Chipping away at convex sets

Sydney, 2004

by Bill Casselman

This picture has nothing whatsoever to do with the rest of my talk

Outline

Part I. Euclidean space

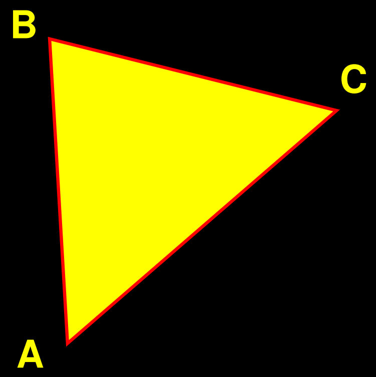
Part II. Arithmetic quotients

This talk can be found at

http://www.math.ubc.ca/~cass/sydney/colloquium.pdf

Part I. Euclidean space
A homework exercise

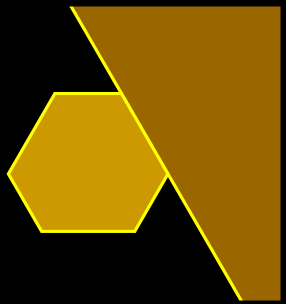
Find the Fourier transform $\widehat{\chi}(s)$ of the characteristic function χ of a triangle with vertices at A, B, C.



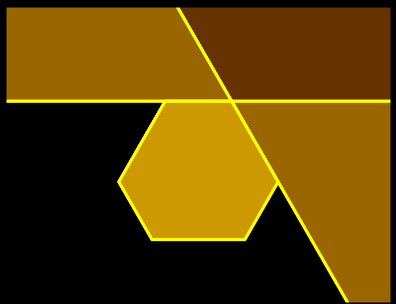
Your answer should make manifest the inherent symmetry of the problem. Check it by verifying that evaluation at $s = 0$ gives the area of the triangle.

Truncation for convex polyhedra

Let C be a closed convex polyhedron, $C^\circ \neq \emptyset$.



- If F is a face of codimension one of C, let E_F^C be the open exterior half-plane bounded by F.

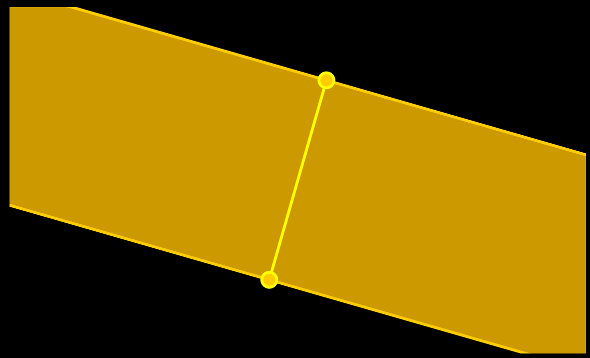


- If F is any other proper face

$$E_F^C = \bigcap_{F \subseteq F_* \subseteq C} E_{F_*}^C$$

- If $F = C$ let $E_F^C = V$.

If $C^\circ = \emptyset$, replace it by its product with the vector space perpendicular to it in these definitions.



Let $\mathcal{E}_F^C$ be the characteristic function of E_F^C.

The following theorem is due to Ishida for cones, Brion and Vergne for nondegenerate bounded polyhedra. The result for arbitrary convex polyhedra seems to be new, even in two dimensions.

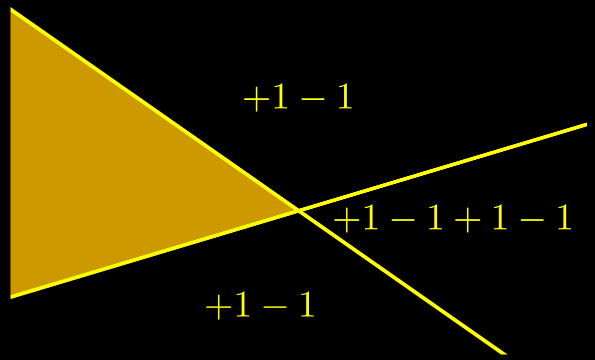
Theorem C. *We have*

$$\sum_{F \preceq C} (-1)^{\operatorname{codim} F} \mathcal{E}_F^C = \mathfrak{char}_C \, .$$

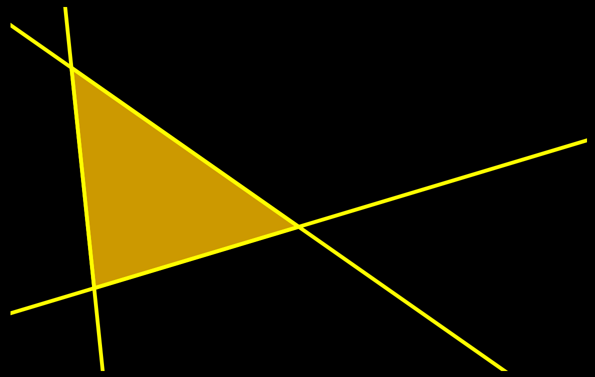
Minkowski or Weyl might have discovered it.

A few simple cases:

- Coordinate octant or simplicial cone



- Simplex



Another formulation:

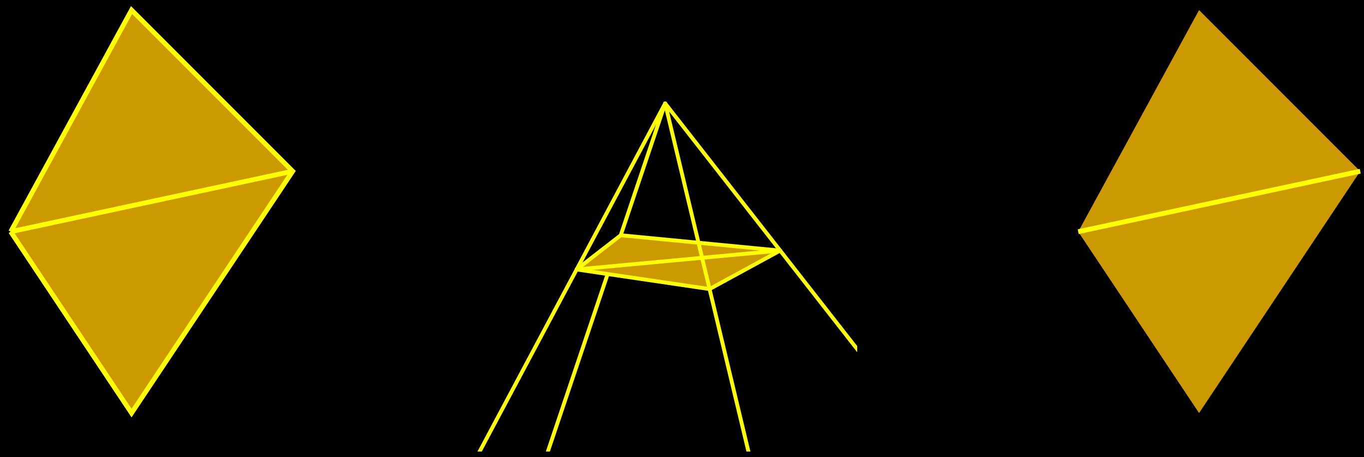
For any P in V

$$\sum_{P \in E_F^C} (-1)^{\operatorname{codim} F} = \begin{cases} 1 & P \in C \\ 0 & \textit{otherwise}. \end{cases}$$

This suggests that cohomological Euler-Poincaré characteristics are going to play a role:

$$\sum_{F \preceq C} (-1)^{\operatorname{codim} F} = \begin{cases} (-1)^{\dim C} & \textbf{if } C \textbf{ is closed and bounded} \\ 0 & \textbf{if } C \textbf{ is a closed cone} \\ 1 & \textbf{if } C \textbf{ is open} \end{cases}$$

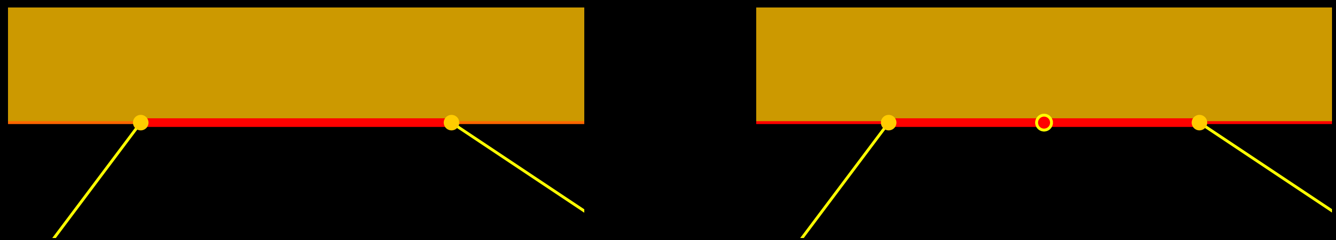
The first two are equivalent, since a suitable slice through a cone is a bounded polyhedron.



Two faces are allowed to have the same affine support. In effect, the Theorem is about a cell decomposition of C compatible with its convex structure. If a number of faces F_* partition an open geometric face F°, their total contribution is just

$$\sum_{F_* \subseteq F^\circ} (-1)^{\operatorname{codim} F_*} E^C_{F_*} = E^C_F$$

since all $E^C_{F_*} = E^C_F$ here and the Euler-Poincaré characteristic of the open face is 1.



What’s going on?

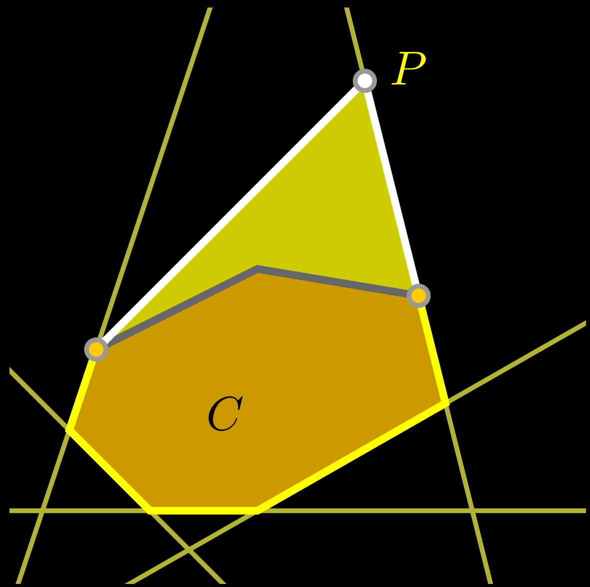
Bounded polyhedra

following Brion & Vergne

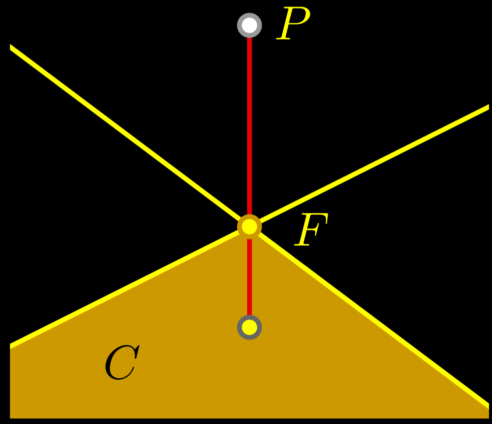
We want to prove that for any point P

$$\sum_{P \in E_F^C} (-1)^{\operatorname{codim} F} = \begin{cases} 1 & P \in C \\ 0 & \text{otherwise} \end{cases}$$

If P is a point of C this is immediate. If P is not in C, let H be the convex hull of C and P.

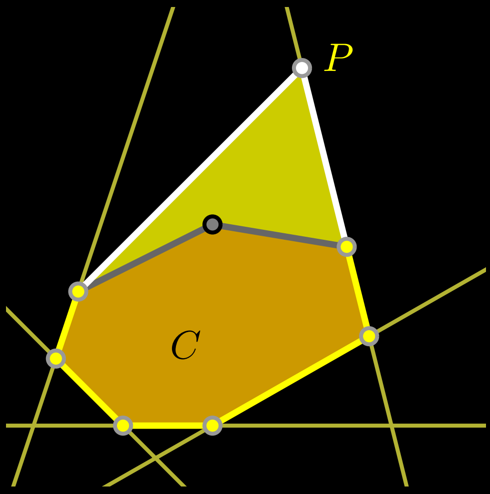


Then H **is the union of all segments** $[P, Q]$ **with** Q **in** C**, and** H° **is the union of all** $[Q, P)$ **with** Q **in** C°**.**



Proposition. *If* F *is a face of* C *then* $F \subset H^\circ$ *if and only if* $P \in E_F^C$.

Corollary. *A face* F *of* C *is one of the cells in the boundary of* H *if and only if* $P \notin E_F^C$.



$$\sum_{F \preceq H} (-1)^{\mathrm{codim} F} = EP(H) = (-1)^{\dim C}$$

$$\sum_{F \preceq H, P \in F} (-1)^{\mathrm{codim} F} = EP(\textbf{cone}) = 0$$

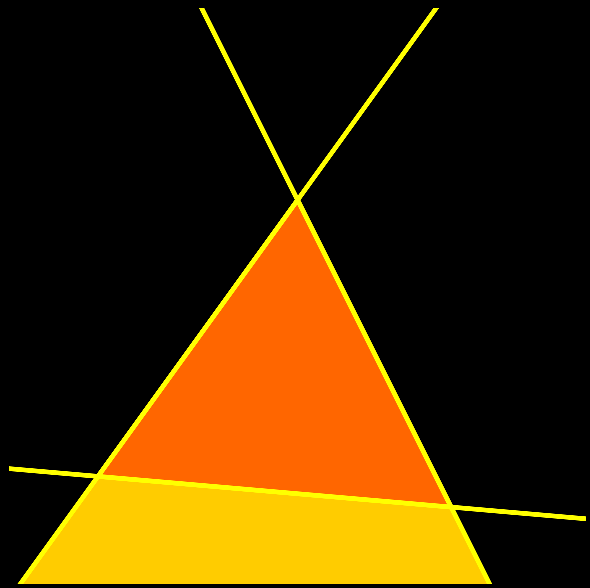
$$\sum_{F \preceq H, P \notin F} (-1)^{\mathrm{codim} F} = \sum_{F \preceq C, P \notin E_F^C} (-1)^{\mathrm{codim} F} = (-1)^{\dim C}$$

$$\sum_{F \preceq C} (-1)^{\mathrm{codim} F} = EP(C) = (-1)^{\dim C}$$

$$\sum_{P \in E_F^C} (-1)^{\mathrm{codim} F} = 0 \qquad Q.E.D.$$

Cones

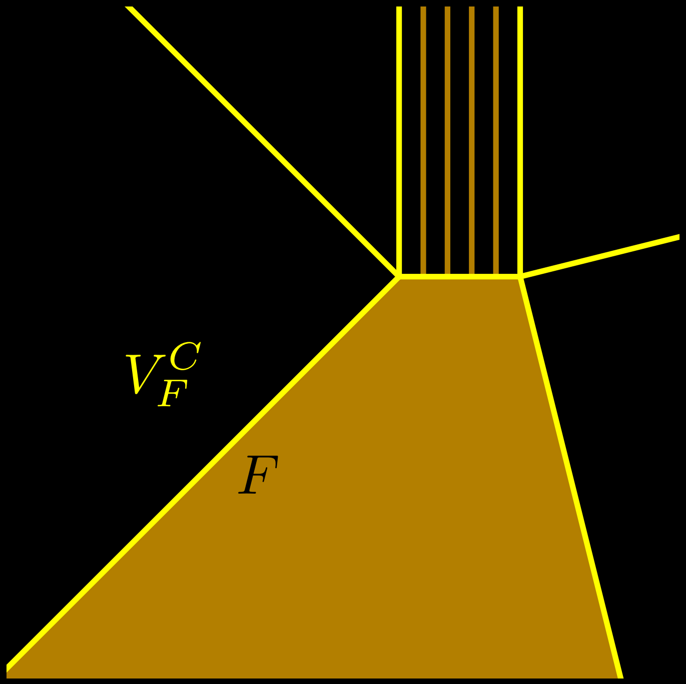
The formula for cones reduces to that for bounded convex sets by taking slices.



Above the slice, the two configurations are the same.

A local version

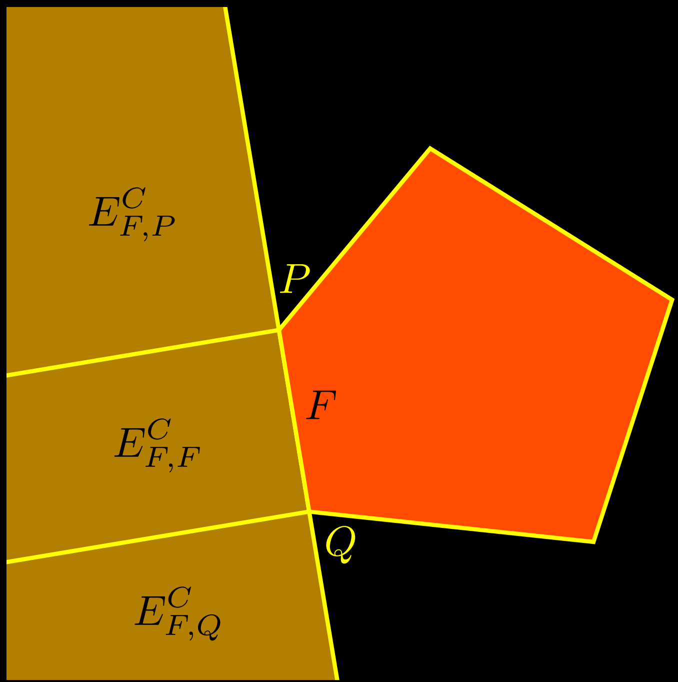
For each face F of C, let V_F^C be the set of points in V for which the point of C nearest to it lies in F°.



It possesses an obvious product structure $F^\circ \times T_F^C$.

For each couple of faces $F_* \preceq F$ let

$$E^C_{F,F_*} = E^C_F \cap V^F_{F_*} \ .$$



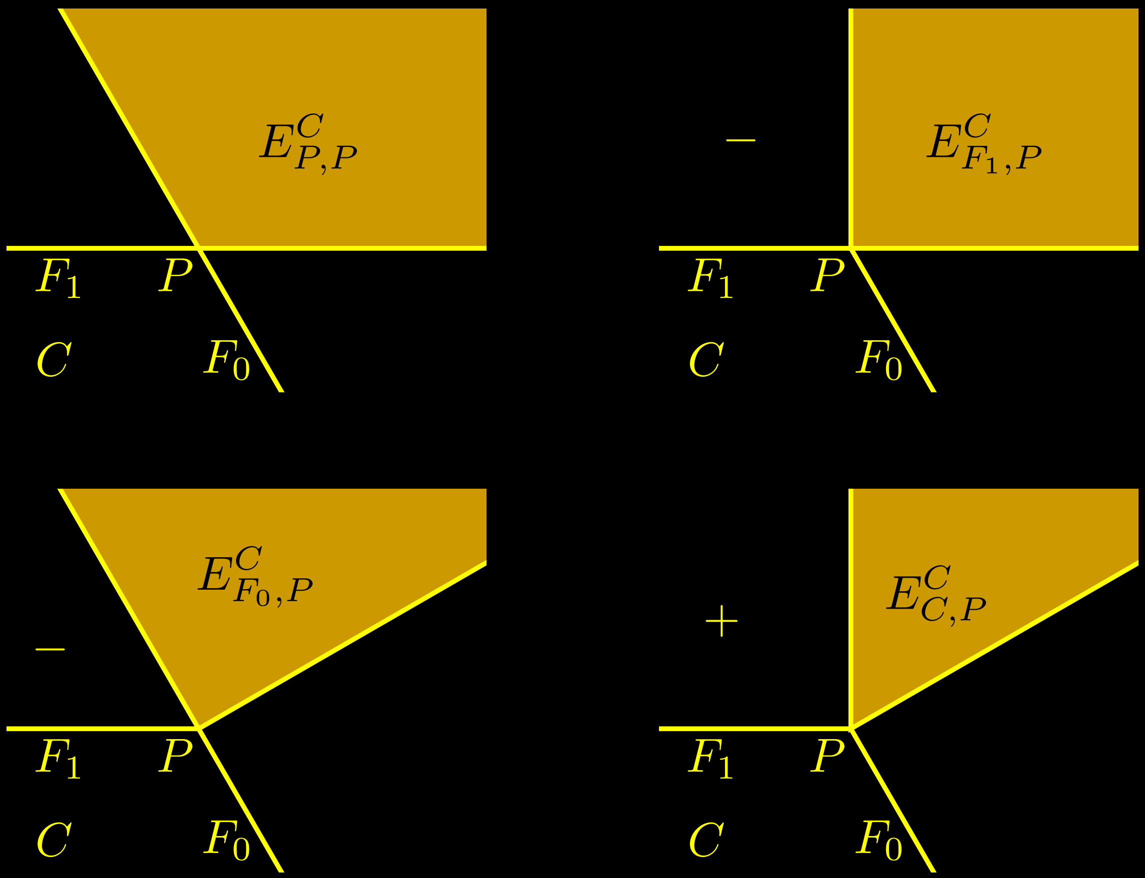
Thus a point v of E^C_F lies in E^C_{F,F_*} if and only if the point of F closest to it lies in F_*.

Theorem L. *For each face F_* of C*

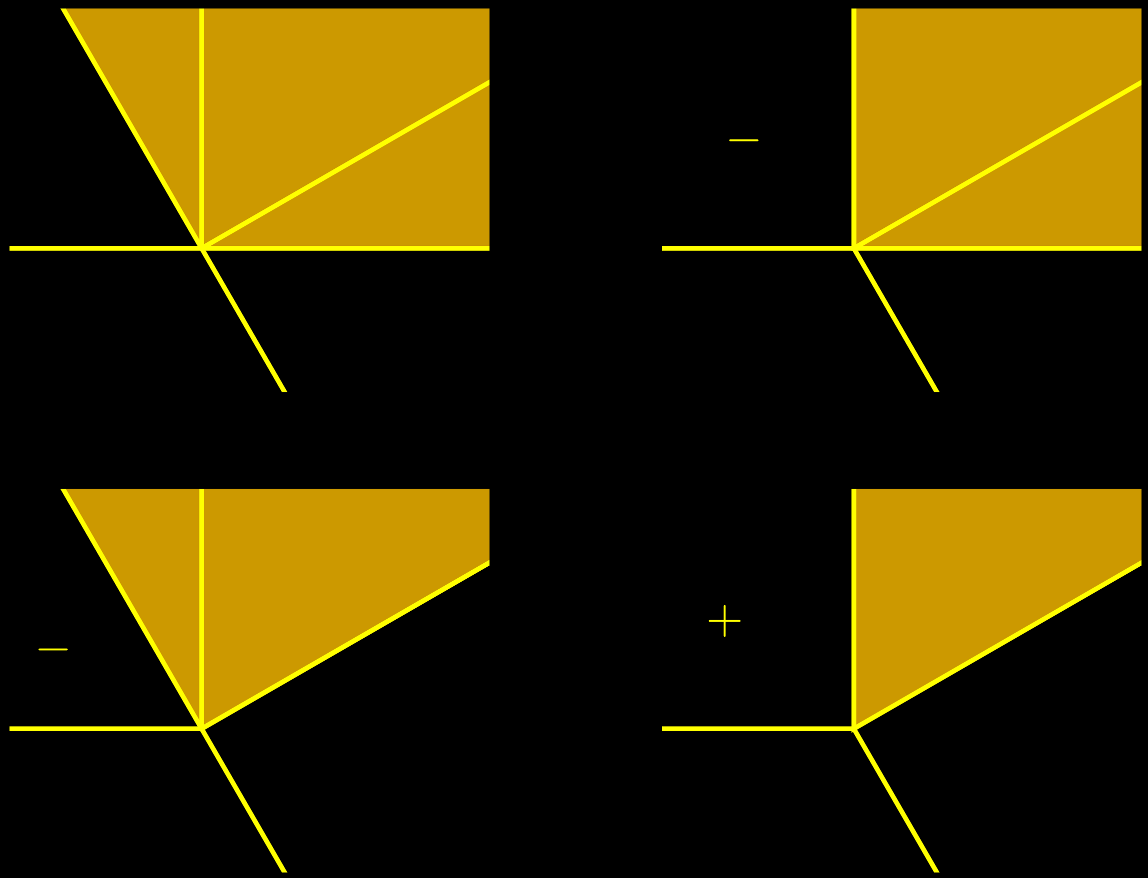
$$\sum_{F \,|\, F_* \preceq F} (-1)^{\operatorname{codim} F} \, \mathcal{E}^C_{F,F_*} = \begin{cases} 0 & F_* \neq C \\ \chi_C & F_* = C \end{cases}$$

This is one variation of Langlands' combinatorial lemma.

L in two dimensions is covered by these images . . .



. . . whose secret is given away by these:



C for cones implies L.

Theorem L asserts that

$$\sum_{F \mid F_* \preceq F} (-1)^{\operatorname{codim} F} \, \mathcal{E}^C_{F,F_*} = \begin{cases} 0 & F_* \neq C \\ \chi_C & F_* = C \end{cases}$$

In this, C may be replaced by its tangent cone at F_*. At any face but a vertex, the tangent cone at that face has a simple product structure, and induction proves the claim. The formula for the full cone can be rearranged to give it for the vertex.

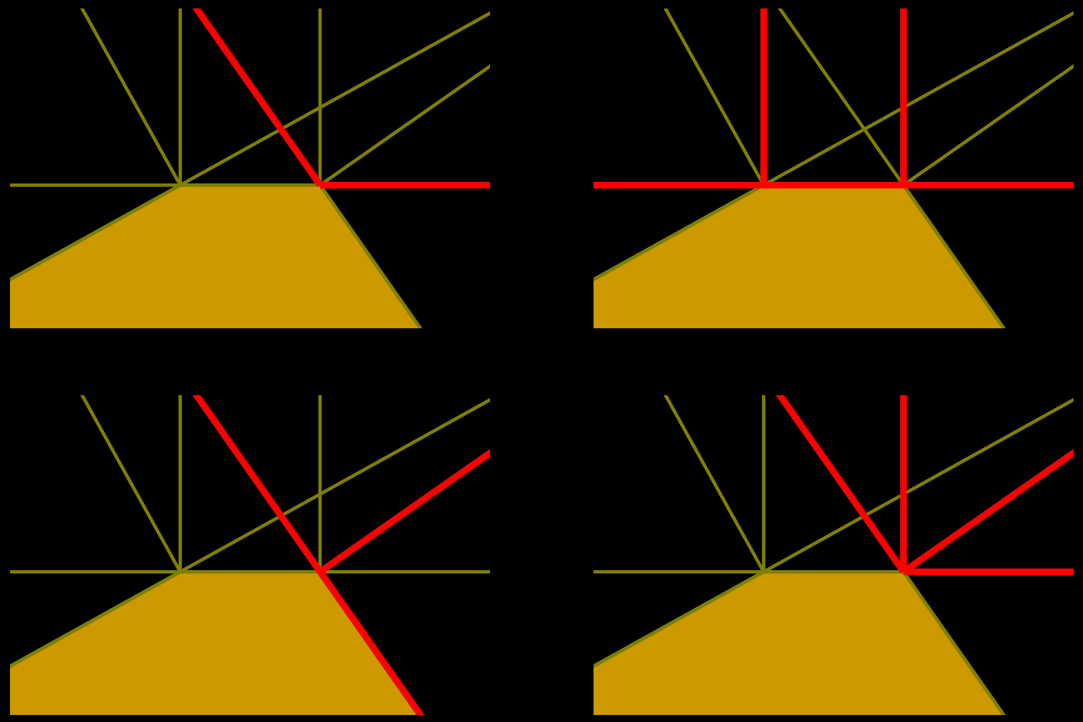
Conclusion of the proof of C

C follows from L by introducing the partition

$$\mathcal{E}_F^C = \sum_{F_* \preceq F} \mathcal{E}_{F,F_*}^C$$

and then rearranging the sum:

$$\begin{aligned}\sum_F (-1)^{\operatorname{codim} F}\, \mathcal{E}_F^C &= \sum_{F,F_* | F_* \preceq F} (-1)^{\operatorname{codim} F}\, \mathcal{E}_{F,F_*}^C \\ &= \sum_{F_*} \sum_{F | F_* \preceq F} (-1)^{\operatorname{codim} F}\, \mathcal{E}_{F,F_*}^C \\ &= \chi_C\ .\end{aligned}$$



Langlands' combinatorial lemma

If F is a face of C, let T_F^C be the translation of V_F^C by the support of F, and τ_F^C its characteristic function. Thus $V_F^C = T_F^C \times F^\circ$. When C is an obtuse simplicial cone the following result is the same as the original combinatorial lemma of Langlands.

Theorem. *For any face F of C*

$$\sum_{F \preceq F_* \preceq C} (-1)^{\mathrm{codim} F} \tau_{F_*}^C \mathcal{E}_F^{F_*} = \begin{cases} 1 & \textit{if } F = C \\ 0 & \textit{otherwise} \end{cases}$$

The case $F = C$ is trivial. The proof for other F uses the partition of V into the V_F^C, and goes by induction.

The original applied to simplicial cones and was announced by Langlands without proof in his 1965 Boulder talk on Eisenstein series, and a result apparently equivalent to this one is contained in the appendices to a recent paper by Goresky et al.

Fourier analysis

One curious application of the result of Brion and Vergne is a useful formula for the Fourier transform of the characteristic function of a bounded convex polyhdron C with $C^\circ \neq \emptyset$. This is the entire function of the complex variable s

$$\widehat{\chi}_C(s) = \int_C e^{-\langle s,x\rangle}\, dx\ .$$

and the formula asserts that

$$\widehat{\chi}_C(s) = (-1)^{\dim C} \sum_P \widehat{\mathcal{E}}_P^C(s)$$

where the right hand sum is over the vertices of C, and the expression is taken to be the analytic continuation of the obvious integral.

Applying the fundamental theorem again to the exterior cones, this can be rewritten

$$\widehat{\chi}_C(s) = \sum_P \widehat{\mathcal{I}}_P^C(s)$$

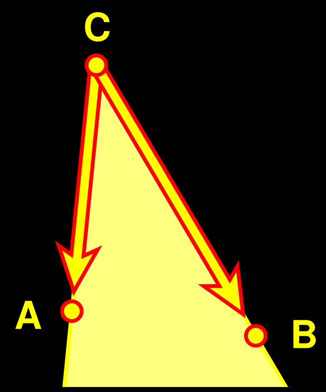
where $\mathcal{I}$ **is the characteristic function of the** *interiors* **(tangent cones) of the vertices.**

These integrals are easy to compute when the cones are simplicial, but as far as I know there is no simple formula otherwise.

The 'formal' Fourier transform (i.e. Laplace transform) of the cone with vertex C and whose edges pass through A and B is

$$\int \chi_C(x,y)\,dx\,dy = \det \textbf{Jacobian} \int_0^\infty \int_0^\infty e^{-\langle s,\eta\rangle}\,d\eta$$
$$= \frac{2 \cdot \textbf{triangle area} \cdot e^c}{(a-c)(b-c)},$$

where $a = \langle s, A\rangle$ etc.



For the case of a triangle with vertices A, B, C this is the product of twice the area A of the triangle with

$$\frac{e^c}{(a-c)(b-c)} + \frac{e^b}{(a-b)(c-b)} + \frac{e^a}{(b-a)(c-a)}\ ,$$

where $a = \langle s, A \rangle = s_x x_A + s_y y_A$, etc. Why in heck is this equal to the area at $s = 0$? Why is it even an entire function?

Part II. Arithmetic quotients

SL_2

To start

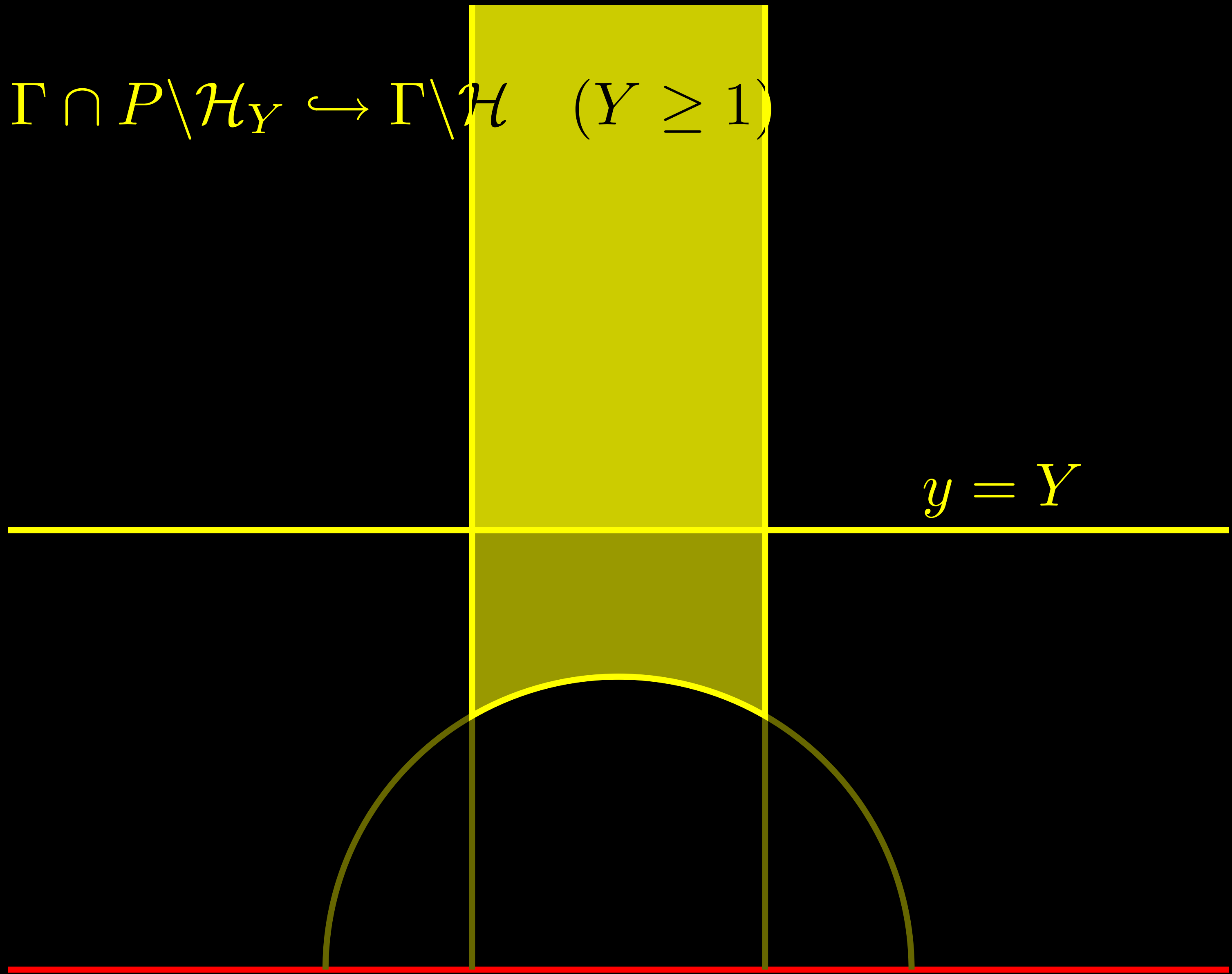
$\Gamma = \mathrm{SL}_2(\mathbb{Z})$

$\mathcal{H} =$ **upper half plane**

$P =$ **Borel subgroup of upper triangular matrices**

$N =$ **unipotent matrices in** P

$$\Gamma \cap P \backslash \mathcal{H}_Y \hookrightarrow \Gamma \backslash \mathcal{H} \quad (Y \geq 1)$$



Thus for $Y \geq 1$ the quotient $\Gamma \cap P \backslash \mathcal{H}_Y$ may be identified with a subset of $\Gamma \backslash \mathcal{H}$. We have the map

$$\Gamma \cap P \backslash \mathcal{H}_Y \longrightarrow (0, \infty) \colon \; z = x + iy \longmapsto y \; .$$

If χ_Y is the characteristic function of the region $y > Y$ truncation at Y is the operator

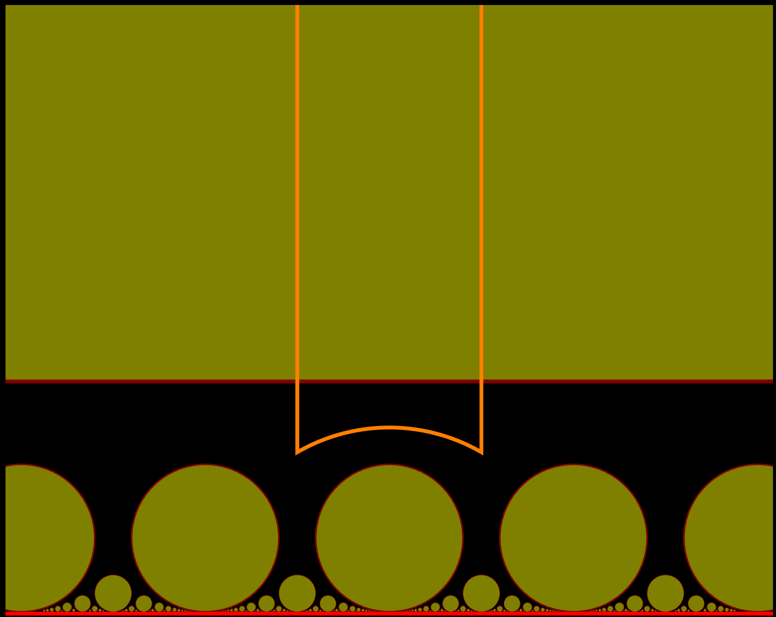
$$\Lambda^Y F = F - \chi_Y F_0$$

where

$$F_0(y) = \int_0^1 F(x + iy)\, dx$$

is the constant term of F.

The quotient $\Gamma\backslash\mathcal{H}$ may be compactified by adding a cusp at infinity, and truncation chops away the constant term of a function in the neighbourhood of the cusp.



As a Γ-invariant function on $\mathcal{H}$

$$\Lambda^Y F(z) = F(z) - \sum_{\Gamma\cap P\backslash\Gamma} \chi(y) F_0(y(\gamma z)) \,,$$

The most important property of Λ^Y is that under a mild growth condition on F its truncation $\Lambda^Y F$ is rapidly decreasing at ∞.

It is for this reason that truncation plays a role in the meromorphic continuation of Eisenstein series and in proving the Selberg trace formula.

In particular, if E_s is the Eisenstein series of Maass then $\Lambda^Y E_s$ is square-integrable, and the Maass-Selberg formula for $\|\Lambda^Y E_s\|^2$ is important in proving properties of E_s.

All of these features occur in using the truncation operator for groups of higher rank as well.

The Maass-Selberg formula:

$$\langle \Lambda^Y E_s, E_{-t} \rangle = \langle \Lambda^Y E_s, E_{-t} \rangle$$

$$= \int_0^Y (\textbf{const. term of } E_s)(\textbf{const. term of } E_{-t}) \frac{dy}{y^2}$$

$$= \int_0^Y \left(y^s + c(s) y^{1-s}\right)\left(y^{-t} + c(-t) y^{1+t}\right) \frac{dx\,dy}{y^2}$$

Groups of higher rank

Suppose now for simplicity that G is a split group over $\mathbb{Q}$, $\Gamma = G(\mathbb{Z})$, $X = G/K$. Under these conditions, all Borel subgroups are Γ-conjugate.

Fix one, call it $P_\emptyset$. Let Σ be the corresponding set of roots, Δ the basic roots.

For any rational parabolic subgroup P let N_P be its unipotent radical, $M_P = P/N_P$, A_P the connected component of the centre of M_P.

Given the compact subgroup K of G, for any parabolic subgroup of G there exists a unique copy of M_P in P stable under the Cartan involution determined by K.

Parabolic subgroups containing $P_\emptyset$ are parametrized by subsets $\Theta \subseteq \Delta$.

For $\Theta \subseteq \Xi$

$$A_\Xi \subseteq A_\Theta \subseteq M_\Theta \subseteq M_\Xi$$

Every rational parabolic subgroup is Γ-conjugate to exactly one of these.

For any rational parabolic subgroup P and F on $\Gamma\backslash X$ the constant term of F with respect to P is

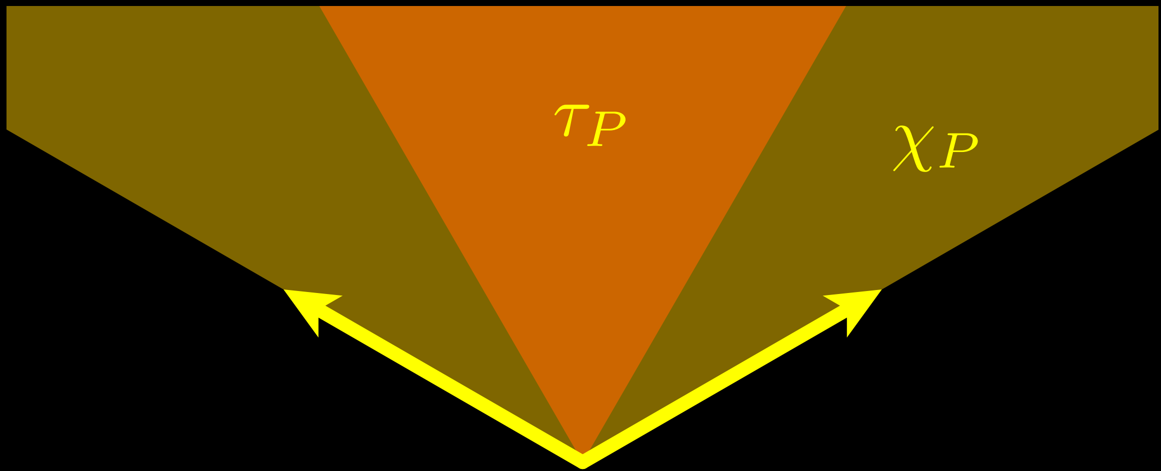
$$F_P(x) = \int_{\Gamma\cap N_P\backslash N_P} f(nx)\,dn\ ,$$

a function on $N_P(\Gamma \cap P)\backslash X$.

Conversely, for F on $N_P(\Gamma \cap P)\backslash X$ define (formally) the Eisenstein series

$$(E_P^G F)(x) = \sum_{\Gamma\cap P\backslash\Gamma} F(\gamma x)\ .$$

In A_P lie two naturally defined cones, one obtuse and one acute. Let χ_P be the characteristic function of the obtuse one, τ_P that of the acute one.



There is a canonical projection

$$N_P(\Gamma \cap P)\backslash X = N_P(\Gamma \cap P)\backslash P/K \cap P \longrightarrow A_P \ .$$

Let χ_P, τ_P be also their lifts back to X, $\chi_{P,p}$ and $\tau_{P,p}$ their shifts by p in P.

Fix T in the positive Weyl chamber in $A_\emptyset$ far away from the walls. Arthur's definition of truncation is this:

$$\Lambda_G^T F = \sum_P (-1)^{\dim A_P - \dim A_G} E_P^G(\chi_{P,T} \cdot F_P)$$

The sum evaluated at any given element is finite.

$$\Lambda_G^T F = \sum_P (-1)^{\dim A_P - \dim A_G} E_P^G(\chi_{P,T} \cdot F_P)$$

If you are familiar with the geometry of $\Gamma\backslash X$ you will likely find this definition puzzling, because there is no longer any obvious relationship between truncation and the geometry of a compactification of X.

For groups of rational rank greater than one, Arthur's truncation is not local on any Satake compactification.

Nonetheless, there is no doubt that Arthur's definition is the correct one. It is again true, but not so simple to prove, that under a mild growth condition on F the truncation $\Lambda^T F$ is rapidly decreasing at infinity.

Truncation is a projection operator, too.

It does not affect functions whose constant term support lies inside a well defined compact subset of $\Gamma\backslash X$. In particular it does not affect cusp forms.

Truncation is defined on every M_P as well as G itself. There is an equivalent recursive definition of truncation that defines it for G in terms of truncation on the other M_P.

Theorem. *We have an orthogonal decomposition*

$$F = \sum_P E_P^G(\tau_{P,T} \cdot \Lambda_{M_P}^T F_P) \; .$$

This is proven by means of a purely geometric lemma about obtuse simplicial cones, originally due to Langlands.

The rest of this talk will try to explain why the definition of truncation is reasonable, and why this theorem holds. *Without, however, proving either of them!*

A simple model for Eisenstein series

When trying to understand Arthur's calculations, I find it helpful to see what's going on in a much simpler situation, one where combinatorial difficulties are isolated from analytical ones.

Suppose given a split algebraic torus A and a root system Σ associated to it.

Let G be the algebraic group generated by the torus and the Weyl group W of Σ. In some sense, this is a reductive group in which the unipotent groups are reduced to shadows.

The parabolic subgroups in this scheme are parametrized by the faces of Weyl chambers.

Given a face F, the associated group of rational points is the subgroup generated by the torus and the subgroup W_P of W whose elements fix the points on the face.

The group Γ is just W.

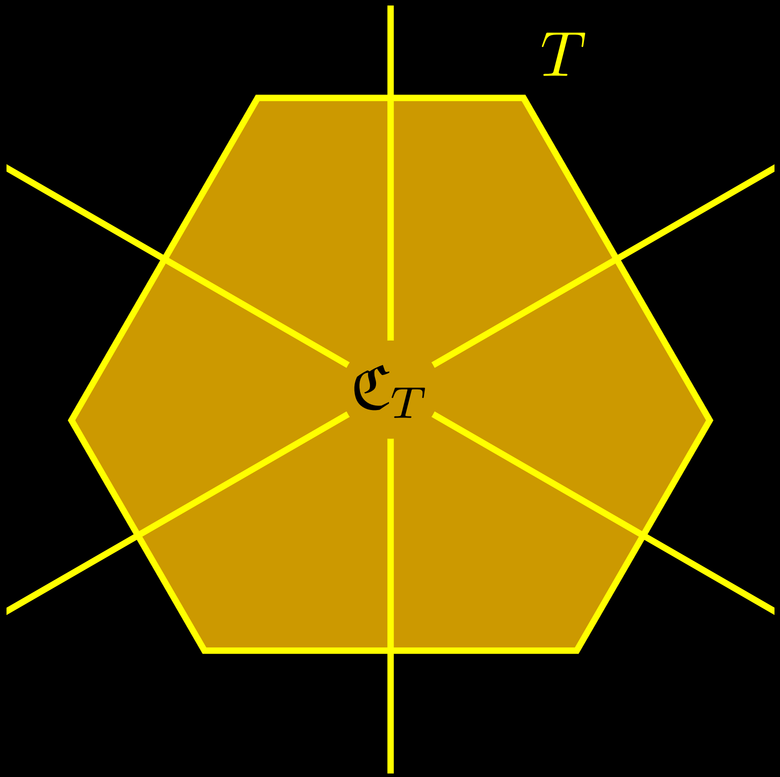
The space X may be identified with the quotient of A by its torsion subgroup, which I will identify with the vector space V in which the (co)roots live.

Automorphic functions are the characters of V that are W-invariant, and the analogue of an Eisenstein series is the finite sum

$$(E_P^G F)(v) = \sum_{W_P \backslash W} F(wv)$$

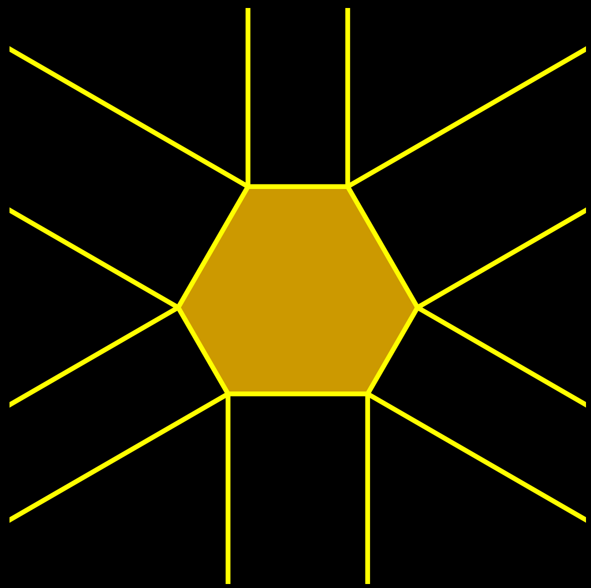
that maps a function in V^{W_P} to one in V^W.

The natural definition of truncation associated to a point T in V is multiplication of a function F by the characteristic function of the convex hull of T.

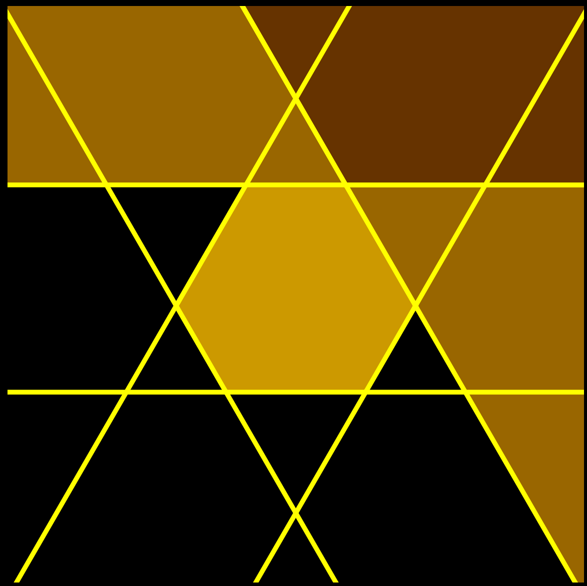


If T is non-singular, then the faces of its convex hull $\mathfrak{C}_T$ are parametrized by the ‘parabolic subgroups’.

The orthogonal decomposition is that corresponding to the partition of V **according to the** *nearest face of the convex set* $\mathfrak{C}_T$**.**



That this agrees with Arthur’s definition is not obvious.



The agreement of the two definitions is a actually a special case of a much more general result about convex polyhedra.

Back to Arthur's truncation

Arthur's truncation is, as I believe I have worked out this week, a generalization of this theory to the building of the rational group G.

References

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M. Brion and M. Vergne, Lattice points in simple polytopes, *Journal of the American Mathematical Society* 10 (1997), 371–392.

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(1997), 477–554. *Appendix B is the first place I am aware of where Langlands' combinatorial lemma is formulated for general cones.*

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`http://sunsite.ubc.ca/scans/lemma/cl.html`

———————, Eisenstein series, *Proceedings of Symposia in Pure Mathematics* **IX, 1965.** *This was the Boulder conference. The relevant section is picturesquely called '*L^2 *as the bed of Procrustes'.*