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INTRODUCTION TO THE SCHWARTZ SPACE OF $\Gamma\backslash G$

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Introduction. Let G be the group of $\mathbf{R}$-rational points on a reductive group defined over $\mathbf{Q}$ and Γ an arithmetic subgroup. The aim of this paper is to describe in some detail the Schwartz space $\mathcal{S}(\Gamma\backslash G)$ (whose definition I recall in Section 1) and in particular to explain a decomposition of this space into constituents parametrized by the Γ-associate classes of rational parabolic subgroups of G. This is analogous to the more elementary of the two well known decompositions of $L^2(\Gamma\backslash G)$ in **[20]** (or **[17]**), and a proof of something equivalent was first sketched by Langlands himself in correspondence with A. Borel in 1972. (Borel has given an account of this in **[8]**.) Langlands' letter was in response to a question posed by Borel concerning a decomposition of the cohomology of arithmetic groups, and the decomposition I obtain here was motivated by a similar question, which is dealt with at the end of the paper. The decomposition given here is not deep, in contrast to the L^2-decomposition of Langlands involving the residues of Eisenstein series. Nonetheless, on the one hand it does not seem to be in the literature, and on the other it will be required for subsequent and more important results characterizing functions in $\mathcal{S}(\Gamma\backslash G)$ by their integrals against automorphic forms (the Paley-Wiener theorems for $\Gamma\backslash G$), so that in spite of its elementary character I feel it is worthwhile to give a detailed exposition.

I include along the way a number of results in analysis on $\Gamma\backslash G$ which I have also been unable to find in the literature in exactly the form I need them, and in fact, I take the opportunity to provide a relatively self-contained introduction to the topic. In other words, I hope to make up for the lack of systematic exposition available in the literature by starting from scratch. In Section 1 I define the Schwartz space $\mathcal{S}(\Gamma\backslash G)$ of any quotient $\Gamma\backslash G$, where G is the group of $\mathbf{R}$-valued points on any affine algebraic group defined over $\mathbf{R}$ and Γ is an arbitrary discrete subgroup, prove a number of very basic properties, and introduce a few important ancillary notions. Most of these elementary results are modeled on the classical case $G = \mathbf{R}$. In Section 2 I restrict myself to the case where G is reductive and Γ is arithmetic, and prove just one substantial result: namely, that the analogue of the Poincaré series defines a surjective map

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from $\mathscr{S}(G)$ to $\mathscr{S}(\Gamma\backslash G)$. I must say that although this answers a natural question, it plays no further role in this paper. In Section 3 I introduce Eisenstein series as defining a map from certain representations induced from parabolic subgroups of G to $\mathscr{S}(\Gamma\backslash G)$. This generalizes to certain spaces of rapidly decreasing functions a well known construction concerning functions of compact support. I also make the observation that automorphic forms as defined classically (in **[4]** for example) are simply the $Z(\mathfrak{g})$-finite, K-finite distributions of $\Gamma\backslash G$ which are tempered in the sense that they extend to continuous functionals defined on the Schwartz space. This small point will turn out to be crucial in subsequent cohomological investigations. Further in Section 3 I define cuspidal Schwartz functions and prove that they make up a continuous summand of $\mathscr{S}(\Gamma\backslash G)$. In Section 4 I prove the main result. In Section 5, as mentioned already, I show how the parabolic decomposition of the Schwartz space implies a corresponding direct sum decomposition of the cohomology of Γ. This decomposition is almost the same as Langlands' earlier one, but the particular version I prove here will turn out to be important in more strongly relating the cohomology of Γ to automorphic forms.

It should be clear from this outline that there are indeed no deep results in this paper. Many arguments are only technical modifications of earlier ones. Prominent among previous expositions I have used extensively as a model are the lecture notes **[17]**, the first two chapters of which have much in common with this paper. But I hope it will be apparent also that the idea (due, I believe, to Godement) of beginning the theory of automorphic forms by looking at the Schwartz space of $\Gamma\backslash G$ (or the idea, more roughly put, of considering automorphic forms as tempered distributions) adds coherence to the subject.

This paper is the first one which I have produced from start to finish with computer word processing. Since my handwriting is notoriously difficult for secretaries to read, this has turned out to be especially satisfying. I wish to thank the National Science and Engineering Research Council of Canada for supporting this project financially, and also to express my gratitude to Donald Knuth for inventing incredible T_EX.

1. Growth conditions. *In this section, let G be the group of* **R**-*rational points on any affine algebraic group defined over* **R**.

There exists on G a distinguished class of norms, obtained in any one of several ways, which I will call *algebraic*. Let σ be an algebraic, finite-dimensional, complex representation of G with finite kernel. Assume further that $\sigma(G)$ is closed in the matrix ring of the representation space V. This is no serious restriction since if σ doesn't satisfy this condition then the direct sum of σ and the one-dimensional representation $\det^{-1}(\sigma)$ will. Define a norm on G by choosing a Banach norm on V and specifying

$$||g|| := \text{corresponding Banach norm of } \sigma(g).$$

Any two norms on V will determine norms on G which are equivalent in the sense that either is bounded by some multiple of the other. Fix from now on a maximal compact subgroup K of G. Then one can choose a Hilbert norm on V so that $\sigma(K)$ is unitary, and the norm on G will be right and left K-invariant, as I shall assume from now on.

One could also assign to G the norm inherited from any Banach norm on the vector space $\mathrm{End}_{\mathbf{C}}(V)$, for example by setting $||g|| = \sup|\sigma_{i,j}(g)|$. As the choices of norm and σ are now allowed to vary, all the norms one obtains will be equivalent in the weaker sense that any two will be bounded by a multiple of a power of the other. An *algebraic norm* is defined to be any norm on G equivalent, in this weak sense, to one of those obtained in one of these ways. In this paper, only the weak equivalence class of a norm will matter.

Any algebraic norm will satisfy the condition $||gh|| \leqq C||g||\,||h||$ with C independent of g and h. I will assume for convenience, as I may, that $C = 1$. Furthermore, since G is closed in $\mathrm{End}_{\mathbf{C}}(V)$ the norm on G will be bounded from below. By choosing σ suitably one may as well assume:

(1.1a) $||1|| = 1$

(1.1b) $||g|| \geqq 1$

(1.1c) $||g^{-1}|| = ||g||$

(1.1d) $||g||\,||h||^{-1} \leqq ||gh|| \leqq ||g||\,||h||$.

for all g and h, and stipulate as well that an algebraic norm be continuous.

Sets in G which are bounded with respect to an algebraic norm are relatively compact.

If G is an algebraic torus, any σ will be a sum of characters. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ form a basis of the rational characters of G, one may take as norm

$$||x|| = \sup_i(\,|\alpha_i(x)\,|,\ |\alpha_i(x)\,|^{-1}).$$

For example, if G is the one-dimensional split torus over $\mathbf{R}$, then

$$||x|| = \sup(\,|x|,\ |x|^{-1}).$$

Keep in mind that although G is isomorphic to $\{\pm 1\} \times \mathbf{R}$ this is an analytic but not algebraic isomorphism, since the logarithm is not an algebraic function.

If G is reductive then $G = KAK$ for a suitable maximal $\mathbf{R}$-split torus A. In this case an algebraic norm on G is determined by one on A which is invariant under the relative Weyl group.

Any affine function on G and in particular any matrix coefficient of a finite-dimensional algebraic representation of G will be bounded by some algebraic norm. If G is the direct product of two subgroups U and H then the product norm $\|uh\| = \|u\|\,\|h\|$ is a norm on G (arising from a tensor product of representations). If G contains closed algebraic subgroups U and H such that $G = U \times H$ as an algebraic variety then the product norm, although not obtained from a representation of G, will still be equivalent in the weak sense to one which is.

A function on a subset of G will be said to vanish *rapidly at infinity* if it is of order less than the inverse of any algebraic norm, or equivalently if it is of order less than any negative power of a fixed algebraic norm. In other words, the function f vanishes rapidly at infinity when the norm

$$\|f\|_{-n} := \sup_{g\in G} \|g\|^n |f(g)|$$

is finite for each positive integer n. Define the Schwartz space of G:

$$\mathscr{S}(G) := \{f \in C^\infty(G) |\text{ for all } X \in U(\mathfrak{g}),\ R_X f \text{ vanishes rapidly at infinity}\}.$$

In other words, a C^∞ function f on G lies in $\mathscr{S}(G)$ if and only if for every $X \in U(\mathfrak{g})$ and integer $n > 0$

$$\|f\|_{X,-n} := \sup_{g\in G} \|g\|^n |R_X f(g)| < \infty.$$

Using the formula

$$L_X f(g) = R_{\mathrm{Ad}(g^{-1})X} f(g) \quad (X \in U(\mathfrak{g}))$$

and remarking that the adjoint representation of G on $U_n(\mathfrak{g})$ is algebraic, one can see that the left derivatives of $f \in \mathscr{S}(G)$ also vanish rapidly at infinity. Therefore the group $G \times G$ acts on $\mathscr{S}(G)$ by means of the left and right regular representations.

1.1. PROPOSITION. *The semi-norms $\|f\|_{X,-n}$ make $\mathscr{S}(G)$ into a nuclear Fréchet space on which this representation of $G \times G$ is smooth.*

Proof. This is straightforward except, apparently, for nuclearity. Since this involves ideas quite different from those used elsewhere, and does not play an important role in this paper, I shall only sketch the proof. Recall that we have G embedded as a closed subspace in some matrix ring over $\mathbf{R}$, hence certainly in some real vector space M. Let $\overline{G}$ be the closure of G in the associated projective space $\mathbf{P}(M)$. Thus $\overline{G}$ is a compact but in general singular real algebraic manifold. According to a well known result of Hironaka, one can find a desingularization of $\overline{G}$; that is to say, a compact algebraic variety G^* and an algebraic map from G^* to $\overline{G}$ which is an isomorphism over G itself, which may hence be identified with a Zariski-open subset of G^*. I claim now that the Schwartz space of G may be identified with the space of all smooth functions on G^* which vanish of infinite order at each point of the complement of G in G^*. This follows without trouble from the fact that the vector fields on G associated to the differential operators R_X ($X \in U(\mathfrak{g})$) are meromorphic on G^* and linearly independent at every point of G, so that if x is any point in the complement of G then local partial derivatives may be expressed in terms of the R_X with coefficients holomorphic on G, meromorphic on G^*. The proposition follows now from the relatively elementary fact that the space of smooth functions on a compact manifold is nuclear, since any closed subspace of a nuclear subspace is again nuclear (according to [**24**, Proposition 50.1], for example).

Warning. This Schwartz space of G has also been used in [**25**]. It is not generally the same as the one defined in [**18**] since in the case of a reductive group the functions defined here are required to vanish much more rapidly than his. When G is the multiplicative group, for example, the space $\mathscr{S}(G)$ may be identified with the subspace of $\mathscr{S}(\mathbf{R})$ of all functions vanishing of infinite order at 0, while for Harish-Chandra the condition of rapid decrease means, in multiplicative coordinates, $O(\log^{-n}|x|)$ for all n. The Schwartz functions I define here vanish so rapidly that for reductive groups they are almost as good as those in $C_c^\infty(G)$. If $G = \mathbf{R}^\times$, for example, their Fourier transforms are entire. It is curious that although the terminology conflicts with Harish-Chandra's, the notation does not, since his Schwartz space is generally written as $\mathscr{C}(G)$.

1.2. LEMMA. *There exists $m > 0$ and a constant $C > 0$ such that the volume of the subset $\{ \|g\| \leqq t\}$ is at most Ct^m, for all $t > 0$.*

Proof. Suppose G to be the semi-direct product of U by H. Then $dg = du dh$ (all right-invariant measures), and as norm on G one may take the product of norms on U and H. Thus if the lemma is true for U and H it is true for G. Since it holds for the additive group $\mathbf{R}$ it is true for all unipotent groups. In characteristic 0, every algebraic group is the semi-direct product of its unipotent radical and a reductive group, so that it remains to prove the lemma only for reductive groups. In this case $G = KAK$ and one has [Helgason: X.1.17] the volume formula

$$dg = |\Delta(a)|\, dk_1 da dk_2$$

where $\Delta(a)$ is an affine function on A.

1.3. COROLLARY. *If m is chosen as in Lemma 1.2 then the function $\|g\|^{-n}$ is integrable for all $n > m + 1$.*

Proof. We have

$$\int_G \|g\|^{-n} dg = \sum_{k=1}^{\infty} \int_{k \leqq \|g\| < k+1} \|g\|^{-n} dg \leqq \sum k^{-n}(k + 1)^m.$$

1.4. PROPOSITION. *Integration over G is a continuous functional on $\mathcal{S}(G)$. More generally, the inclusion of $\mathcal{S}(G)$ in each $L^p(G)$ ($p \geqq 1$) is continuous.*

Proof. For $p = \infty$ the assertion is clear. Otherwise, if $1 \leqq p < \infty$ and $\|g\|^{-m}$ is integrable then for f in $\mathcal{S}(G)$

$$\left|\int_G |f(g)|^p dg\right| = \left|\int_G |f(g)|^p \|g\|^{mp} \|g\|^{-mp} dg\right|$$
$$\leqq \|f\|^p_{-m} \int_G \|g\|^{-mp} dg.$$

If (π, V) is any continuous representation of G on a Fréchet space V, I call it *of moderate growth* if for every semi-norm ρ on V there exists an integer n, a constant C, and another semi-norm ν such that

$$\|\pi(g)v\|_\rho \leqq C\|g\|^n \|v\|_\nu$$

for all $g \in G$, $v \in V$. Every algebraic finite-dimensional representation is of moderate growth, and it is straightforward to prove that $\mathcal{S}(G)$ is also a representation of moderate growth. It is also well known (see [**25**, 2.2], following [**26**, Example, p. 282]) that every continuous representation of G on a Banach space is of moderate growth. Now if Φ is a continuous linear functional on V then there exists a semi-norm ρ bounding Φ. In other words:

1.5. LEMMA. *If (π, V) is a representation of G of moderate growth and Φ a continuous linear functional on V, then for some semi-norm ρ and integer n*

$$|\Phi(\pi(g)v)| \leqq \|g\|^n \|v\|_\rho.$$

Suppose (π, V) to be any representation of G of moderate growth. For $f \in C_c^\infty(G)$ and a semi-norm ρ, we have from

$$\pi(f)v = \int_G f(g)\pi(g)v dg$$

the estimate

$$\|\pi(f)v\|_\rho \leqq \int_G |f(g)| \, \|\pi(g)v\|_\rho dg$$
$$\leqq (\text{constant}) \int_G |f(g)| \, \|g\|^n \|v\|_\nu dg$$
$$\leqq (\text{constant}) \, \|f\|_{-m} \|v\|_\nu \int_G \|g\|^{n-m} dg$$

for large m. Therefore the representation of $C_c^\infty(G)$ on V can be extended continuously to one of $\mathcal{S}(G)$, and the associated map of tensor products from $C_c^\infty(G) \otimes V$ to V extends continuously to one from $\mathcal{S}(G) \hat{\otimes} V$ to V.

From now on until the end of this section, let Γ be an arbitrary discrete subgroup of G. To a norm on G is associated one on the quotient $\Gamma\backslash G$:

$$\|g\|_{\Gamma\backslash G} := \inf_{\gamma\in\Gamma} \|\gamma g\|.$$

Sometimes I will write $\|g\|_G$ for $\|g\|$ to avoid confusion. Of course

(1.2a) $\|g\|_{\Gamma\backslash G} \leqq \|g\|_G$

and more generally

(1.2b) $\|gh\|_{\Gamma\backslash G} \leqq \|g\|_{\Gamma\backslash G}\|h\|_G$.

1.6. PROPOSITION. *The norm* $\|g\|_{\Gamma\backslash G}$ *is continuous.*

Proof. Let $g \in G$ be given. Let Ξ be the finite subset of Γ and $\epsilon > 0$ such that

$$\|g\|_{\Gamma\backslash G} = \|\xi g\|$$

for $\xi \in \Xi$ and

$$\|g\|_{\Gamma\backslash G} < \|\gamma g\|$$

for $\gamma \in \Gamma - \Xi$. Choose $\epsilon > 0$ so that

$$\|g\|_{\Gamma\backslash G}(1 + \epsilon) < \|\gamma g\|$$

for all $\gamma \in \Gamma - \Xi$. Since $\|g\|_G$ is continuous and $\|1\| = 1$ by (1.1a), for each $\eta > 0$ we may choose a neighborhood U of 1 so small that $\|u\| < (1 + \eta)$ in U. By (1.1d)

$$\|\gamma gu\| > (1 + \epsilon)(1 + \eta)^{-2}\|\xi gu\|$$

or

$$\|\gamma gu\| \geqq \|\xi gu\|$$

if $\gamma \in \Gamma - \Xi$, $\xi \in \Xi$, $u \in U$ and η is small enough. Thus

$$\|x\|_{\Gamma\backslash G} = \inf_{\xi\in\Xi} \|\xi x\| \quad \text{for } x \in gU,$$

and since Ξ is finite the proposition is proved.

1.7. PROPOSITION. *There exists an open set* Ω *in* G *such that*

(a) *The image of* Ω *covers* $\Gamma\backslash G$;

(b) *On* Ω, $\|g\|_{\Gamma\backslash G}$ *and* $\|g\|_G$ *are comparable.*

If $C > 0$ *is such that* $\|g\|_G \leqq C\|g\|_{\Gamma\backslash G}$ *on* Ω, *then whenever* $\|\gamma\omega\|_G \leqq R (\gamma \in \Gamma, \omega \in \Omega)$

$$\|\omega\|_G \leqq CR, \quad \|\gamma\|_G \leqq CR^2.$$

Proof. Let $x \in \Gamma\backslash G$ be given. Choose $g \in x$ (that is to say, so that $x = \Gamma g$) with $\|g\|_G = \|x\|_{\Gamma\backslash G}$. Given $\epsilon > 0$ there exists by Proposition 1.6 a neighborhood U of g such that

$$|\,\|u\|_{\Gamma\backslash G} - \|g\|_{\Gamma\backslash G}| < \epsilon/2, \quad |\,\|u\| - \|g\|\,| < \epsilon/2$$

for all $u \in U$. Consequently

$$| \, \|u\|_{\Gamma\backslash G} - \|u\|_G | < \epsilon$$

for all $u \in U$. Thus we have

$$\|u\|_{\Gamma\backslash G} \leqq \|u\|_G < \|u\|_{\Gamma\backslash G} + \epsilon \leqq 2\|u\|_{\Gamma\backslash G}$$

as long as $\epsilon \leqq 1$. Let Ω be the union of the sets U obtained in this way as x ranges over $\Gamma\backslash G$.

For the last claim, note that

$$\|\omega\|_G \leqq C\|\omega\|_{\Gamma\backslash G} = \|\gamma\omega\|_{\Gamma\backslash G} \leqq CR,$$

$$\|\gamma\|_G \leqq \|\gamma\omega\|_G\|\omega\|_G \leqq CR^2.$$

A domain Ω with the properties in Proposition 1.7 is a weak kind of fundamental domain for Γ. Weak, since in the case of $\Gamma = SL_2(\mathbf{Z})$, for example, one could take Γ to be the region

$$x^2/y \leqq \epsilon$$

for some $\epsilon > 0$.

Define the Schwartz space $\mathcal{S}(\Gamma\backslash G)$ to be that of all smooth functions f on $\Gamma\backslash G$ such that for every $X \in U(\mathfrak{g})$ and integer $n > 0$

$$\|f\|_{X,-n} := \sup_{g \in G} \|g\|^n_{\Gamma\backslash G} | R_X f(g) | < \infty.$$

A little more generally, define for every open set Ω in $\Gamma\backslash G$ the Schwartz space $\mathcal{S}(\Omega)$ to be that of all smooth functions f such that

$$\sup_{g \in \Omega} \|g\|^n_{\Gamma\backslash G} | R_X f(g) | < \infty.$$

If a finite set of Ω_i cover $\Gamma\backslash G$ then f lies in $\mathcal{S}(\Gamma\backslash G)$ if and only if the restriction of f to each Ω_i lies in $\mathcal{S}(\Omega_i)$.

1.8. Proposition. *Through the right regular representation of G the space $\mathcal{S}(\Gamma\backslash G)$ becomes a smooth Fréchet representation of G of moderate growth.*

This is clear. We shall see later important cases where $\mathcal{S}(\Gamma\backslash G)$ is nuclear, but I do not know whether this is always so.

The proof of Proposition 1.4 together with Proposition 1.7 gives:

1.9. Proposition. *The function $\|g\|^{-m}_{\Gamma\backslash G}$ is integrable on $\Gamma\backslash G$ whenever $\|g\|^{-m}$ is integrable on G, and integration defines a continuous linear functional on $\mathcal{S}(\Gamma\backslash G)$. For any $p \geqq 1$ the inclusion of $\mathcal{S}(\Gamma\backslash G)$ in $L^p(\Gamma\backslash G)$ is continuous.*

Since Γ might be $\{1\}$, this cannot be improved. But of course the larger Γ is the weaker a statement it becomes.

1.10. LEMMA. *The series* $\sum_{\gamma\in\Gamma}\|\gamma\|^{-m}$ *converges whenever* $\|g\|^{-m}$ *is integrable on G. In fact for such m there exists a constant* $C = C_m > 0$ *such that*

$$\sum \|\gamma g\|^{-m} \leqq C$$

for all $g \in G$.

Proof. Choose a relatively compact neighborhood of the identity U such that $U\gamma \cap U = \emptyset$ for $\gamma \neq 1 \in \Gamma$. Then for $u \in U$

$$\|u\gamma g\| \leqq C\|\gamma g\|$$

where $C = \sup_{u\in U}\|u\|$. Thus for $m > 0$

$$\begin{aligned}\sum \|\gamma g\|^{-m} &\leqq C^{-m}(\text{meas } U)^{-1} \sum_{\gamma\in\Gamma} \int_U \|u\gamma g\|^{-m} du \\ &= (\text{constant}) \sum_{\gamma\in\Gamma} \int_{U\gamma g} \|x\|^{-m} dx \\ &\leqq (\text{constant}) \int_G \|x\|^{-m} dx.\end{aligned}$$

It follows that for f in $\mathscr{S}(G)$ the series

$$\Theta(f|\Gamma) := \sum_{\gamma\in\Gamma} f(\gamma)$$

converges absolutely. Extend the definition of Θ to obtain according to Lemma 1.10 a bounded function on G:

$$\Theta(f|\Gamma)(g) := \Theta(R_g f|\Gamma) = \sum f(\gamma g).$$

I shall frequently drop the reference to Γ. This map commutes with right translations, hence right $U(\mathfrak{g})$-derivations. So $\Theta(f)$ is certainly smooth for all $f \in \mathscr{S}(G)$. Better:

1.11. PROPOSITION. *If f lies in* $\mathscr{S}(G)$ *then* $\Theta(f)$ *lies in* $\mathscr{S}(\Gamma\backslash G)$. *The map* Θ *from* $\mathscr{S}(G)$ *to* $\mathscr{S}(\Gamma\backslash G)$ *is continuous.*

Proof. By Lemma 1.10, if $\|g\|^{-m}$ is integrable on G then

$$\begin{aligned}\|g\|^n_{\Gamma\backslash G}|\Theta f(g)| &\leqq \sum \|g\|^n_{\Gamma\backslash G}|f(\gamma g)| \\ &\leqq \sum \|\gamma g\|^n_G |f(\gamma g)| \\ &\leqq \sum \|\gamma g\|^{n+m}_G |f(\gamma g)|\,\|\gamma g\|^{-m}_G \\ &\leqq \|f\|_{-(m+n)} \sum \|\gamma g\|^{-m}_G.\end{aligned}$$

Suppose now that Γ_1 and Γ_2 are two discrete subgroups of G. If the first is contained in the second, the proof of Lemma 1.11 may be modified slightly to prove:

1.12. Lemma. *If* $\|g\|^{-m}$ *is integrable on* $\Gamma_1\backslash G$ *then for any* $n > 0$ *the series*

$$F(g) = \sum_{\gamma\in\Gamma_1\backslash\Gamma_2} \|\gamma g\|_{\Gamma_1\backslash G}^{-(m+n)}$$

converges for all $g \in G$ *to a function on* $\Gamma_2\backslash G$ *satisfying the inequality*

$$|F(g)| \leqq C\|g\|^{m-n}$$

for some constant C.

Still assuming that $\Gamma_1 \subseteq \Gamma_2$, as a generalization of Proposition 1.11 we have:

1.13. Proposition. *For any* $f \in \mathscr{S}(\Gamma_1\backslash G)$ *the series*

$$\Theta(f|\Gamma_1, \Gamma_2)(g) := \sum_{\gamma\in\Gamma_1\backslash\Gamma_2} f(\gamma g)$$

converges to a function in $\mathscr{S}(\Gamma_2\backslash G)$. *The map* $\Theta(\Gamma_1, \Gamma_2)$ *is continuous from* $\mathscr{S}(\Gamma_1\backslash G)$ *to* $\mathscr{S}(\Gamma_2\backslash G)$.

If both Γ_1 and Γ_2 are contained in the discrete subgroup Γ, and $\Gamma_2 = \gamma\Gamma_1\gamma^{-1}$ for some $\gamma \in \Gamma$, then $f \mapsto L_\gamma f$ is an isomorphism of $\mathscr{S}(\Gamma_1\backslash G)$ with $\mathscr{S}(\Gamma_2\backslash G)$ fitting into this commutative diagram:

$$\begin{array}{ccc} \mathscr{S}(\Gamma_1\backslash G) & \xrightarrow{\ L_\gamma\ } & \mathscr{S}(\Gamma_2\backslash G) \\ \Theta(\Gamma_1, \Gamma)\searrow & & \swarrow\Theta(\Gamma_2, \Gamma) \\ & \mathscr{S}(\Gamma\backslash G) & \end{array} \tag{1.3}$$

Since the representation of G on $\mathscr{S}(\Gamma\backslash G)$ is of moderate growth, it becomes a module over $\mathscr{S}(G)$, as remarked earlier. In this case, more explicitly, we have for a compactly supported f

$$\begin{aligned} R_f F(g) &= \int_G F(gx)f(x)dx \\ &= \int_{\Gamma\backslash G} F(x) \sum_\Gamma f(g^{-1}\gamma x)dx \\ &= \int_{\Gamma\backslash G} F(x)K_f(g, x)dx \end{aligned}$$

where the kernel of the operator R_f is

$$\begin{aligned} K_f(g, x) &= \sum_\Gamma f(g^{-1}\gamma x) \\ &= \Theta(L_g f)(x). \end{aligned}$$

Here a left-invariant measure on G is used. This expansion actually makes sense for $f \in \mathscr{S}(G)$, and we have a good estimate of the kernel in this case as a function of g and x:

1.14. LEMMA. *If* $\|g\|^{-m}$ *is integrable on* G *then for some* $C > 0$ *and all* $n > 0$

$$\|y\|^n_{\Gamma\backslash G}|K_f(y, x)| \leqq C\|x\|^{m+n}\|f\|_{-(m+n)}.$$

The proof is the same calculation as that in the proof of Proposition 1.11.

Define

$$A(\Gamma\backslash G) := \text{the strong topological dual of } \mathscr{S}(G).$$

I call it the space of *tempered distributions* or *distributions of moderate growth* on $\Gamma\backslash G$. (Note that when $\Gamma = 1$ this terminology conflicts with Harish-Chandra's.) Define further

$A_{\mathrm{int}}(\Gamma\backslash G) :=$ the space of locally integrable functions F such that for some $m > 0$

$$\int_{\|g\| \leqq t} |F(g)|\,dg = O(t^m)$$

$A_{\mathrm{mg}}(\Gamma\backslash G) := \{f \in C^\infty(\Gamma\backslash G) \mid$ for all $X \in U(\mathfrak{g})$ there exists some positive integer $r > 0$ such that $\|f\|_{X,r}$ is finite$\}$

$A_{\mathrm{umg}}(\Gamma\backslash G) := \{f \in A_{\mathrm{mg}}(\Gamma\backslash G) \mid$ the constant r can be chosen independently of $X\}$.

One can define similarly sets $A_{\mathrm{int}}(U)$, $A_{\mathrm{mg}}(U)$ and $A_{\mathrm{umg}}(U)$ when U is any open set in $\Gamma\backslash G$. (I follow here terminology introduced by Borel. The functions in A_{mg} and A_{umg} are of moderate growth and uniform moderate growth respectively.) Incidentally, note that for $r > 0$ and a Γ-invariant function F the conditions

$$F(x) = O(\|x\|^r) \quad \text{and} \quad F(x) = O(\|x\|^r_{\Gamma\backslash G})$$

are equivalent.

The following result will not be necessary in this paper, but it will, I imagine, turn out occasionally to be convenient. The proof is a variant of a clever and classical argument found in **[19]** (and referred to in **[14**, p. 40], where it came to my attention).

1.15. PROPOSITION. *Any function in* $A_{\mathrm{mg}}(\Gamma\backslash G)$ *which is rapidly decreasing at infinity on* $\Gamma\backslash G$ *lies in* $\mathscr{S}(\Gamma\backslash G)$.

Proof. Let f be a rapidly decreasing function in $A_{\mathrm{mg}}(\Gamma\backslash G)$. The proof will continue by induction, once it has been shown that $Xf = O(\|g\|^{-m}_{\Gamma\backslash G})$ for $X \in U(\mathfrak{g})$ and $m > 0$. Let r be such that $X^2f = O(\|g\|^r)$ and choose $n > 2m + r$. Let C_0 and C_2 be such that for all $x \in G$

$$|f(x)| \leqq C_0 \|x\|^n_{\Gamma\backslash G}, \quad |(X^2 f)(x)| \leqq C_2 \|x\|^r_{\Gamma\backslash G}.$$

For every real number t set $x_t = x \exp(tX)$ and define the complex-valued function $F(t) = f(x_t)$. Thus $F'(t) = (Xf)(x_t)$ and $F''(t) = (X^2 f)(x_t)$. The mean-value theorem implies that

$$Xf(x) = F'(0) = \frac{1}{2t}(F(t) - F(-t)) + \frac{t}{2}(F''(\xi_2) - F''(\xi_1))$$

for some ξ_1, ξ_2 in the interval $(-t, t)$. Choose a relatively compact neighborhood U of the identity in G, and choose $\epsilon > 0$ such that $\exp(tX)$ lies in U for $t < \epsilon$. Let

$$K = \sup_{u \in U} \|u\|.$$

Then for all $t < \epsilon$

$$|F(t)| = |f(x_t)| \leqq C_0 K^n \|x\|^{-n}_{\Gamma\backslash G}, \quad |F''(t)| \leqq C_2 K^r \|x\|_{\Gamma\backslash G},$$

hence

$$|(Xf)(x)| \leqq \frac{C_0 K^n}{t} \|x\|^{-n}_{\Gamma\backslash G} + \frac{1}{2} C_2 K^r \|x\|^r_{\Gamma\backslash G}$$

$$= \frac{A}{t} + Bt$$

if

$$A = C_0 K^n \|x\|^{-n}_{\Gamma\backslash G}, \quad B = \frac{1}{2} C_2 K^r \|x\|^r_{\Gamma\backslash G}.$$

The function $A/t + Bt$ achieves its minimum value

$$\sqrt{2AB} = K_1 \|x\|^{(r-n)/2}_{\Gamma\backslash G}$$

when

$$(1.4) \qquad t = \sqrt{A/B} = K_2 \|x\|^{-(n+r)/2}_{\Gamma\backslash G}.$$

Therefore for

$$\|x\|_{\Gamma\backslash G} \geqq (K_2/\epsilon)^{2/(n+r)}$$

we have

$$|(Xf)(x)| \leqq K_1 \|x\|^{-m}_{\Gamma\backslash G}.$$

If F lies in $A_{\text{int}}(\Gamma\backslash G)$ then the proof of Proposition 1.4 shows that for some $n > 0$ the function $F(g)\|g\|^{-n}$ is integrable. More explicitly if for all $R > 0$

$$\int_{\|g\| \leqq R} |F(g)| dg \leqq C_F R^m$$

then

$$\int_G |F(g)|\,\|g\|^{-n}dg \leqq C_F \sum k^{m-n}$$

and is bounded if $n > m + 1$. Hence for $F \in A_{\text{int}}(\Gamma\backslash G)$ and $f \in \mathscr{S}(\Gamma\backslash G)$ the product Ff is integrable, and one obtains thus an embedding of $A_{\text{int}}(\Gamma\backslash G)$ in $A(\Gamma\backslash G)$ (which depends of course only up to a positive scalar on the choice of right-invariant Haar measure on G). Naturally $A_{\text{int}}(\Gamma\backslash G)$ contains in it the spaces $A_{\text{mg}}(\Gamma\backslash G)$ and $A_{\text{umg}}(\Gamma\backslash G)$, as well as all the $L^p(\Gamma\backslash G)$.

1.16. THEOREM. *The canonical image of $A_{\text{umg}}(\Gamma\backslash G)$ is the Gårding subspace of $A(\Gamma\backslash G)$.*

Proof. The deeper but easier half is that $A_{\text{umg}} = A_{\text{umg}}(\Gamma\backslash G)$ lies in the Gårding subspace. Something slightly stronger is even true: for any $F \in A_{\text{umg}}$ there exist $F_i \in A_{\text{umg}}$ and $\varphi_i \in C_c^\infty(G)$ such that $F = \sum R(\varphi_i)F_i$. Since A_{umg} is the union of the spaces $A_{\text{umg},n}$ on which $\|f\|_{X,n} < \infty$ for all $X \in U(\mathfrak{g})$, it will suffice to prove this claim for each of these. However, these semi-norms make $A_{\text{umg},n}$ into a Fréchet space on which G acts smoothly. Hence what we want is implied by the main result (Théorème 3.3) of **[13]**.

For the other half, it must be shown that if F lies in $A(\Gamma\backslash G)$ and f in $C_c^\infty(G)$ then R_fF lies in $A_{\text{umg}}(\Gamma\backslash G)$. Certainly R_fF is a smooth function on $\Gamma\backslash G$ since it may be considered as the convolution on G of a smooth function with a distribution. Formally,

$$(1.5a)\quad R_fF(g) = \int_G F(gx)f(x)dx,$$

which since F is Γ-invariant is to be interpreted as

$$(1.5b)\quad \int_{\Gamma\backslash G} F(x)\sum_{\Gamma} f(g^{-1}\gamma x)dx = \langle F, \Theta(L_gf)\rangle.$$

Since F is continuous on $\mathscr{S}(\Gamma\backslash G)$ and the map $\Theta: \mathscr{S}(G) \to \mathscr{S}(\Gamma\backslash G)$ is also continuous according to Lemma 1.5 there exists $n > 0$ and for each f a constant C_f such that

$$|\langle F, \Theta(L_gf)\rangle| \leqq C_f\|g\|^n.$$

It holds equally, with the same n, for all derivatives of f, hence gives what we want.

The same argument implies that any F in $L^{p,\infty}(\Gamma\backslash G)$ lies in $A_{\text{umg}}(\Gamma\backslash G)$. An improved argument leads to:

1.17. PROPOSITION. *For each $p \geqq 1$ the natural inclusion of $L^p(\Gamma\backslash G)$ in $A(\Gamma\backslash G)$ is continuous. For each $p \geqq 1$ there exists a single positive n and a semi-norm ρ on $L^{p,\infty}$ such that $\|F\|_n \leqq \|F\|_\rho$, and in particular $\|F\|_n < \infty$, for all $F \in L^{p,\infty}(\Gamma\backslash G)$.*

Proof. The first assertion follows from Hölder's inequality together with Proposition 1.9 applied to the inclusion of $\mathcal{S}(G)$ in $L^q(\Gamma\backslash G)$, where $1/p + 1/q = 1$. It follows equally from the remarks above about $A_{\mathrm{int}}(\Gamma\backslash G)$, but that amounts to the same. Now I will show that there exists a positive integer n with $\|F\|_n < \infty$ for all $F \in L^{p,\infty}(\Gamma\backslash G)$. By **[13]** again it suffices to look at elements in $L^{p,\infty}(\Gamma\backslash G)$ of the form $R_f F$ with $F \in L^p(\Gamma\backslash G)$, $f \in C_c^\infty(G)$. Use again Equation (1.5):

$$\begin{aligned} |R_f F(g)| &= |\langle F, \Theta(L_g f)\rangle| \\ &\leqq |||F|||_p |||\Theta(L_g f)|||_q \\ &= |||F|||_p |||L_g \Theta(f)|||_q \\ &\leqq |||F|||_p \|L_g \Theta(f)\|_{-n} \\ &\leqq |||F|||_p \|g\|^n \|f\|_{-n} \end{aligned}$$

if n is chosen so that for all $\varphi \in \mathcal{S}(G)$

$$|||\Theta(\varphi)|||_q \leqq (\text{constant})\|\varphi\|_{-n}.$$

Here $|||F|||_p$ is the usual L^p-norm. It is possible to choose such an n by Propositions 1.9 and 1.11.

To get the last assertion: since the injection of $L^{p,\infty}(\Gamma\backslash G)$ into $A(\Gamma\backslash G)$ is continuous, and factors through some fixed $A_{\mathrm{umg},n}$ by what has just been proven, it is implied by a variant of the Closed Graph Theorem **[24**, Proposition 17.2, Corollary 4 on p. 173].

2. Arithmetic subgroups of reductive groups. From now on in this paper I take

$G :=$ **R**-rational points on a Zariski-connected, reductive, **Q**-rational group

$\Gamma :=$ an arithmetic subgroup.

If P is an arbitrary **Q**-rational parabolic subgroup of G let

$N_P :=$ the unipotent radical of P
$M_P :=$ the reductive component P/N_P
$A_P :=$ the (topological) connected component of the **Q**-split centre of M_P
$\Sigma_P :=$ the roots of the adjoint action of A_P on $\mathfrak{n}_P$
$\Gamma(P) := \Gamma \cap P$
$\Gamma(N_P) := \Gamma \cap N_P$
$\Gamma(M_P) :=$ the image of $\Gamma(P)$ modulo N_P
$\delta_P :=$ the rational modulus character of $P: p \mapsto |\det \mathrm{Ad}_{\mathfrak{n}}(p)|$.

I shall drop subscripts when advisable. Thus $\Gamma(N)$ is an arithmetic subgroup of N and the quotient $\Gamma(N)\backslash N$ is compact. The group $\Gamma(M)$ is an arithmetic subgroup of M. For each $t > 0$ let

$$A^{++}(t) := \{a \in A \mid |\alpha(a)| > t \text{ for all } \alpha \in \Sigma_P\}.$$

If P is minimal then the quotient of M/A by $\Gamma(M)$ is compact as well. Reduction theory **[2]** says that as P ranges over a set of representatives of the Γ-conjugacy classes of minimal $\mathbf{Q}$-rational parabolic subgroups of G (which is a finite set) then (a) if t is small enough and (b) Ω_P is for each P a relatively compact subset of P covering $\Gamma(P)\backslash P/A$, then the quotient $\Gamma\backslash G$ is covered by the images of the Siegel sets

$$\mathfrak{S}(\Omega_P, t) := \Omega_P A_P^{++}(t)K.$$

2.1. Lemma. *If P is a minimal $\mathbf{Q}$-rational parabolic subgroup of G then on any Siegel set $\mathfrak{S}(\Omega_P, t)$ the norms $\|g\|_G$ and $\|g\|_{\Gamma\backslash G}$ are equivalent.*

This is a weak version of a more substantial result in **[5]** which in turn generalizes to arbitrary semi-simple groups a well known result of C. L. Siegel concerning SL_n. This stronger theorem asserts that the two norms are in fact comparable (i.e., each bounded by a multiple of the other, rather than a multiple of a power) on Siegel sets, and is more difficult to prove. The argument I present below was suggested by the demonstration of Lemma 5.1 in **[1]**.

For the proof, let P be any minimal rational parabolic subgroup. If π is the irreducible finite-dimensional representation of G with highest weight δ_P, and v is a weight vector, then since Γ preserves a lattice containing v, one may choose a norm on V such that for any $\gamma \in \Gamma$

$$\|\pi(\gamma^{-1})v\| \geqq 1.$$

However, one may also choose $C, n > 0$ such that for any $x \in G$ and $\gamma \in \Gamma$

$$\begin{aligned}\|\pi(\gamma^{-1})v\| &= \|\pi(x)\pi(\gamma x)^{-1}v\| \\ &\leqq C\|x\|^n\|\pi(\gamma x)^{-1}v\| \\ &= C\|x\|^n\delta_P(p(\gamma x))^{-1}\|v\|\end{aligned}$$

(assuming $\|v\| = 1$ and the norm K-invariant) where for any $g \in G$, $p(g)$ is an element of P (determined modulo $K \cap P$) such that $p(g)^{-1}g \in K$. But then for any $x \in G$, $\gamma \in \Gamma$, and for some new constant C

$$|\delta_P(p(\gamma x))| \leqq C\|x\|_G^n$$

which in turn implies that

$$(2.1) \qquad |\delta_P(p(\gamma x))| \leqq C\|x\|_{\Gamma\backslash G}^n.$$

Of course any rational character of G is also bounded by an algebraic norm. This and Equation (2.1) together imply that on any Siegel set for a minimal rational parabolic subgroup that for some C

$$\|x\|_G \leqq C\|x\|^n_{\Gamma\backslash G},$$

which proves the lemma.

2.2. THEOREM. *Let P be a* **Q**-*rational parabolic subgroup of G. Then*

$$\Theta: \mathscr{S}(G) \to \mathscr{S}(\Gamma(P)\backslash G)$$

is surjective, and even possesses a continuous linear splitting.

The proof is rather long. To get some idea of what is involved, consider the case when G is anisotropic, hence the quotient $\Gamma\backslash G$ is compact and $P = G$. Cover $\Gamma\backslash G$ by a finite number of relatively compact subsets $\{U_i\}$ over which $G \to \Gamma\backslash G$ has a splitting, and let the subsets $\{\Omega_i\}$ of G be their images under some splitting. Let $\{\psi_i\}$ be a partition of unity subordinate to the given covering $\{U_i\}$. In this case the space $\mathscr{S}(\Gamma\backslash G)$ is the same as $C^\infty(\Gamma\backslash G)$, so for any f in $\mathscr{S}(\Gamma\backslash G)$ we may write $f = \sum \psi_i f$. Each $\psi_i f$ will lift to a unique function f_i with support in Ω_i, and the map $f \to \sum f_i$ is the splitting required.

This argument suggests looking at a compactification of $\Gamma\backslash G$ in order to interpret the conditions of rapid decrease at infinity as local conditions on the compactification. Suppose for a while that G is semi-simple. Let $\mathfrak{X}$ be the associated symmetric space, which may be identified with G/K. The norms $\|g\|_G$ and $\|g\|_{\Gamma\backslash G}$ may be considered functions on $\mathfrak{X}$ and $V = \Gamma\backslash\mathfrak{X}$.

The group Γ is called *neat* if $\Gamma(P)$ has no torsion for any rational parabolic subgroup P. Inside every Γ there always exists a normal subgroup of finite index which is neat [**4**, Section 17]. If Γ is neat then the compactification V^* of V constructed in [**10**] is a manifold with corners whose interior is V. Recall that a smooth function on a manifold with corners is one which extends locally to a smooth function in the neighborhood of every corner. Note that a smooth function on V^* determines by lifting a right K-invariant smooth function on G.

2.3. LEMMA. *Assuming that Γ is neat*:

(a) *If f lies in $C^\infty(V^*)$ then its restriction to V lies in $A_{\mathrm{mg}}(\Gamma\backslash G)$;*

(b) *Any point of V^* possesses a neighborhood U such that $U \cap V$ lifts to a subset* of $\mathfrak{X}$ *on which $\|g\|_G$ and $\|g\|_{\Gamma\backslash G}$ are comparable.*

The second property will turn out to be just a translation of Lemma 2.1 into new terminology, but the first requires some work. (It was apparently first observed by Borel.)

The Borel-Serre compactification V^* is the union of V with smaller boundary submanifolds V_P parametrized by Γ-conjugacy classes of rational parabolic subgroups P. Given P and K, according to [**10**, 1.1.9] there exists a unique copy of M in P, which I will still call M, stable under the Cartan involution of G associated to K. The group M contains $K \cap P = K(P)$. If

$$P(1) := \cap_{P \subseteq Q} \operatorname{Ker}(\delta_Q)$$

$$M(1) := M \cap P(1),$$

then P is the semi-direct product of $P(1)$ and A.

Continuing the same type of abusive notation, let A be the corresponding copy of A in P. If Δ is for the moment the basis of the set of roots of A acting on $N = N_P$, then A may be identified with the positive quadrant of $\mathbf{R}^{\Delta}: a \mapsto (\alpha^{-1}(a))$ $(\alpha \in \Delta)$. The closure of A in $\mathbf{R}^{\Delta}$ will be the closed positive quadrant, and for $t > 0$ the closure $\overline{A^{++}(t)}$ of $A^{++}(t)$ will be a neighborhood of the origin in this closed quadrant.

The set V_P may be identified with $\Gamma(P)\backslash P/K(P)A$, which may be identified in turn with $\Gamma(P)\backslash P(1)/K(P)$. Let η be the canonical projection from $\Gamma(P)\backslash P/K(P)$ to V_P, ϵ that from $\Gamma(P)\backslash P/K(P)$ to A. Given a set U in V_P, define

$$U^{++}(t) := \eta^{-1}(U) \cap \epsilon^{-1}(A^{++}(t))$$

which is isomorphic to the product of U and $A^{++}(t)$. If U is relatively compact, this will inject into V for $t \gg 0$. Adjoining points at infinity one obtains a neighborhood of U in V^*, isomorphic to $U \times \overline{A^{++}(t)}$. A smooth function on this is one obtained by restriction from $U \times \mathbf{R}^{\Delta}$.

Property (b) in Lemma 2.3 is now clear, since any such neighborhood is contained in some Siegel set.

To prove property (a), the derivatives in local coordinates on $U^{++}(t)$ and the operators $R_X (X \in U(\mathfrak{g}))$ must be compared. Choose U so small that (a) the projection

$$P/K(P)A \rightarrow \Gamma(P)\backslash P/K(P)A$$

splits over a neighborhood of the closure of U, and (b) that on this enlarged neighborhood one has a coordinate system (z_i). On A, let (y_i) be the coordinate system (α^{-1}) $(\alpha \in \Delta)$. Then on $U^{++}(t)$ we have the coordinate system (z, y). The smooth functions on $U^{++}(t)$ are smooth functions f on $U^{++}(t)$ with the property that f and all its derivatives in the (z, y) coordinates are bounded.

Choose now a parabolic subgroup P_0 of G contained in P and minimal of $\mathbf{R}$. Let

$$Q = N_0 A_0 \cap P(1).$$

Then G is the direct product QAK and $P(1)/K(P) \cong Q$. Therefore a basis of $\mathfrak{q}$ corresponds to linearly independent vector fields on $P(1)/K(P)$, hence in particular on U, spanning the tangent space at each point. Instead of using partial derivatives in the z-coordinates we may use the differential operators ∂_Z ($Z \in U(\mathfrak{q})$) arising from the left action of Q on $P(1)/K(P)$. Instead of the vector fields $\partial/\partial y_i$ we shall use the operators ∂_Y ($Y \in U(\mathfrak{a})$) arising from the right action of A on $P(1)/K(P)$ (among which are the $y_i\partial/\partial y_i$). More explicitly:

$$\partial_Z f(qak) = \text{left derivative with respect to } Z \text{ of } q \mapsto f(qak)$$

$$\partial_Y f(qak) = \text{right (or left) derivative with respect to } Y \text{ of } a \mapsto f(qak)$$

Note that each ∂_Z will commute with each ∂_Y, and that both commute with the right action of K.

2.4. LEMMA. *For any $X \in U(\mathfrak{g})$, the operator R_X may be expressed as a sum of products*

$$f(\kappa, Z, Y)R_\kappa \partial_Z \partial_Y$$

($\kappa \in U(\mathfrak{k})$, $Z \in U(\mathfrak{q})$, $Y \in U(\mathfrak{a})$) *with affine functions $f(\kappa, Z, Y)$ on G as coefficients. Conversely, any product of these operators can be expressed as a linear combination of the R_X with affine functions for coefficients.*

This follows from a double application of:

2.5. LEMMA. *Let B be any Lie group isomorphic as a manifold with the product CD of subgroups C and D. Then any R_X ($X \in U(\mathfrak{b})$) may be expressed as a linear combination of products of operators $L_Y R_Z$($Y \in U(\mathfrak{c})$, $Z \in U(\mathfrak{d})$ with affine functions on B as coefficients, and conversely.*

Proof. Let $X \in U(\mathfrak{b})$ be given. By Poincaré-Birkhoff-Witt X may be written as a sum of products YZ, with $Y \in U(\mathfrak{c})$, $Z \in U(\mathfrak{d})$, so we may as well assume $X \in U(\mathfrak{c})$, and we may assume given a basis $\{X_i\}$ of $U(\mathfrak{b})$ of this form. For any $c \in C$ we have

$$\mathrm{Ad}(c)X = \sum f_i(c)X_i$$

where the f_i are affine functions. In other words,

$$X = \sum f_i(c)\,\mathrm{Ad}(c)^{-1}\,X_i. \tag{2.2}$$

On the other hand we also have for any $d \in D$

$$X = \sum \varphi_i(d)\,\mathrm{Ad}(d)^{-1}\,X_i \tag{2.3}$$

where the φ_i are affine functions on D. If $X_i = Y_i Z_i$ then

$$\mathrm{Ad}(d)^{-1}(X_i) = \mathrm{Ad}(d)^{-1}(Y_i)\,\mathrm{Ad}(d)^{-1}(Z_i).$$

The second factor lies again in $U(\mathfrak{d})$, while to the first we may apply Equation (2.2) to write X as a sum of terms of the form $f(c)\varphi(d)(\mathrm{Ad}(cd)^{-1}Y)Z$. But

$$R_{\mathrm{Ad}(cd)^{-1}Y}R_Z f(cd) = L_Y R_Z f(cd).$$

This proves the first half of the lemma, and the other half is similar. This concludes the proofs of both Lemma 2.5 and Lemma 2.4.

It follows immediately from Lemma 2.4 that if f is a smooth function on $U^{++}(t)$ all of whose derivatives in the coordinate system are of moderate growth, then so are all the $R_X f$ $(X \in U(\mathfrak{g}))$. This concludes the proof of Lemma 2.3.

Theorem 2.2 in turn follows from Lemma 2.3 when G is semi-simple and $P = G$, or even when G is the direct product of a semi-simple group with a compact factor. One can simply follow the proof I gave above for $\Gamma\backslash G$ compact, but choosing the partition of unity to be by functions smooth on the Borel-Serre compactification. Note that we may as well assume Γ is neat. If G is reductive but not necessarily semi-simple, then one may write $G = G_0 \times A_G$ where G_0 is as above and A_G is isomorphic to a product of copies of $\mathbf{R}^{\mathrm{pos}}$. The $\mathscr{S}(G)$ is isomorphic to $\mathscr{S}(G_0) \hat{\otimes} \mathscr{S}(A_G)$ and similarly $\mathscr{S}(\Gamma\backslash G)$ is isomorphic to $\mathscr{S}(\Gamma\backslash G_0) \hat{\otimes} \mathscr{S}(A_G)$ where since $\mathscr{S}(A_G)$ is nuclear [**24**, Theorem 50.1] $\hat{\otimes}$ means here virtually any topological tensor product. Then Theorem 2.2 follows in this case (still with $P = G$) according to [**24**, Proposition 50.1].

If P is now any rational parabolic subgroup of G, then since $\Gamma(N)\backslash N$ is compact, simple technical manipulations allow one to deduce from what has just been done, applied to its reductive component, that the canonical Θ-map from $\mathscr{S}(P)$ to $\mathscr{S}(\Gamma(P)\backslash P)$ is surjective. But since $P\backslash G$ is compact, $\mathscr{S}(G)$ may be identified with the representation of G smoothly induced from $\mathscr{S}(P)$ and $\mathscr{S}(\Gamma(P)\backslash G)$ with that smoothly induced from $\mathscr{S}(\Gamma(P)\backslash P)$ (see Proposition 3.1 later on, which is completely independent of this section). This concludes the proof of Theorem 2.2.

Although, as we have just seen, smooth functions on the Borel-Serre compactification are of moderate growth, it may happen that they are not of uniform moderate growth. For example, any smooth function on $SL_2(\mathbf{Z})\backslash\mathscr{H}$ which is equal to $f(x)$ for $y \gg 0$ will be of uniform moderate growth if and only if $f(x)$ is constant.

2.6. COROLLARY. *The space* $\mathscr{S}(\Gamma(P)\backslash G)$ *is nuclear.*

Proof. This follows from Proposition 1.1, since quotients of nuclear spaces are nuclear.

3. Eisenstein series and the cuspidal summand. If P is a rational parabolic subgroup of G, then the G-module $\mathscr{S}(\Gamma(P)\backslash G)$ is an induced representation. More precisely, for a given $g \in G$ and $f \in \mathscr{S}(\Gamma(P)\backslash G)$ the function $p \mapsto f(pg)$ lies in $\mathscr{S}(\Gamma(P)\backslash P)$, so f corresponds to a function

$$F_f : G \to \mathscr{S}(\Gamma(P)\backslash P)$$

satisfying the condition

$$F_f(pg) = R_p F_f(g)$$

for all $g \in G$, $p \in P$. Since $P\backslash G$ is compact:

3.1. PROSPOSITION. *The correspondence* $f \to F_f$ *is an isomorphism of* $\mathcal{S}(\Gamma(P)\backslash G)$ *with* $\mathrm{Ind}^{sm}(\mathcal{S}(\Gamma(P)\backslash P)|P, G)$. *Dually, the pairing*

$$\langle f, F_f^* \Phi \rangle = \int_{G/P} \langle F_f(g), \delta_P \Phi(g) \rangle$$

induces an isomorphism of $\mathrm{Ind}^{sm}(A(\Gamma\backslash G)|P, G)$ *with* $A(\Gamma(P)\backslash G)$.

Here Ind^{sm} means the smooth induced representation, which is a smooth representation of G with the obvious topology. Recall also that the factor δ_P is necessary since one can only integrate densities over G/P. Following this proposition, I shall often confound f and F_f.

Since $N = N_P$ is normal in P and $\Gamma(N)\backslash N$ is compact, $\Gamma(P)N$ is closed in G. Define the space $\mathcal{S}(\Gamma(P)N\backslash G)$ to be the closed subspace of left N-invariants in $\mathcal{S}(\Gamma(P)\backslash G)$. It is in fact the summand corresponding to the projection operator which I write in various ways

$$\Pi_{P,G} f(g) = \Pi_P f(g) = \Pi_N f(g) := \frac{1}{\mathrm{meas}(\Gamma(N)\backslash N)} \int_{\Gamma(N)\backslash N} f(ng)dn. \tag{3.1}$$

For convenience, I shall generally assume $\Gamma(N)\backslash N$ to have measure 1. This projection of f is called the *constant term* of f with respect to N. Since the image is a summand of $\mathcal{S}(\Gamma(P)\backslash G)$ one has also a corresponding projection on the space of tempered distributions on $\Gamma(P)\backslash G$. For $F \in A_{\mathrm{mg}}(\Gamma(P)\backslash G)$ this projection is given by the analogue of Equation (3.1).

3.2. *Remark*. Suppose that Γ_* is a subgroup of Γ such that $\Gamma_*(N)$ is of finite index in $\Gamma(N)$. Then the constant term defined by Equation (3.1) is the same as that defined by the formula

$$\frac{1}{\mathrm{meas}(\Gamma_*(N)\backslash N)} \int_{\Gamma_*(N)\backslash N} f(ng)dn.$$

After all, either of these constant term maps is just the (unique) projection from a certain space of functions of $\Gamma(P)\backslash G$ onto the subspace of its N-invariants.

3.3. COROLLARY. *The G-representation* $\mathcal{S}(\Gamma(P)N\backslash G)$ *may be identified with*

$$\mathrm{Ind}^{sm}(\mathcal{S}(\Gamma(M)\backslash M)|P, G).$$

Here the group P acts on $\mathcal{S}(\Gamma(M)\backslash M)$ through the canonical surjection $P \to M$.

$\mathscr{A}(\Gamma(P)N\backslash G) :=$ the space of $Z(\mathfrak{g})$-finite, K-finite tempered distributions on $\Gamma(P)N\backslash G$.

3.4. COROLLARY. *The representation of* $(\mathfrak{g}, K)$ *on* $\mathscr{A}(\Gamma(P)N\backslash G)$ *may be identified with* $\mathrm{Ind}(\mathscr{A}(\Gamma(M)\backslash M)|P, G)$.

Let $P = G$. If v is a K-finite, $Z(\mathfrak{g})$-finite vector in any smooth representation (π, V) of G, then there always exists [**6**, Théorème 3.18] a function $f \in C_c^\infty(G)$ such that $\pi(f)v = v$. Hence $\mathscr{A}(\Gamma\backslash G)$ is contained in the Gårding subspace of $A(\Gamma\backslash G)$, thus in $A_{\mathrm{umg}}(\Gamma\backslash G)$ by Lemma 1.16. In other words, every element of $\mathscr{A}(\Gamma\backslash G)$ is a C^∞ function on $\Gamma\backslash G$ of moderate growth. In fact, ellipticity arguments [**6**, Proposition 3.14] imply that it will even be analytic. In other words, according to the standard definition (in [**3**] or [**9**], for example):

3.5. THEOREM. *The elements of* $\mathscr{A}(\Gamma\backslash G)$ *are precisely the automorphic forms on* $\Gamma\backslash G$.

3.6. *Remark*. This result suggests that one plausible, and perhaps useful, extension of the notion of automorphic form would be to include functions in $A_{\mathrm{umg}}(\Gamma\backslash G)$ which are $Z(\mathfrak{g})$-finite but not necessarily K-finite. The known Paley-Wiener theorems for $\mathscr{S}(\Gamma\backslash G)$ together with results of Wallach and myself (partly explained in [**25**] and to appear in [**12**]) indicate that this is so.

Recall the map $\Theta(\Gamma(P), \Gamma)$ from $\mathscr{S}(\Gamma(P)\backslash G)$ to $\mathscr{S}(\Gamma\backslash G)$, defined in Section 1 by the formula

$$\Theta(f|\Gamma(P), \Gamma)(g) = \sum_{\gamma \in \Gamma(P)\backslash\Gamma} f(\gamma g)$$

for every $f \in \mathscr{S}(\Gamma(P)\backslash G)$. Define the *Eisenstein series map* $E(P, G)$ to be the restriction of this to the subspace $\mathscr{S}(\Gamma(P)N_P\backslash G)$. Dual to it is one on tempered distributions, from $A(\Gamma\backslash G)$ to $A(\Gamma(P)N_P\backslash G)$. This is very simple when restricted to $A_{\mathrm{int}}(\Gamma\backslash G)$: it takes the function F on $\Gamma\backslash G$ to its constant term on $\Gamma(P)N_P\backslash G$, as can be seen easily. In other words, the maps $E(P, G)$ and $\Pi_{P,G}$ are adjoint. Summarizing:

3.7. PROPOSITION. *If D is a tempered distribution on $\Gamma\backslash G$ then its constant term D_P is the tempered distribution on $\Gamma(P)\backslash G$ obtained by duality from the Eisenstein series map*:

$$\langle D_P, f\rangle = \langle D, E(f|P, G)\rangle \tag{3.2}$$

for all $f \in \mathscr{S}(\Gamma(P)N_P\backslash G)$. *For* $F \in A_{\mathrm{int}}(\Gamma\backslash G)$ *its constant term is the function*

$$\prod_P F(g) = \int_{\Gamma(N_P)\backslash N_P} F(ng)\,dn \tag{3.3}$$

in $A_{\mathrm{int}}(\Gamma(P)N_P\backslash G)$. *If* F *is in* $A_{\mathrm{umg}}(\Gamma\backslash G)$ *then* $\Pi_P F \in A_{\mathrm{umg}}(\Gamma(P)N_P\backslash G)$.

For functions F in $A_{\mathrm{mg}}(\Gamma\backslash G)$ and f in $\mathcal{S}(\Gamma(P)N_P\backslash G)$, Equation (3.2) therefore becomes

$$(3.4)\qquad \int_{\Gamma(P)N_P\backslash G} \prod_P F(g)f(g)dg = \int_{\Gamma\backslash G} F(g)E(f|P, G)dg,$$

where measures on G, N_P, and $N_P\backslash G$ are chosen compatibly.

Next I shall prove the well known result that, roughly speaking, the constant term controls the asymptotic behaviour of an element of $A_{\mathrm{umg}}(\Gamma\backslash G)$. I shall follow, essentially, the treatment in the notes **[17]**, where an equivalent result is formulated as Theorem 1.10. Another version can be found in **[21]**. First of all I will formalize the main step. If N is a unipotent subgroup of G, F be a smooth function on G, Φ a positive function on $N\backslash G$, and U an open subset of G, the function F will be called *Φ-uniform* on U if for any X in $U(\mathfrak{g})$

$$\|F\|_{X,\Phi} = \sup|R_X F(g)/\Phi(g)| < \infty$$

on U.

In the next result, which is rather technical but important, let

N = a unipotent subgroup of G
N_* = a normal subgroup of N with N/N_* one-dimensional
A = a subgroup of G conjugating both N and N_* into themselves, such that the adjoint action of A on N/N_* is through a single character α
Ω = a compact subset of G.

3.8. LEMMA. *Suppose F to be a Φ-uniform function on an open N-stable subset U of $\Gamma(N)N_*\backslash G$. Then for every positive integer $m > 0$ there exists a constant $C = C_{m,\Omega} > 0$ and a finite set Ξ of elements of $U(\mathfrak{g})$ such that*

$$\left|F(g) - \prod_N F(g)\right| \leqq C \inf_{X\in\Xi} \|F\|_{X,\Phi}\alpha(a)^{-m}\Phi(g)$$

for all $g = a\omega$ in $U \cap A\Omega$.

Proof. Since $\Gamma(N)N_*\backslash N$ is compact and one-dimensional, we have the Fourier expansion

$$F(g) - \prod_N F(g) = \sum_{\chi\neq 1} \Lambda_\chi(R_g F)$$

where the sum is over all non-trivial characters χ of $\Gamma(N)N_*\backslash N$, and Λ_χ is the Fourier coefficient functional

$$\Lambda_\chi F = \int_{\Gamma(N)N_*\backslash N} \chi^{-1}(x)F(x)dx.$$

These Fourier coefficients can be estimated uniformly in the following way. If X lies in $U(\mathfrak{n})$ then on the one hand

$$\Lambda_\chi(R_a R_X R_\omega F) = \Lambda_\chi(R_{\mathrm{Ad}(a)X} R_{a\omega} F) = d\chi(\mathrm{Ad}(a)X)\Lambda_\chi(R_g F)$$

and on the other

$$\Lambda_\chi(R_a R_X R_\omega F) = \Lambda_\chi(R_{a\omega} R_{\mathrm{Ad}(\omega)^{-1}X} F) = \Lambda_\chi(R_g R_{\mathrm{Ad}(\omega)^{-1}X} F)$$

so that in combination

$$\Lambda_\chi(R_g F) = (d\chi(\mathrm{Ad}(a)\ X))^{-1}\Lambda_\chi(R_g R_{\mathrm{Ad}(\omega)^{-1}X} F)$$

whenever $d\chi(\mathrm{Ad}(a)\ X)$ is not equal to 0. Because F is Φ-uniform and Φ is left-N-invariant,

$$|\Lambda_\chi(R_g R_{\mathrm{Ad}(\omega)^{-1}X} F)| = \left|\int_{\Gamma(N)N_*\backslash N} \chi(x)^{-1} \sum_i c_i(\omega) R_{X_i} F(xg)dx\right|$$
$$\leqq \sup_{\omega\in\Omega} |c_i(\omega)| \sup_{X_i} \|F\|_{X_i,\Phi}\Phi(g)$$

with the functions $c_i(\omega)$ defined by the formula $\mathrm{Ad}(\omega)^{-1}X = \sum c_i(\omega)X_i$. Here the X_i form a basis of $\mathrm{Ad}(K)X$. Hence for $X \in U(\mathfrak{n})$, $g = a\omega \in A\Omega \cap U$

$$|\Lambda_\chi(R_g F)| = d\chi(\mathrm{Ad}(a)\ X)^{-1}\Lambda_\chi(R_g R_{\mathrm{Ad}(\omega)^{-1}X} F|$$
$$\leqq C|d\chi(\mathrm{Ad}(a)X|^{-1}\Phi(g)$$

for suitable C. Choose X to be ν^m, where ν is any non-zero element of $\mathfrak{n} - \mathfrak{n}_*$, and sum over the possible χ, which are all integral multiples of some fixed χ_0 to get

$$\left|F(g) - \prod_N F(g)\right| \leqq C \sum (1/|n|)^m |d\chi_0(v)|^{-m}|\alpha(a)|^{-m}\Phi(g).$$

If P is a rational parabolic subgroup of G then the parabolic subgroups containing P are indexed by the subsets of the set $\mathrm{Max}(P, G)$ of maximal proper parabolic subgroups of G containing P: to the subset $\mathcal{M}$ corresponds $\cap_{R\in\mathcal{M}} R$. If $P \subseteq Q$ are two rational parabolic subgroups of G, then the image of P modulo N_Q is a parabolic subgroup of M_Q, and this establishes a bijective correspondence between the rational parabolic subgroups of M_Q and those of G contained in Q. Therefore the set $\mathrm{Par}(P, Q)$ of rational parabolic subgroups of G lying in between P and Q is parametrized by the subsets of the set $\mathrm{Max}(P, Q)$ of rational parabolic subgroups R containing P, strictly contained in Q, maximal for this property. For $P \subseteq S \subseteq Q$ let $r(S, Q)$ be the cardinality of the R in $\mathrm{Max}(P, Q)$ containing S.

If $P \subseteq Q$ are two rational parabolic subgroups of G, define a *Siegel subset of $N_Q\backslash G$ with respect to the pair (P, Q)* to be one of the form $\Omega A_{P,Q}^{++}(t)K$, where Ω is a compact subset of Q and

$$\Sigma^+_{P,Q} := \text{the eigencharacters of the adjoint action of } A_P \text{ on } \mathfrak{n}_P/\mathfrak{n}_Q$$

$$A^{++}_{P,Q}(t) := \{a \in A_P | \alpha(a) > 0 \text{ for all } \alpha \in \Sigma^+_{P,Q}\}.$$

This subset may also be expressed as the image in $N_Q\backslash G$ of some $\mathfrak{S}K$ where $\mathfrak{S}$ is a Siegel subset of M_Q with respect to the image of P in M_Q. (The Siegel subsets defined earlier were with respect to pairs (P, G).) Further let

$$\delta_{P,Q}(a) := |\det(\mathrm{Ad}_{\mathfrak{n}_P/\mathfrak{n}_Q}(a))|.$$

3.9. THEOREM. *Let $P \subseteq Q$ be two rational parabolic subgroups of G, $\mathfrak{S} = \Omega AK$ a Siegel subset of G with respect to $(P_\emptyset, Q)$ where $P_\emptyset$ is a minimal rational parabolic subgroup contained in P. Suppose F to be Φ-uniform on $\Gamma(N_P)N_Q\backslash G$. Then for some constant $C > 0$*

$$\left|\sum_{S \in \mathrm{Par}(P,Q)} (-1)^{r(S,Q)} \prod\nolimits_S F\right| \leqq C\delta^{-m}_{P,Q}(a)\Phi(g)$$

for all $g = \omega ak$ in $\mathfrak{S}$, where C is bounded by some finite set of $\|F\|_{X,\Phi}$.

Proof. Suppose R to be in $\mathrm{Max}(P, Q)$. Filter N_R by subgroups

$$N_Q = N_0 \subseteq N_1 \ldots \subseteq N_n = N_R$$

where each N_i is normal in AN_R and each quotient N_i/N_{i-1} is one-dimensional and corresponds to a root space of the adjoint action of A on $\mathfrak{n}_R/N_Q$. Then by applying Lemma 3.8 successively and summing we have for each $m > 0$ a suitable constant C such that

$$\left|F(g) - \prod_R F(g)\right| \leqq C\delta^{-m}_{R,Q}(a)\Phi(g)$$

on $\mathfrak{S}_{P,Q}$. Apply this argument again to $F - \Pi_R F$ and a second parabolic subgroup S to get

$$\left|F(g) - \prod_R F(g) - \prod_S F(g) + \prod_{R\cap S} F(g)\right|$$
$$\leqq C\delta_{R\cap S,Q}(a)^{-m}\Phi(g).$$

And so on. Keep in mind that $\Pi_R \Pi_S = \Pi_{R\cap S}$, and keep track of the constants.

3.10. THEOREM. *If F lies in $A_{\mathrm{umg}}(\Gamma\backslash G)$, then in order for F to lie in $\mathscr{S}(\Gamma\backslash G)$ it is necessary and sufficient that for some $r > 0$ these two conditions hold*

(a) *Whenever* (i) *P is a minimal rational parabolic subgroup of G;* (ii) *$\mathfrak{S}$ a Siegel subset of G associated to P; and* (iii) *Q a rational parabolic subgroup of G containing P, all the derivatives of the function F satisfy for all $m > 0$ the growth estimate*

$$\prod_Q (R_X F) = O(\delta_Q^{-m}\|g\|^r)$$

on $\mathfrak{S}$;

(b) *For all rational characters* χ *of* G

$$\chi R_X F = O(\|g\|^r)$$

on all of $\Gamma\backslash G$.

Note that condition (a) is superfluous if $\Pi_Q F = 0$, and that condition (b) is superfluous if G has no rational characters; in particular, if G is semi-simple.

Proof. If F lies in $\mathcal{S}(\Gamma\backslash G)$ then conditions (a) and (b) are immediate. If, conversely, (a) and (b) hold then Theorem 3.9 together with Proposition 1.15 implies inductively that each constant term $\Pi_Q F$, and eventually F itself, lies in $\mathcal{S}(\mathfrak{S})$.

A distribution D on $\Gamma\backslash G$ is said to be *cuspidal* if $D_P = 0$ for all proper rational parabolic subgroups P of G. If D is tempered then by Proposition 3.7 this means that

$$\langle D, E(f|P, G)\rangle = 0$$

for all such P and $f \in \mathcal{S}(\Gamma(P)N\backslash G)$. Let $\mathcal{S}_{\text{cusp}}(\Gamma\backslash G)$, etc. be the subspaces of cuspidal elements in $\mathcal{S}$, etc. These cuspidal subspaces are all closed and G-stable.

3.11. LEMMA. *The subspace* $\mathcal{S}_{\text{cusp}}(\Gamma\backslash G)$ *is dense in each of* $L^2_{\text{cusp}}(\Gamma\backslash G)$ *and* $A_{\text{cusp}}(\Gamma\backslash G)$.

Proof. Suppose D to be in $A_{\text{cusp}}(\Gamma\backslash G)$. Then for every $f \in C_c^\infty(G)$ the convolution $R_f D = D * f$ also lies in $A_{\text{cusp}}(\Gamma\backslash G)$. By Theorem 1.16 the distribution $D * f$ lies in $A_{\text{umg}}(\Gamma\backslash G)$. Since D may be arbitrarily well approximated by such $D * f$, it suffices now to assume $D = F \in A_{\text{umg}}(\Gamma\backslash G)$. Let G_* be the derived group of G, Z_* the quotient G/G_*. For each $T \gg 0$ let φ_T be a C_c^∞ function on Z_* which is equal to 1 for $\|z\| \leqq T$ and 0 for $\|z\| \geqq T + 1$. Then $F_T = F\varphi_T$ will again be cuspidal and in A_{umg}, and F_T converges as a tempered distribution to F as T goes to infinity. However by Corollary 3.10 each F_T lies in $\mathcal{S}_{\text{cusp}}$.
The same argument works for $F \in L^2$ as well.

This proof also shows a little more generally:

3.12. LEMMA. *If* $A_* \subseteq A_{\text{cusp}}(\Gamma\backslash G)$ *is a closed G-stable subspace with the property that whenever* $F \in A_*$ *and* $\varphi \in C_c^\infty(Z_*)$ *then* $F\varphi$ *also lies in* A_*, *then* $A_* \cap \mathcal{S}_{\text{cusp}}$ *is dense in* A_*.

Define

$L^2_{\rm Eis}(\Gamma\backslash G) :=$ closure in $L^2(\Gamma\backslash G)$ of the sum of the images of the spaces $\mathscr{S}(\Gamma(P)N\backslash G)$ under the maps $E(P, G)$ as P ranges over all proper rational parabolic subgroups

$$\mathscr{S}_{\rm Eis}(\Gamma\backslash G) := \mathscr{S}(\Gamma\backslash G) \cap L^2_{\rm Eis}(\Gamma\backslash G).$$

Since $L^2_{\rm cusp}$ is the annihilator in L^2 of the functions $E(f|P, G)$, $L^2_{\rm Eis}$ is also the orthogonal complement of $L^2_{\rm cusp}$. In other words, essentially by definition,

$$(3.7) \quad L^2(\Gamma\backslash G) = L^2_{\rm cusp}(\Gamma\backslash G) \oplus L^2_{\rm Eis}(\Gamma\backslash G).$$

Not quite so trivially:

3.13. PROPOSITION. *The Schwartz space decomposes similarly*:

$$(3.8) \quad \mathscr{S}(\Gamma\backslash G) = \mathscr{S}_{\rm cusp}(\Gamma\backslash G) \oplus \mathscr{S}_{\rm Eis}(\Gamma\backslash G).$$

In other words, if an element of $\mathscr{S}(\Gamma\backslash G)$ is decomposed according to (3.7) then each component also lies in $\mathscr{S}(\Gamma\backslash G)$.

Proof. Suppose

$$f = f_{\rm cusp} + f_{\rm Eis}$$

according to (3.7), with $f \in \mathscr{S}(\Gamma\backslash G)$. In the decomposition (3.7) it is immediate that if f is in $L^{2,\infty}(\Gamma\backslash G)$, each of its components also lies in $L^{2,\infty}$. Therefore by **[13]** and Proposition 1.16, the component $f_{\rm cusp}$ lies at least in $A_{\rm umg}$. Condition 3.10 (a) is vacuous, so it remains to check Condition 3.10 (b).

Since f is in $\mathscr{S}(\Gamma\backslash G)$, so is χf for every rational character χ of G. For each such χ, let χ_T be a smooth truncation of χ; i.e., equal to χ where $\|z\| \leqq T$ and 0 where $\|z\| \geqq T + 1$. Then the decomposition of $\chi_T f$ according to (3.7) is

$$\chi_T f = \chi_T f_{\rm cusp} + \chi_T f_{\rm Eis}$$

and (with some mild assumption on how truncation is carried out)

$$\|_\chi F\|^2 \geqq \|\chi_T f\|^2 = \|\chi_T f_{\rm cusp}\|^2 + \|\chi_T f_{\rm Eis}\|^2$$

so that letting T go to infinity we see that $\chi f_{\rm cusp}$ also lies in $L^{2,\infty}(\Gamma\backslash G)$. Apply Proposition 1.17 to conclude.

Recall that by definition $A_{\rm cusp}(\Gamma\backslash G)$ is the annihilator in $A(\Gamma\backslash G)$ of all the $E(f|P, G)$, $f \in \mathscr{S}(\Gamma(P)N\backslash G)$.

3.14. LEMMA. *The space $A_{\rm cusp}(\Gamma\backslash G)$ is the annihilator of $\mathscr{S}_{\rm Eis}(\Gamma\backslash G)$.*

Proof. By definition, $\mathscr{S}_{\rm Eis}$ is $\mathscr{S} \cap L^2_{\rm Eis}$, so that it contains all the $E(f|P, G)$, and the annihilator of $\mathscr{S}_{\rm Eis}$ is contained in $A_{\rm cusp}$. At least according to Proposition 3.13

$$A = \mathrm{Ann}(\mathscr{S}_{\mathrm{Eis}}) \oplus \mathrm{Ann}(\mathscr{S}_{\mathrm{cusp}})$$

so that

$$A_{\mathrm{cusp}} = \mathrm{Ann}(\mathscr{S}_{\mathrm{Eis}}) \oplus A_*$$

where $A_* = A_{\mathrm{cusp}} \cap \mathrm{Ann}(\mathscr{S}_{\mathrm{cusp}})$. But this satisfies the conditions of Lemma 3.12, so that $A_* \cap \mathscr{S}_{\mathrm{cusp}}$, which is just $\{0\}$, is also dense in A_*.

3.15. PROPOSITION. *Dual to the decomposition* (3.8),

$$A(\Gamma\backslash G) = A_{\mathrm{cusp}}(\Gamma\backslash G) \oplus A_{\mathrm{Eis}}(\Gamma\backslash G).$$

3.16. *Remark.* If Γ_* is a subgroup of finite index in Γ, then there exist two maps relating the two corresponding Schwartz spaces; the projection $\Theta(\Gamma_*, \Gamma)$ from $\mathscr{S}(\Gamma_*\backslash G)$ onto $\mathscr{S}(\Gamma\backslash G)$ and also the inclusion of $\mathscr{S}(\Gamma\backslash G)$ in $\mathscr{S}(\Gamma_*\backslash G)$. These are clearly both completely compatible with the direct sum decompositions of Proposition 3.13, in view of Remark 3.2.

3.17. *Remark.* It is known that if G is semi-simple, then $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ is a discrete direct sum of irreducible unitary representations, each with finite multiplicity. The usual proof of this (see, for example, **[15]** of **[16]**) is illuminated somewhat by the decomposition of Theorem 3.13. Consider $f \in \mathscr{S}(G)$ and the operator R_f on $\mathscr{S}(\Gamma\backslash G)$. This is, as we have seen in Section 1, defined by integration against a kernel K_f in $A_{\mathrm{umg}}(\Gamma\backslash G \times \Gamma\backslash G)$. Thus K_f decomposes into a sum of terms K_{cusp} and K_{Eis}, and K_{cusp} turns out to be the kernel of the restriction of the operator R_f to cusp forms. But since G is semi-simple, K_{cusp} is itself in $\mathscr{S}(\Gamma\backslash G \times \Gamma\backslash G)$, and certainly of trace class. Incidentally, it seems likely to me that all of Arthur's work on the Selberg trace formula carries through for Schwartz functions on G as well as smooth compactly supported ones. This is useful to know, if true, since Schwartz functions (in the sense meant here) are much easier to come by. For example, as Wallach has pointed out, if D is any compactly supported right- and left-K-finite distribution on G and Δ is a Casimir element of $U(\mathfrak{g})$ then $\exp(-t\Delta)\, D$ is a Schwartz function. This observation should make it possible to do with fewer references to Paley-Wiener theorems in applying the trace formula.

4. The parabolic decomposition. In this section the decomposition of $\mathscr{S}(\Gamma\backslash G)$ into $\mathscr{S}_{\mathrm{cusp}}(\Gamma\backslash G)$ and $\mathscr{S}_{\mathrm{Eis}}(\Gamma\backslash G)$ will be refined to obtain a further decomposition of $\mathscr{S}_{\mathrm{Eis}}$. This result, as I have mentioned in the Introduction, will be analogue of a well known decomposition of $L^2(\Gamma\backslash G)$ due to Langlands; the one he calls the elementary decomposition, not the one which uses the residual Eisenstein series. Langlands himself has obtained, in correspondence with Borel dated 1972, the corresponding decomposition for A_{umg}. The preprint **[8]** fills in details of Langlands' argument and at the end derives from the earlier result the decomposition I give here. (I believe that Borel's motivation for including the result on Schwartz spaces was a query of mine. The essential idea, seen from the perspective of this paper, is that the Schwartz space is the Gårding subspace of the topological dual of A_{umg}.) Their techniques are overall somewhat different.

I begin with an elementary result. If $P \subseteq Q$ are two rational parabolic subgroups then the image of P modulo N_Q is a rational parabolic subgroup R of M_Q, which I will call P mod N_Q or sometimes $\text{mod}_Q(P)$, whose unipotent radical is $N_R \cong N_P/N_Q$ and whose Levi component is isomorphic to M_P. This sets up a bijective correspondence mod_Q between the rational parabolic subgroups of M_Q and those of G contained in Q. Corresponding to the inclusion of R in M_Q is an Eisenstein series map $E(R, M)$ from $\mathscr{S}(\Gamma(R)N_R\backslash M)$ to $\mathscr{S}(\Gamma(M)\backslash M)$, hence by induction, according to Remark 3.2, to a map $\text{Ind}(E(R, M))$ from $\mathscr{S}(\Gamma(P)N_P\backslash G)$ to $\mathscr{S}(\Gamma(Q)N_Q\backslash G)$. The following is trivial:

4.1. LEMMA (Transitivity of Eisenstein series). *For parabolic subgroups $P \subseteq Q$, this diagram commutes*:

$$\begin{array}{ccc} \mathscr{S}(\Gamma(P)N_P\backslash G) & \xrightarrow{\ \text{Ind}(E(R,M))\ } & \mathscr{S}(\Gamma(Q)N_Q\backslash G) \\ & {\scriptstyle E(P,G)}\searrow \quad \swarrow{\scriptstyle E(Q,G)} & \\ & \mathscr{S}(\Gamma\backslash G) & \end{array}$$

If P is a rational parabolic subgroup of G then the subspace $\mathscr{S}_{\text{cusp}}(\Gamma(M)\backslash M)$ is a summand of the Schwartz space $\mathscr{S}(\Gamma(M)\backslash M)$ according to Theorem 3.13. Define $\mathscr{S}_{\text{cusp}}(\Gamma(P)N\backslash G)$ to be the corresponding summand of $\mathscr{S}(\Gamma(P)N\backslash G)$, $A_{\text{cusp}}(\Gamma(P)N\backslash G)$ that of $A(\Gamma(P)N\backslash G)$. Then of course

$$\mathscr{S}_{\text{cusp}}(\Gamma(P)N\backslash G) \cong \text{Ind}^{sm}(\mathscr{S}_{\text{cusp}}(\Gamma(M)\backslash M)\,|P, G),$$

$$A_{\text{cusp}}(\Gamma(P)N\backslash G) \cong \text{Ind}^{sm}(A_{\text{cusp}}(\Gamma(M)\backslash M)\,|P, G).$$

As temporary notation, let $\mathscr{S}_{\text{cusp}}(P, G)$ be the image of $\mathscr{S}_{\text{cusp}}(\Gamma(P)N\backslash G)$ in $\mathscr{S}(\Gamma\backslash G)$ with respect to $E(P, G)$.

4.2. LEMMA. *The sum of the spaces $\mathscr{S}_{\text{cusp}}(P, G)$, as P ranges over all rational parabolic subgroups of G, is dense in $\mathscr{S}(\Gamma\backslash G)$ and also in $L^2(\Gamma\backslash G)$.*

The proof proceeds by induction on the **Q**-rank of the derived group of G. If this is zero, then the derived group of G is **Q**-anisotropic, the quotient $\Gamma\backslash G$ is compact, and all of $\mathscr{S}$ is cuspidal so there is nothing to be proven.

In the general case, suppose D to be a tempered distribution which annihilates all of the spaces $\mathscr{S}_{\text{cusp}}(P, G)$ including $\mathscr{S}_{\text{cusp}}(\Gamma\backslash G) = \mathscr{S}_{\text{cusp}}(G, G)$. According to Proposition 3.7 this means that $\langle D_P, f\rangle = 0$ whenever P is a rational parabolic subgroup and f lies in

$\mathscr{S}(\Gamma(P)N\backslash G)$. By the induction hypothesis and the transitivity of Eisenstein series we see that $D_P = 0$ for all proper rational parabolic subgroups P. But since $\langle D, f\rangle = 0$ as well for cuspidal f, Proposition 3.15 implies that $D = 0$. This implies the lemma, by Hahn-Banach.

More generally, if P and Q are any two rational parabolic subgroups then the image of their intersection $P \cap Q$ modulo N_Q is a rational parabolic subgroup of M_Q whose unipotent radical is the image of $N_P \cap Q$. I shall call P and Q *immediate associates* if the image of $P \cap Q$ modulo N_P is all of M_P and that modulo N_Q is all of M_Q. In this case the two projections from $P \cap Q$ onto M_P and M_Q identify either one of these with the Levi component of $P \cap Q$, so that in particular M_P and M_Q are isomorphic. If it is only assumed that the image of $P \cap Q$ modulo N_Q is all of M_Q, then

$$\operatorname{mod}_P^{-1}(P \cap Q \bmod N_P)$$

is an immediate associate of Q, so that P must at least contain an immediate associate of Q. The two groups are called simply *associates* if they possess rational conjugates which are immediate associates, and more particularly associated by the element g if $g^{-1}Pg$ is an immediate associate of Q. This is equivalent to the more usual definition, which is that they possess Levi components which are conjugate. Roughly speaking, if $g^{-1}Pg$ is an immediate associate of Q then conjugation by g^{-1} combined with the projection onto M_Q induces an isomorphism of M_P with M_Q. (These things are discussed in **[17]**, around Lemma 29 on page 34. See also Section 4 of **[11]**.)

What I shall do next is to calculate, at least in a weak sense, the constant terms of Eisenstein series. The calculation will follow, roughly, **[17**, pp. 38-39].

4.3. Proposition. *When f is cuspidal in $\mathscr{S}(\Gamma(P)N_P\backslash G)$ then*

$$\prod_Q E(f|P, G) = 0$$

unless Q contains an associate of P.

Proof. Given P, let f be in $\mathscr{S}(\Gamma(P)N_P\backslash G)$. Then the constant term of $E(f|P, G)$ with respect to the parabolic subgroup Q is by definition the function

$$\prod_Q E(f|P, G)(g) = \int_{\Gamma\cap N_Q/N_Q} E(f|P, G)(xg)dx.$$

From now on I shall set $g = 1$ for convenience, which is no loss since we can always replace f by $R_g f$ at the end. Thus this becomes

$$\int_{\Gamma(N_Q)\backslash N_Q} \sum_{\Gamma(P)\backslash\Gamma} f(\gamma x)dx = \sum_{\Gamma(P)\backslash\Gamma/\Gamma(N_Q)} \Delta_\gamma(f)$$

where

$$\begin{aligned}\Lambda_\gamma(f) &= \int_{\Gamma(N_Q)\backslash N_Q} dx \sum_{\delta\in\Gamma(P)\backslash\Gamma(P)\gamma\Gamma(N_Q)} f(\delta x)\\ &= \int_{\Gamma(N_Q)\backslash N_Q} dx \sum_{\delta\in\Gamma(N_Q)\cap\gamma^{-1}P\gamma\backslash\Gamma(N_Q)} f(\delta x)\\ &= \int_{\Gamma(N_Q)\cap\gamma^{-1}P\gamma\backslash N_Q} f(\gamma x)dx\\ &= \int_{N_Q\cap\gamma^{-1}P\gamma\backslash N_Q} dx \int_{\Gamma(N_Q)\cap\gamma^{-1}P\gamma\backslash N_Q\cap\gamma^{-1}P\gamma} f(\gamma y x)dy.\end{aligned}$$

For the moment let $h = \gamma x$. Then the inner integral becomes, after a change of variables $z = \gamma^{-1}y\gamma$,

$$(4.1)\qquad \int_{\gamma\Gamma(N_Q)\gamma^{-1}\cap P\backslash\gamma N_Q\gamma^{-1}\cap P} R_hf(z)dz.$$

Since f is a function on $\Gamma(P)N_P\backslash G$, it is left-N_P-invariant. From the remarks made just before, the intersection $\gamma Q\gamma^{-1} \cap P$ has as image modulo N_P a rational parabolic subgroup R of M_P, whose unipotent radical N_R is the image of $\gamma N_Q\gamma^{-1} \cap P$. By Remark 3.2, if $\Gamma(N_R)$ is the image of $\Gamma \cap N_Q\gamma^{-1} \cap P$ in N_R the integral above is the same as

$$(4.2)\qquad \int_{\Gamma(N_R)\backslash N_R} R_hf(z)dz \quad (h = \gamma x)$$

which is the constant term of R_hf with respect to the parabolic subgroup R in M_P, if $m \mapsto R_h(m)$ is considered a function on $\Gamma(M)\backslash M$. Suppose f to be cuspidal. Then the constant term in Equation (4.2) vanishes unless N_R is trivial; i.e., unless R is all of M_P. In other words, $\Lambda_\gamma(f) = 0$ unless the image of $\gamma Q\gamma^{-1} \cap P$ modulo N_P is all of M_P, which means by remarks made earlier that $\gamma Q\gamma^{-1}$ contains an immediate associate of P.

Suppose f to be cuspidal in $\mathcal{S}(\Gamma(P)N_P\backslash G)$ and φ cuspidal in $\mathcal{S}(\Gamma(Q)N_Q\backslash G)$. By Proposition 3.7 we see that

$$\langle E(f|P, G), E(\varphi|Q, G)\rangle = \langle f, E(\varphi|Q, G)_P\rangle = \langle E(f|P, G)_Q, \varphi\rangle.$$

Applying the conclusion above to φ as well as f, we see:

4.4. Corollary. *When f is cuspidal in $\mathcal{S}(\Gamma(P)N_P\backslash G)$ and φ cuspidal in $\mathcal{S}(\Gamma(Q)N_Q\backslash G)$ then the inner product of $E(f|P, G)$ and $E(\varphi|Q, G)$ is zero unless P and Q are Γ-associates.*

For a class $\mathcal{P}$ of Γ-associate rational parabolic subgroups of G, define $L^2_{\mathcal{P}}(\Gamma\backslash G)$ to be the closure in $L^2(\Gamma\backslash G)$ of the sum of the spaces $\mathcal{S}(P, G)$ (the images under the Eisenstein series maps of the spaces $\mathcal{S}_{\text{cusp}}(\Gamma(P)N_P\backslash G)$) as P ranges over $\mathcal{P}$. The calculation just made shows that the spaces corresponding to two distinct associate classes are orthogonal, while Lemma 4.2 shows that the sum of the spaces $L^2_{\mathscr{P}}(\Gamma\backslash G)$, as $\mathscr{P}$ ranges over all such sets, is all of $L^2(\Gamma\backslash G)$. Applying the Closed Graph Theorem, we have proven:

4.5. PROPOSITION. *We have the continuous direct sum decomposition*

$$L^2(\Gamma\backslash G) = \oplus L^2_{\mathscr{P}}(\Gamma\backslash G)$$

with $\mathscr{P}$ ranging over all classes of Γ-associate rational parabolic subgroups of G.

When $\mathscr{P}$ is a Γ-associate class of parabolic subgroups of G, or slightly more generally a union of Γ-associate classes, define

$$\mathscr{S}_{\mathscr{P}}(\Gamma\backslash G) := \mathscr{S}(\Gamma\backslash G) \cap L^2_{\mathscr{P}}(\Gamma\backslash G)$$

$A_{\mathscr{P}}(\Gamma\backslash G) :=$ the annihilator in $A(\Gamma\backslash G)$ of the spaces $\mathscr{S}_{\mathscr{Q}}$ with $\mathscr{Q}$ different from $\mathscr{P}$.

The inclusions of $A_{\mathscr{P}}(\Gamma\backslash G)$ in $A(\Gamma\backslash G)$ and $\mathscr{S}_{\mathscr{P}}(\Gamma\backslash G)$ in $\mathscr{S}(\Gamma\backslash G)$ induce a canonical map from $A_{\mathscr{P}}(\Gamma\backslash G)$ to the dual of $\mathscr{S}_{\mathscr{P}}(\Gamma\backslash G)$.

If the parabolic subgroup Q contains parabolic subgroups in the Γ-associate class $\mathscr{P}$, the images modulo N_Q of those subgroups will make up a union of $\Gamma(M_Q)$-associate classes of parabolic subgroups in M_Q, which I express as $\mathscr{P}$ mod N_Q.

4.6. THEOREM. *The inclusion of the spaces $\mathscr{S}_{\mathscr{P}}(\Gamma\backslash G)$ in $\mathscr{S}(\Gamma\backslash G)$ induces a topological isomorphism*

$$\mathscr{S}(\Gamma\backslash G) \cong \oplus \mathscr{S}_{\mathscr{P}}(\Gamma\backslash G)$$

where $\mathscr{P}$ ranges over the Γ-associate classes of rational parabolic subgroups in G.

Proof. Before beginning the proof, I must carry out some preliminaries. An equivalent way to formulate the theorem is to say that if f in $\mathscr{S}(\Gamma\backslash G)$ is decomposed into orthogonal components according to Theorem 4.5, then each component lies in $\mathscr{S}(\Gamma\backslash G)$. This will be proven by induction on the semi-simple rank of G, but in order to do this I shall require several direct consequences of the theorem to be used inductively. Assume for the moment that the theorem is known for the group G.

Then the density Lemma 4.2 and the orthogonality of the components in Theorem 4.5 imply that the subspace $\mathscr{S}_{\mathscr{P}}(\Gamma\backslash G)$ is the same as the closure in $\mathscr{S}(\Gamma\backslash G)$ of the sums of the spaces $\mathscr{S}_{\text{cusp}}(P, G)$. Another consequence is that the canonical projection from $A(\Gamma\backslash G)$ to the dual of $\mathscr{S}_{\mathscr{P}}(\Gamma\backslash G)$ induces an isomorphism of $A_{\mathscr{P}}(\Gamma\backslash G)$ with that dual. This in turn demonstrates:

4.7. COROLLARY. *The space $A(\Gamma\backslash G)$ is isomorphic to the direct sum of its subspaces $A_{\mathscr{P}}(\Gamma\backslash G)$. The elements of $A_{\mathscr{P}}(\Gamma\backslash G)$ may be characterized as* *those tempered distributions F on $\Gamma\backslash G$ such that $(\Pi_Q F)_{\text{cusp}} = 0$ whenever Q is not in $\mathscr{P}$. If F in $A(\Gamma\backslash G)$ is expressed as $\sum F_{\mathscr{P}}$ then the component $\Phi = F_{\mathscr{P}}$ may be characterized by the equation*

$$\left(\prod_P \Phi\right)_{\text{cusp}} = \begin{cases} 0 & P \notin \mathscr{P} \\ \left(\prod_P F\right)_{\text{cusp}} & P \in \mathscr{P}. \end{cases}$$

If $\mathscr{Q}$ is a union of Γ-associate classes in G then for any F in $A(\Gamma(P)N_P\backslash G)$, let $F_{\mathscr{Q}\bmod N_{\mathscr{P}}}$ be the sum of components F_Q with Q in $\mathscr{Q}$ mod N_P given by Theorem 4.6 for $\Gamma(M)\backslash M$.

4.8. Corollary. *If $F \in A(\Gamma\backslash G)$ is expressed as $\sum F_{\mathscr{P}}$ and P is any rational parabolic subgroup of G, then*

$$\text{(4.3)} \qquad \prod_P (F_{\mathscr{P}}) = \begin{cases} 0 & P \text{ does not contain a member of} \\ \left(\prod_P (F)\right)_{\mathscr{P}\bmod N_P} & \mathscr{P} \text{ otherwise.} \end{cases}$$

Proof. This follows from the observation that for Ψ in $A(\Gamma(P)N_P\backslash G)$ the component $\Psi_{\mathscr{Q}}$ is characterized by the equations

$$\left(\prod\nolimits_Q \Psi_Q\right)_{\text{cusp}} = \begin{cases} 0 & Q \text{ does not contain a} \\ \left(\prod\nolimits_{P,\mathfrak{a}} (\Psi)\right)_{\mathscr{Q}\bmod N_Q} & \text{member of } \mathscr{Q} \text{ otherwise.} \end{cases}$$

Now I commence the proof of Theorem 4.6 itself. If the semi-simple rational rank of G is one, then the theorem is the same as Theorem 3.13. Assume then that it is true for groups of smaller semi-simple rank. Given $f \in \mathscr{S}(\Gamma\backslash G)$, decomposed into components $f_{\mathscr{P}}$ according to Theorem 4.5, it must be shown that each $f_{\mathscr{P}}$ lies in $\mathscr{S}(\Gamma\backslash G)$. First of all, if $\mathscr{P}$ is the class of G itself the $f_{\mathscr{P}}$ is just the cuspidal component f_{cusp} of f, so that it lies in $\mathscr{S}(\Gamma\backslash G)$ by Theorem 3.13. For the rest of the components, the criterion of Theorem 3.10 may be applied.

4.9. Lemma. *Under the induction assumption, if F is any function in $L^2(\Gamma\backslash G)$ and expressed as the sum of components $F_{\mathscr{P}}$, the constant terms of each $F_{\mathscr{P}}$ are given by* (4.3).

This should be clear. Note that the induction assumption is needed to make sense of the formula in (4.3), since all we can say about the constant term of a square-integrable function is that it is a tempered distribution.

Now to finish the proof of Theorem 4.6. It only remains to use induction, the formula in Lemma 4.9, and the criterion of Theorem 3.10, which is easily verified since f itself lies in the Schwartz space, and Equation (4.3) identifies the constant terms of the components $f_{\mathscr{P}}$ with components of constant terms of f.

The proof I've given is at the same time somewhat sketchy as well as unnecessarily complicated. Since the formula for the constant terms for associate groups is much simpler than for larger ones, the proof becomes correspondingly simpler if one uses the following result of Paul Ringseth: If F lies in $L^{2,\infty}_{\mathcal{P}}(\Gamma\backslash G)$ and every χF is square-integrable (χ a character of G) then F lies in the Schwartz space if and only if $\prod_Q F$ is rapidly vanishing at infinity for every rational parabolic subgroup Q in $\mathcal{P}$.

Let $P_1, \ldots, P_n$ form a set of representatives of Γ-conjugacy classes in $\mathcal{P}$.

4.10. PROPOSITION. *The map*

$$F \mapsto \left((\prod_P F)_{\text{cusp}} \right) \quad (P \in \mathcal{P})$$

annihilates the subspaces $A_{\mathcal{Q}}(\Gamma\backslash G)$ *for* $\mathcal{Q}$ *not equal to* $\mathcal{P}$ *and embeds* $A_{\mathcal{P}}(\Gamma\backslash G)$ *into* $\prod_i A_{\text{cusp}}(\Gamma(P_i)N_i\backslash G)$.

Remark 3.17 implies that the representation of $(\mathfrak{g}, K)$ on $\mathcal{A}_{\text{cusp}}(\Gamma\backslash G)$ has the property that for a given finite-dimensional representation τ of K and an ideal I in $Z(\mathfrak{g})$ of finite codimension, the subspace of the τ-component of $A_{\text{cusp}}(\Gamma\backslash G)$ annihilated by I is finite-dimensional. The same is true of the induced representation $\mathcal{A}_{\text{cusp}}(\Gamma(P)N_P\backslash G)$, and by Proposition 4.10 true also for all of $\mathcal{A}(\Gamma\backslash G)$.

4.11. *Remark*. Following Remark 3.16, the decompositions in 4.6 and 4.8 are compatible with the two canonical maps between $\mathcal{S}(\Gamma\backslash G)$ and $\mathcal{S}(\Gamma_*\backslash G)$ when Γ_* is a subgroup of Γ of finite index.

5. Growth conditions and cohomology. Suppose that G is semi-simple. Identify the symmetric space $\mathfrak{X}$ with G/K. Let $V = \Gamma\backslash\mathfrak{X}$. The space $\mathcal{S}(\Gamma\backslash G)$ may be understood as the space of sections of a certain sheaf on the Borel-Serre compactification V^* of V. If the map

$$\mathrm{pr}_K: \Gamma\backslash G \to V = \Gamma\backslash\mathfrak{X}$$

is the canonical projection then this sheaf will be the one associated to the pre-sheaf which assigns to any open set U in V^* the space $\mathcal{S}(\mathrm{pr}_K^{-1} U)$ of smooth functions on the inverse image of U in $\Gamma\backslash G$ which along with all their right $U(\mathfrak{g})$-derivatives are rapidly decreasing. It has been proven in Section 2 that this sheaf is fine. The space $A_{\text{umg}}(\Gamma)$ can also be interpreted by a sheaf on V^*, but by the remarks made in the last paragraphs of Section 2 it does not turn out to be fine. There are, however, other compactifications on which A_{umg} does correspond to a fine sheaf; roughly, those obtained from the Borel-Serre compactification by collapsing at least the unipotent fibres. These include those defined in **[23]**, but include also one which is in some sense the largest such quotient of the Borel-Serre compactification, and which as far as I know was first constructed in **[27]**, where the sheaf corresponding to $A_{\rm umg}$ as well as other spaces defined by growth conditions are proved to be fine. Zucker's compactification is the same as the largest Satake compactification when, say, G is absolutely simple, but, for example, when $G = SL_2(\mathbf{F})$ with $\mathbf{F}$ a totally real field of degree $d > 1$ then it is obtained by adding $(d - 1)$-dimensional real tori at infinity rather than simply points. What Zucker shows is that on this compactification as well as several others there exist arbitrarily fine partitions of unity by smooth functions f with the property that all $R_X f$ $(X \in U(\mathfrak{g}))$ are bounded.

It follows from Lemma 2.4 that the smooth functions on the Borel-Serre compactification V^* which vanish of infinite order along the boundary are the same as the functions in $\mathcal{S}(\Gamma\backslash G)$ fixed by the right action of K. This generalizes to a result essentially due to Borel about differential forms with values in certain local coefficient systems, which I will now explain.

Suppose E to be a finite-dimensional representation of G. It corresponds to a locally constant coefficient system $\mathcal{E}$ on V, defined according to the rule that for U open in V the sections of the system over U are to be identified with the Γ-invariant locally constant functions from the inverse image of U on $\mathfrak{X}$ with values in E. If i is the inclusion of V in V^*, then the direct image sheaf $i_*\mathcal{E}$ is still a locally trivial coefficient system on V^* locally isomorphic to E. Define the space of *Schwartz forms* on V^* with values in $i_*\mathcal{E}$ to be that of the smooth forms on V^* with values in this system such that in local coordinates the coefficients of the form vanish of infinite order on the boundary $V^* - V$.

On the other hand, there is a canonical isomorphism, which I will call Ψ, between the de Rham complex of smooth forms on V with values in this system and the Koszul complex of the $(\mathfrak{g}, K)$-module $C^\infty(\Gamma\backslash G) \otimes E$. Explicitly, a form ω is lifted back to $\Gamma\backslash G$ and then identified with an element of

$$\mathrm{Hom}_K(\Lambda^\cdot(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G)).$$

5.1. Proposition. *The Schwartz forms on V^* with values in E are those forms on V which correspond under Ψ to elements of the Koszul complex of $\mathcal{S}(\Gamma\backslash G) \otimes E$.*

This is immediate from Lemma 2.4, since the Schwartz forms are those taking exterior products of ∂_Z, ∂_Y $(Y \in \mathfrak{a}, Z \in \mathfrak{q})$ to functions all of whose derivatives with respect to products of the operators ∂_Z and ∂_Y (Y and Z now in the corresponding universal enveloping algebras) are rapidly decreasing in the coordinates y_i.

5.2. Lemma. *If M is any manifold with corners, then the inclusion of smooth differential forms with support in the interior of M into the complex of Schwartz forms on M induces an isomorphism of cohomology.*

In other words, the cohomology of the complex of Schwartz forms on M may be canonically indentified with the cohomology of compact support of its interior.

Proof. The space of forms supported in the interior of M is the space of sections of the sheaf which is the de Rham sheaf in the interior and null on the boundary of M. According to a standard sheaf-theoretic argument, it must hence be shown that at any point on the boundary of M that the cohomology of the complex of local Schwartz forms is null. This follows from the constructions in the usual proof of Poincaré's Lemma.

The map Ψ carries forms with compact support in V to the Koszul complex with values in $C_c^\infty(\Gamma\backslash G)$.

5.3. THEOREM. *Let E be a finite-dimensional representation of G, $\mathscr{E}$ the corresponding locally trivial coefficient system on X. The inclusion of $C_c^\infty(\Gamma\backslash G)$ into $\mathscr{S}(\Gamma\backslash G)$ induces an isomorphism*

$$H_c^\cdot(V, \mathscr{E}) \cong H^\cdot(\mathfrak{g}, K, \mathscr{S}(\Gamma\backslash G) \otimes E).$$

Dually, applying this to the contragredient of E, since ordinary cohomology is calculated by currents:

5.4. COROLLARY. *The inclusion of $A(\Gamma\backslash G)$ into the space of all distributions on $\Gamma\backslash G$ induces an isomorphism*

$$H^\cdot(V, \mathscr{E}) \cong H^\cdot(\mathfrak{g}, K, A(\Gamma\backslash G) \otimes E).$$

Another way to say this is that the inclusion of the complex of tempered currents into that of all currents induces an isomorphism of cohomology. At any rate, from Theorem 4.6:

5.5. COROLLARY. *The decomposition $A = \oplus A_{\mathscr{P}}$ induces a decomposition of the cohomology of V with coefficients in $\mathscr{E}$ into a direct sum of components indexed by the Γ-associate classes of rational parabolic subgroups of G.*

In particular the inclusion of cusp forms induces an injection of cohomology, an old result due to Borel. A conjecture of Borel, which has been verified for groups of rational rank one, asserts in a rough way how the cohomology of each component should be determined by means of Eisenstein series.

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