{H}(K\backslash G/K)$ (that's $\mathfrak{H}$ for $\mathfrak{Hecke}$).

Geometrically speaking, this operator replaces the value of F at a point z by a weighted integral of its average values on non-Euclidean circles concentric at z. The functions f are what Selberg called point-pair invariants.

If f is the characteristic function of the unit disc around i, the operator R_f replaces F at any point z by the average value of F over the unit disc centred at z.

The operator R_f commutes with the actions of G and its Lie algebra $\mathfrak{g}$ on $\mathcal{H}$. The function y^s is, up to scalar, unique with the two properties

$$\frac{\partial y^s}{\partial x} = 0$$
$$y\,\frac{\partial y^s}{\partial y} = s y^s \ .$$

Therefore there exists a homomorphism η_s from $\mathfrak{H}(K\backslash G/K)$ to $\mathbb{C}$ such that

$$R_f y^s = \eta_s(f) y^s \ .$$

Define a truncated form of R_f:

$$\overline{R}_f F = R_f F - \chi(y)\, R_f F_0(y)$$

This is a compact operator, hence by the Fredholm theory its resolvent

$$(\overline{R}_f - \lambda)^{-1}$$

is meromorphic. If $\lambda_s = \eta_s(f)$ then in the region of convergence

$$\begin{aligned}(\overline{R}_f - \lambda_s)E_s &= (R_f E_s - \lambda_s E_s) \\ &\qquad - \chi(y) R_f \left(y^s + c(s) y^{1-s}\right) \\ &= -\lambda_s\, \chi(y)\, \left(y^s + c(s) y^{1-s}\right)\end{aligned}$$

$$(\overline{R}_f - \lambda_s)E_s = -\lambda_s\,\chi(y)\left(y^s + c(s)y^{1-s}\right)$$

If we could solve

$$(\overline{R}_f - \lambda_s)E_s^* = -\lambda_s\chi(y)\,y^s$$

then we could set

$$E_s = E_s^* + c(s)E_{1-s}^*\ .$$

To solve

$$(\overline{R}_f - \lambda_s)E_s^* = -\lambda_s \chi(y)\, y^s \ ,$$

we want to rewrite it as an equation with something in L^2 on the right hand side. Set $E_s^{**} = E_s^* - \chi(y)y^s$, which gives

$$\begin{aligned}(\overline{R}_f - \lambda_s)E_s^{**} &= (\overline{R}_f - \lambda_s)E_s^* - (\overline{R}_f - \lambda_s)\chi(y)y^s \\ &= \overline{R}_f\chi(y)y^s \\ &= R_f\chi(y)y^s - \chi R_f\chi(y)y^s\end{aligned}$$

and this right hand side has compact support.

We can find E_s^{**} as a meromorphic function of s throughout $\mathbb{C}$, then find E^s, and finally set

$$E_s = E_s^* + c(s)E_{1-s}^* \ ,$$

if we can somehow find the meromorphic continuation of $c(s)$. But

$$(R_f - \lambda_s)E_s = 0$$

which means that

$$(R_f - \lambda_s)E_s^* = -c(s)(R_f - \lambda_s)E_{1-s}^*$$

so we can define $c(s)$ as the ratio of the two sides.

Summary:

Spectral analysis of Δ allows us to analytically continue E_s to all but $(1/2, 1]$ in $\Re(s) > 1/2$. It also allows us to get growth estimates, since there is a classical bound on $\|\Delta - \lambda\|^{-1}$.

For meromorphic continuation we use Hecke operators.

From the first, we get $E_s = c(s)E_{1-s}$, hence $c(s)c(1-s) = 1$. Also $|c(s)| = 1$ on $\Re(s) = 1/2$.

The simplicity of the pole at $s = 1$ follows from an explicit formula for $\Lambda_\chi E_s$.

The Plancherel formula . . . Completeness of the discrete spectrum.

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