Harmonic analysis of the Schwartz space of $\Gamma\backslash\mathbf{SL}_2(\mathbb{R})$

by Bill Casselman

To Joe Shalika with fond memories
of a fruitful if brief collaboration

This is the fourth in a series of papers (the earlier ones are [Casselman:1989], [Casselman:1993] and [Casselman:1999]) intended to present eventually a new way of proving, among other things, the well known results of Chapter 7 of [Langlands:1977] on the completeness of the spectrum arising from cusp forms and Eisenstein series. That Langlands' results have lasted for nearly 40 years without major improvements is testimony to their depth, but—to (mis)quote Peter Sarnak, who had some recent work of Joseph Bernstein's in mind—it is time to reconsider the theory.

In the first of this series of papers I attempted to pursue with some force an idea apparently due originally to Godement—that from an analyst's point of view the theory of automorphic forms is essentially the study of the Schwartz space of $\Gamma\backslash G$ and its dual. In the next two, I looked at subgroups of $\mathrm{SL}_2(\mathbb{R})$ in this perspective. In one of these two I attempted to explain in terms of tempered distributions certain features of the theory—integral formulas such as that for the volume of $\Gamma\backslash G$ and the Maass-Selberg formula—which might have seemed up to then coincidental. In the other I found a new derivation of the Plancherel measure in the case of rank one groups.

This is presumably the last of the series in which I try to explain how a few new ideas, tailored principally to the case of higher rank, may be applied in the simplest case, that of $\mathrm{SL}_2(\mathbb{R})$. In this paper, the principal result will be a theorem of Paley-Wiener type for the Schwartz space $\mathcal{S}(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$, from which the completeness theorem (due originally in this case, I imagine, to Selberg) follows easily.

Paley-Wiener theorems of this sort have been proven before. The earliest result that I am aware of can be found in the remarkable paper [Ehrenpreis-Mautner:1959]. Ehrenpreis and Mautner defined the Schwartz space of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ (where $\mathcal{H}$ is the upper half-plane) and characterized functions in it by their integrals against cusp forms and Eisenstein series. Their formulation and their proof both depended strongly on properties of the Riemann ζ-function, and it was not at all apparent how to generalize their results to other than congruence subgroups. In fact, their dependence on properties of $\zeta(s)$ disguised the essentially simple nature of the problem. I was able to find a generalization of their result in [Casselman:1984], in the course of trying to understand the relationship between Paley-Wiener theorems and cohomology. In spite of the title of that paper, the arguments there are valid for an arbitrary arithmetic subgroup acting on $\mathcal{H}$, and indeed only a few slight modifications would be required in order to deal with arbitrary arithmetic groups of rational rank one.

In this paper, I will prove a slightly stronger result than that in [Casselman:1984], but by methods which I have developed in the meantime to apply also to groups of higher rank. The point is not so much to prove the new result itself, which could have been done by the methods of [Casselman:1984], but to explain how the new methods work in a simple case.

What is new here? • In the Paley-Wiener theorem I envisage in general, a crucial role is played by a square-integrability condition on the critical line. In the earlier work, I followed Langlands' argument in shifting contours *in towards* the critical line to deal with square-integrability before I moved contours *out from* the critical line in order to derive estimates on the growth of certain functions near a cusp. This duplication of effort was annoying. Since then, in [Casselman:1999], I have been able to obtain with no contour movement a Plancherel theorem which implies the square-integrability condition directly. • Both here and earlier I move contours in order to evaluate a certain constant term. The most difficult point in this is to take the first nearly infinitesimal step off the critical line. In the earlier paper I used a very special calculation (the Maass-Selberg formula) to do that. That argument, although surprisingly elementary (depending only on the integrability of $1/\sqrt{x}$ near $x = 0$) will unfortunately not work in all situations which arise for groups of higher rank. In this paper I replace that argument by a very general one, one closely related to a more or less standard one in the theory of Laplace transforms of distributions.

Roughly speaking, the arguments of this series differ from Langlands' in that whereas he moved contours to evaluate the inner product of two Eisenstein series of functions of compact support, I move them to evaluate a constant term. There are many virtues to this new technique, but most of them will appear clearly only for groups of higher rank. One virtue, likely to be appreciated by those familiar with Langlands' work, is that in the new arguments each Eisenstein series residue actually contributes to the spectrum, whereas in Langlands' argument (illustrated by his well known example of G_2) there occurs a certain complicating cancellation of residues which makes it difficult to understand their significance. I would like to think that the new arguments will eventually make it possible to calculate residues of Eisenstein series by computer, something not easy to see how to do at the moment.

1. Introduction

Let

$G =$ $\mathrm{SL}_2(\mathbb{R})$
$P =$ the subgroup of upper triangular matrices in G
$N =$ the subgroup of unipotent matrices in P
$A =$ the group of positive diagonal matrices in G, which may be identified with the multiplicative group $\mathbb{R}^{\mathrm{pos}}$
$K =$ the maximal compact subgroup of rotation matrices

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

I assume Γ to be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ of 'arithmetic type'. In addition, a few extra conditions will be put on Γ in order to simplify the argument without significant loss of generality. The precise assumptions we make on Γ are:

- The group Γ has a single cusp at ∞;
- the intersection $\Gamma \cap P$ consists of all matrices of the form

$$\begin{bmatrix} \pm 1 & n \\ 0 & \pm 1 \end{bmatrix}$$

where n varies over all of $\mathbb{Z}$.

The effect of these assumptions is to allow a reasonable simplification in notation, without losing track of the most important ideas. Of course there is at least one group satisfying these conditions, namely $\mathrm{SL}_2(\mathbb{Z})$.

Let $\mathcal{H}$ be the upper half-plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. The group G acts on it on the left:

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$

This action preserves the non-Euclidean metric $(dx^2 + dy^2)/y^2$ and the non-Euclidean measure $dx\,dy/y^2$ on $\mathcal{H}$.

The group $\Gamma \cap P$ stabilizes each domain $\mathcal{H}_T = \{y \geq T\}$, and for T large enough the projection from $\Gamma \cap P\backslash\mathcal{H}_T$ to $\Gamma\backslash\mathcal{H}$ is injective. Its image is a neighbourhood of the cusp ∞. That Γ has a single cusp means that the complement of this image is compact.

The isotropy subgroup of i in $\mathcal{H}$ is the subgroup K, and the quotient G/K may therefore be identified with $\mathcal{H}$. Let G_T be the inverse image of $\mathcal{H}_T$, which consists of those g in G with an Iwasawa factorization

$$g = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} k$$

where $a^2 > T$ and k lies in K. For large T the quotient $\Gamma \cap P\backslash G_T$ embeds into $\Gamma\backslash G$ with compact complement.

In the rest of this paper, T will be assumed to be large enough so that this embedding occurs.

The area of $\Gamma \cap P\backslash\mathcal{H}_T$ can be calculated explicitly, and it is finite. Because the complement is compact, the area of $\Gamma\backslash\mathcal{H}$ and the volume of $\Gamma\backslash G$ are both finite as well. Define δ to be the function pulled back from the y-coordinate on $\mathcal{H}$. In terms of the Iwasawa factorization $G = NAK$, $\delta(nak) = \delta(a) = \left|\det \mathrm{Ad}_{\mathfrak{n}}(a)\right|$, where

$$\delta(a) = x^2 \;\text{ if }\; a = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} .$$

The character δ is also the modulus character of P.

The length of the non-Euclidean circle around i and passing through iy is $(y - y^{-1})/2$. This means that in terms of the Cartan factorization $G = KAK$ we have an integral formula

$$\int_G f(g)\, dg = \int_K \int_A \int_K f(k_1 a k_2) \left(\frac{\delta(a) - \delta^{-1}(a)}{2} \right) dk_1\, da\, dk_2$$

with a suitable measure assigned to K. On G we define the norm

$$\|k_1 a k_2\| = \max |\delta(a)|, |\delta(a)|^{-1} .$$

This is the same as

$$\sup_{\|v\|=1} \|gv\| \qquad (v \in \mathbb{R}^2)$$

and satisfies the inequality

$$\|gh\| \le \|g\|\,\|h\| .$$

The function $\|g\|^{-(1+\epsilon)}$ is then integrable on G for $\epsilon > 0$.

A function f on $\Gamma\backslash G$ is said to be of **moderate growth** at ∞ if $f = O(\delta^m)$ on the regions G_T for some integer $m > 0$, and **rapidly decreasing** at ∞ if it is $O(\delta^{-m})$ for all m. The **Schwartz space** $\mathcal{S}(\Gamma\backslash G)$ is that of all smooth right-K-finite functions f on $\Gamma\backslash G$ with all $R_X f$ $(X \in U(\mathfrak{g}))$ rapidly decreasing at ∞. Because of the condition of K-finiteness, any function in $\mathcal{S}(\Gamma\backslash G)$ may be expressed as a finite sum of components transforming on the right by a character χ of K:

$$\begin{aligned} \mathcal{S}(\Gamma\backslash G) &= \oplus\, \mathcal{S}(\Gamma\backslash G)_\chi \\ \mathcal{S}(\Gamma\backslash G)_\chi &= \{ f \in \mathcal{S}(\Gamma\backslash G) \mid f(gk) = \chi(k) f(g) \text{ for all } k \in K, g \in G \} . \end{aligned}$$

If $\chi = 1$, we are looking at functions on $\Gamma\backslash\mathcal{H}$.

The problem that this paper deals with is how to characterize the functions in the Schwartz space by their integrals against various automorphic forms, and especially Eisenstein series. There are technical reasons why the Paley-Wiener theorem for $\mathcal{S}(\Gamma\backslash\mathcal{H})$ is simpler than the one for $\mathcal{S}(\Gamma\backslash G)$, and for that reason I will discuss the first case in detail, then go back and deal with the extra complications needed to deal with $\mathcal{S}(\Gamma\backslash G)$. But in order to give an idea of what's going on, I'll explain in the next section a few of the simplest possible Paley-Wiener theorems.

2. The simplest Paley-Wiener theorems

It will be useful to keep in mind a few elementary theorems of the kind we are looking for.

(1) Define the Schwartz space $\mathcal{S}$ of the group A, identified here with the multiplicative group of positive real numbers $\mathbb{R}^{\mathrm{pos}}$, to be made up of those smooth functions $f(x)$ on A satisfying the condition that it and all its derivatives vanish rapidly at 0 and ∞ in the sense that

$$\left| f^{(n)}(x) \right| = O(x^m)$$

for all non-negative integers n and all integers m, whether positive or negative. Then for all s in $\mathbb{C}$ we can define the Fourier transform

$$F(s) = \widehat{f}(s) = \int_0^\infty f(x) x^{-s} \, \frac{dx}{x} \, .$$

It turns out to be holomorphic in all of $\mathbb{C}$, and since the Fourier transform of the multiplicative derivative $x\, df/dx$ is $s\widehat{f}$ it satisfies the growth condition

$$F(\sigma + it) = O\left(\frac{1}{1 + |t|^m} \right)$$

for all $m > 0$, uniformly in vertical bands of finite width. Conversely, if $F(s)$ is any entire function satisfying these growth conditions, then for any real σ

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) x^s \, dx$$

will be a function in $\mathcal{S}(A)$, independent of σ, whose Fourier transform is F. The proof depends on a clearly justifiable shift of contour of integration.

(2) A second result will turn out to look even more similar to that for arithmetic quotients. Define $L^{2,\infty}_{\mathcal{S}}(A)$ to be the space of all smooth functions f on $(0, \infty)$ such that (a) f and all its derivatives vanish of infinite order at 0; (b) f and all its multiplicative derivatives are square-integrable on A. Condition (a) implies that the Fourier transform is defined and holomorphic in the region $\Re(s) > 0$. On the other hand, condition (b) implies that the Fourier transform of f on the line $\Re(s) = 0$ exists as a square-integrable function. The relationship between the two definitions of $F(s)$ on $\Re(s) = 0$ and $\Re(s) > 0$ is that uniformly on bounded horizontal strips the function $F(\sigma + it)$ approaches the function $F(it)$ in the L^2-norm. In these circumstances we have the following result:

Theorem. *If $f(x)$ lies in $L^{2,\infty}_{\mathcal{S}}$ then its Fourier transform $F(s)$ satisfies the following conditions:*

- *$F(s)$ is holomorphic in the half plane $\Re(s) > 0$;*
- *it satisfies the condition*

$$F(\sigma + it) = O\left(\frac{1}{\sqrt{\sigma}(1 + |t|)^n} \right)$$

for $\sigma > 0$ and all $n > 0$, uniformly on horizontally bounded vertical strips;
- *the restriction of F to $\Re(s) = 0$ is square-integrable, and the weak limit of the distributions*

$$F_\sigma(it) = F(\sigma + it)$$

as $\sigma \to 0$.

Conversely, if $F(s)$ is a function satisfying these conditions then it is the Fourier transform of the function

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) x^s \, ds$$

which does not depend on the choice of $\sigma > 0$, and which lies in $L^{2,\infty}_{\mathcal{S}}$.

The natural proof of this relies on results from the last section of this paper, and much of the argument duplicates what I shall say about the analogous (and more difficult) result for the upper half plane. I leave it as an exercise.

3. Quotients of the upper half-plane

For the group $\mathbb{R}^{\mathrm{pos}}$, the results stated in the previous section are just some of many analogous results, most notably one characterizing functions of compact support by their Fourier transforms. But for quotients of symmetric spaces by arithmetic subgroups I do not know whether a result for functions of compact support is possible even in principle. A Paley-Wiener theorem for functions of rapid decrease may be the only natural one to consider.

In this section I'll explain the Paley-Wiener theorem for $\mathcal{S}(\Gamma\backslash\mathcal{H})$. The definition of the space $\mathcal{S}(\Gamma\backslash\mathcal{H})$ involves lifting a function f on $\Gamma\backslash\mathcal{H}$ to a function F on $\Gamma\backslash G$ and then considering the right derivatives $R_X F$. But the functions in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ may be more concretely identified with those smooth functions on $\Gamma\backslash\mathcal{H}$ satisfying the condition that

$$\Delta^n f = O(y^{-m})$$

for all positive integers n and m, where Δ is the non-Euclidean Laplace operator

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) .$$

This definition of the Schwartz space is the one used by Ehrenpreis and Mautner, and in my 1984 paper (Proposition 2.3) I showed that this notion is equivalent to the one given earlier. That equivalence will not play a role here except in so far as it ties the result of Ehrenpreis and Mautner to mine.

Any smooth function $f(z)$ on $\Gamma\backslash\mathcal{H}$ may be expanded in a Fourier series

$$f(x+iy) = \sum_{-\infty}^{\infty} f_n(y) e^{2\pi i n x} .$$

If f is any smooth function on $\Gamma\backslash\mathcal{H}$ which is of **uniform moderate growth** in the sense that for some fixed m

$$\Delta^n f = O(y^m)$$

for all $n > 0$, then all coefficient functions $f_n(y)$ for $n \neq 0$ vanish rapidly as $y \to \infty$, and more generally the difference between $f(y)$ and $f_0(y)$ also vanishes rapidly. In other words, the asymptotic behaviour of $f(y)$ as $y \to \infty$ is controlled by the **constant term** $f_0(y)$. Furthermore, as we shall see later, the Schwartz space decomposes into a sum of two large pieces—the **cuspidal** component, that of functions whose constant terms vanish identically, and the **Eisenstein** component orthogonal to the cuspidal one. The cuspidal component is a discrete sum of eigenspaces of the Laplace operator, and is of no particular interest in this discussion.

The spectrum of Δ is continuous on the Eisenstein component. The functions which for $\Gamma\backslash\mathcal{H}$ play the role of the characters x^s on $\mathbb{R}^{\mathrm{pos}}$ are the **Eisenstein series**. For every s with $\Re(s) > 1$ the series

$$E_s(z) = \sum_{\Gamma\cap P\backslash\Gamma} y(\gamma(z))^s$$

converges to an eigenfunction of Δ on $\Gamma\backslash\mathcal{H}$, with eigenvalue

$$\Delta(s) = s(s-1) = (s-1/2)^2 - 1/4 .$$

When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, for example, this series was first defined by Maass, and can be expressed more explicitly as

$$E_s(z) = \sum_{c>0,\, \gcd(c,d)=1} \frac{y^s}{|cz+d|^{2s}} .$$

For all Γ, the function E_s continues meromorphically in s to all of $\mathbb{C}$. In the right-hand half-plane $\Re(s) \geq 1/2$ there is always a simple pole at $s = 1$, and there may be a few more simple poles on $(1/2, 1)$. The constant term of E_s is of the form

$$y^s + c(s) y^{1-s}$$

where $c(s)$ is a meromorphic function on $\mathbb{C}$. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

$$c(s) = \frac{\xi(2s-1)}{\xi(2s)}, \quad \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \ .$$

In this case the behaviour of E_s for $\Re(s) < 1/2$ is therefore related to the Riemann hypothesis, and ought to be considered, whenever possible, as buried inside an impenetrable box. The function E_s satisfies the functional equation

$$E_s = c(s) E_{1-s}$$

so that s and $1-s$ contribute essentially the same automorphic forms to $\Gamma\backslash\mathcal{H}$. From this equation for E_s it follows that $c(s)$ satisfies the functional equation

$$c(s)c(1-s) = 1 \ .$$

In the region $\Re(s) > 1/2$, $s \notin (1/2, 1]$ the Eisenstein series can be constructed by a simple argument relying only on the self-adjointness of the operator Δ on $\Gamma\backslash\mathcal{H}$. The rough idea is this:

Let $\chi(y)$ be a function on $(0, \infty)$ which is identically 1 for large y, and non-vanishing only for large y. The product $\chi(y)y^s$ may be identified with a function Y_s on $\Gamma\backslash\mathcal{H}$. Choose s such that $\Re(s) > 1/2$, $s \notin (1/2, 1]$, and let $\lambda = s(s-1)$. Then $X_s = (\Delta - \lambda)Y_s$ will have compact support on $\Gamma\backslash\mathcal{H}$, since $\Delta y^s = \lambda y^s$. For $\lambda \notin (-\infty, 0]$ (the spectrum of Δ) let

$$F_s(z) = -(\Delta - \lambda)^{-1} X_s \ .$$

Then

$$E_s(z) = F_s(z) + Y_s(z) \ .$$

In other words, for s in this region the function E_s is uniquely determined by the conditions that (a) $\Delta E_s = \lambda E_s$ and (b) $E_s - y^s$ is square-integrable near ∞. (This is explained in more detail in [Colin de Verdière:1981].) The theory of self-adjoint operators also guarantees that

$$\begin{aligned} \|\Delta - \lambda\|^2 &= \|\Delta - \Re(\lambda)\|^2 + |\Im(\lambda)|^2 \\ \|\Delta - \lambda\| &\geq |\Im(\lambda)| \\ &= |2\sigma t| \quad (s = 1/2 + \sigma + it) \\ \|\Delta - \lambda\|^{-1} &\leq |2\sigma t|^{-1} \ , \end{aligned}$$

which implies that $\|F_s\| = O(|2\sigma t|^{-1})$.

For T large enough we can define the **truncation** of an automorphic form $F(z)$ in the region $y \geq T$. On the quotient $\Gamma\backslash\mathcal{H}_T$ the truncation $\Lambda^T F$ is the difference between F and its constant term. Because the asymptotic behaviour of F is controlled by its constant term, this is always square-integrable. For Eisenstein series there exists the explicit **Maass-Selberg formula** for the inner product of two truncations. For generic s and t it asserts that

$$\langle \Lambda^T E_s, \Lambda^T E_t \rangle = \frac{T^{s+t-1} - c(s)c(t)T^{1-s-t}}{s+t-1} - \frac{c(s)T^{1-s+t} - c(t)T^{1-t+s}}{s-t} \ .$$

Formally, the expression on the right is

$$\int_0^T \left(y^s + c(s)y^{1-s}\right)\left(y^t + c(t)y^{1-t}\right) y^{-2}\, dy \ .$$

This apparent accident is explained in [Casselman:1993]. When $s = 1/2 + \sigma + i\tau$ and $t = \overline{s}$ it becomes

$$\|\Lambda^T E_s\|^2 = \frac{T^{2\sigma} - |c(s)|^2 T^{-2\sigma}}{2\sigma} - \frac{c(s)T^{-2i\tau} - c(t)T^{2i\tau}}{2i\tau} .$$

This formula makes more precise the idea that the behaviour of E_s is determined by that of $c(s)$, and *vice versa*. That this must always be positive, for example, implies that $|c(s)|$ must be bounded at $\pm i\infty$ in the region $\sigma > 0$, and that the poles of E_s and $c(s)$ have to be simple in that region. (See [Langlands:1966], or the proof of Proposition 3.7 in [Casselman:1984] for more detail.)

The **Fourier-Eisenstein transform** of f in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ is

$$F(s) = \widehat{f}(s) = \int_{\Gamma\backslash\mathcal{H}} f(z) E_{1-s}(z)\, \frac{dx\, dy}{y^2} .$$

It follows immediately from properties of E_s that

(PW1) *The function $F(s)$ satisfies the functional equation*

$$F(1-s) = c(s)F(s) .$$

(PW2) *The function $F(s)$ is meromorphic everywhere in $\mathbb{C}$, holomorphic in the half-plane $\Re(s) \leq 1/2$ except for possible simple poles in $[0, 1/2)$ corresponding to those of E_{1-s}.*

There are also a few other significant and more subtle properties of $F(s)$.

(PW3) *The function $F(s)$ is square-integrable on $\Re(s) = 1/2$.*

(PW4) *In any region*

$$\sigma_0 < \Re(s) < 1/2, \quad |\Im(s)| > \tau$$

we have for all $m > 0$

$$|F(s)| = O\left(\frac{1}{|1/2 - \sigma|\, |t|^m}\right) \quad (s = \sigma + it) .$$

The first follows from the following result, a Plancherel formula for the critical line, which is far more basic:

- *For $\Phi(s)$ a function of compact support on the line $\Re(s) = 1/2$, the integral*

$$E_\Phi(z) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) E_s(z)\, ds$$

defines a square-integrable function on $\Gamma\backslash\mathcal{H}$ with

$$\frac{1}{2}\, \|E_\Phi\|^2 = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} |\Phi(s)|^2\, ds .$$

This is well known. The usual proof (as in [Langlands:1966]) relies on contour movement, but in [Casselman:1999] it is proven directly. At any rate, given this, we can verify property (PW3). First of all it implies that E_Φ can be defined as an L^2 limit for arbitrary functions in $L^2(1/2 + i\mathbb{R})$. Second, for f in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ we can calculate that

$$\begin{aligned} \|f\|\, \|E_\Phi\| &\geq \langle f, E_\Phi \rangle \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) \langle f, E_s \rangle\, ds \\ &= \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi(s) \widehat{f}(1-s)\, ds \end{aligned}$$

for any square-integrable Φ, which implies that $\widehat{f}$ itself is square-integrable.

As for property (PW4), it can be proven either from the spectral inequality mentioned above, or from the Maass-Selberg formula. This property is used in moving contours of integration; the second proof, which asserts a more precise result than the other, allows an elementary argument in doing this (see [Casselman:1984]), but in higher rank becomes invalid. The first is therefore preferable. In both proofs, we begin by writing

$$\langle f, E_s \rangle = \langle f, \Lambda^T E_s \rangle + \langle f, C^T E_s \rangle$$

and arguing separately for each term. For the first term, use the spectral construction of E_s described earlier. An estimate for the second term follows easily from an argument about the multiplicative group.

Theorem. *Let $F(s)$ be any function on $\mathbb{C}$ such that all $\Delta^n(s)F(s)$ satisfy conditions (PW1)–(PW4), and for each pole s in $[0, 1/2)$ let $F^\#(s)$ be the residue of F there. Then*

$$f(z) = -\sum F^\#(s)E_s + \frac{1}{2}\,\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(s)E_s(z)\,ds$$

lies in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ and has Fourier-Eisenstein transform F.

Note that because $c(s)c(1-s) = 1$, if c has a pole at $1-s$ then $c(s) = 0$, E_s is well defined, and its constant term is exactly y^s. The integral is to be interpreted as the limit of finite integrals

$$\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} F(s)E_s\,ds$$

which exists as a square-integrable function on $\Gamma\backslash\mathcal{H}$ by the Plancherel formula explained above. In fact, it lies in the space $A_{\mathrm{umg}}(\Gamma\backslash\mathcal{H})$. This is proven directly in [Casselman:1984], but follows easily from extremely general reasoning about $L^{2,\infty}(\Gamma\backslash G)$ (Theorem 1.16 and Proposition 1.17 of [Casselman:1989]). This argument is recalled in a simplified form later in this paper.

What this means is that in order to determine whether $f(x)$ lies in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ we can look at its constant term.

The constant term of the integral is

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left[\frac{F(s)y^s + c(s)F(s)y^{1-s}}{2}\right] ds$$

(suitably interpreted as a limit) which is equal to

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left[\frac{F(s)y^s + c(1-s)F(1-s)y^s}{2}\right] ds = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(s)y^s\,ds$$

by (PW2). The most difficult step in the whole proof is to justify replacing the integral

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} F(s)y^s\,ds$$

by the integral

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)y^s\,ds$$

for some number σ very close to $1/2$. This can be done by the results in the final section of this paper. Once this step has been taken, the growth conditions on $F(s)$ in vertical bands allow us to move arbitrarily far to the left, picking up residues as we go. Recall that the constant term of E_s is y^s at a pole of $F(s)$. These residues cancel out with the residues in the formula for $f(z)$. Therefore the constant term of $f(z)$ is equal to

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)y^s\,ds$$

for arbitrary $\sigma \ll 0$, which implies that it vanishes rapidly as $y \to \infty$. A classical result from the theory of the Laplace transform finishes off the Proposition.

4. The constant term

In this section, I begin consideration of $\Gamma\backslash G$ instead of $\Gamma\backslash\mathcal{H}$. Some points are simpler, and in fact some of the claims for $\Gamma\backslash\mathcal{H}$ are best examined in the current context. The principal complication is that notation is more cumbersome.

For any reasonable function f on $\Gamma\backslash G$ define its **constant term** to be the function on $N(\Gamma\cap P)\backslash G$ defined by the formula

$$f_P(g) = \int_{\Gamma\cap N\backslash N} f\,(xg)\; dx\;.$$

If f lies in $\mathcal{S}(\Gamma\backslash G)$ then f_P will be bounded on all of $N(\Gamma\cap P)\backslash G$ and in addition satisfy an inequality

$$R_X f(g) = O(\delta(g)^{-m})$$

on G_T, for all X in $U(\mathfrak{g})$ and $m > 0$.

Define $A_{\mathrm{umg}}(\Gamma\backslash G)$ to be the space of all functions of **uniform** moderate growth on $\Gamma\backslash G$—those smooth functions F for which there exists a single $m > 0$ with

$$|R_X F(g)| = O(\delta(g)^{m})$$

on G_T, for all X in $U(\mathfrak{g})$.

If F lies in $A_{\mathrm{umg}}(\Gamma\backslash G)$ and f lies in $\mathcal{S}(\Gamma\backslash G)$ then the product Ff will lie in $\mathcal{S}(\Gamma\backslash G)$, and hence may be integrated. The two spaces are therefore in duality. It is shown in [Casselman:1989] that the space $A_{\mathrm{umg}}(\Gamma\backslash G)$ may be identified with the Gårding subspace of the dual of $\mathcal{S}(\Gamma\backslash G)$, the space of **tempered distributions** on $\Gamma\backslash G$.

For large T, the **truncation** $\Lambda^T F$ of a continuous function F on $\Gamma\backslash G$ at T is what you get from F by chopping away its constant term on G_T. More precisely, if Φ is any function on $N(\Gamma\cap P)\backslash G$ define $C^T\Phi$ to be the product of Φ and the characteristic function of G_T, and then for F on $\Gamma\backslash G$ set

$$C^T F(g) = \sum_{\Gamma\cap P\backslash\Gamma} C^T F_P(\gamma g)$$

$$\Lambda^T F = F - C^T F\;.$$

The sum $F = \Lambda^T F + C^T F$ is orthogonal.

One of the basic results in analysis on $\Gamma\backslash G$ is that

- *if F lies in $A_{\mathrm{umg}}(\Gamma\backslash G)$ then $\Lambda^T F$ is rapidly decreasing at ∞.*

5. Analysis on $\mathbf{N(\Gamma\cap P)\backslash G}$

The space $N(\Gamma\cap P)\backslash G$ plays the same role for $\Gamma\backslash G$ that $A \cong N(\Gamma\cap P)\backslash G/K$ plays for $\Gamma\backslash\mathcal{H}$. And analysis on $N(\Gamma\cap P)\backslash G$ still looks much like analysis on the multiplicative group $\mathbb{R}^{\mathrm{pos}}$. One can be phrased literally in terms of the other since we can look at irreducible K-eigenspaces, and $N\backslash G/K \cong A/\{\pm1\} \cong \mathbb{R}^{\mathrm{pos}}$.

For each s in $\mathbb{C}$ define the space

$$I_s = \{f \in C^\infty(G) \mid f \text{ is } K\text{-finite}, f(pg) = \delta^s(p) f(g) \text{ for all } p \in P, g \in G\}\;.$$

Right derivation makes this into the **principal series** representation of $(\mathfrak{g}, K)$ parametrized by the character $p \mapsto \delta^s(p)$. It has a basis made up of functions $f_{n,s}$ where

$$f_{n,s}(pk) = \delta^s(p)\varepsilon^n(k)$$

where

$$\varepsilon\colon \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \longmapsto c + is \,.$$

If

$$\begin{aligned} \kappa &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ X_+ &= (1/2) \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \\ X_- &= (1/2) \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \end{aligned}$$

then on I_s

$$\begin{aligned} R_\kappa f_{n,s} &= ni f_{n,s} \\ R_{X_+} f_{n,s} &= (s + n/2) f_{n+2,s} \\ R_{X_-} f_{n,s} &= (s - n/2) f_{n-2,s} \,. \end{aligned}$$

These are generators of $U(\mathfrak{g})$, and therefore *every element of* $U(\mathfrak{g})$ *acts on* I_s *by a polynomial function of* s.

The representation of $(\mathfrak{g}, K)$ on I_s is irreducible for almost all s, and the Casimir operator $\mathfrak{C}$ acts as the scalar $\Delta(s) = s(s-1)$ on it. Elements of I_0 may be identified with functions on $\mathbb{P}^1(\mathbb{R})$, those of I_1 with smooth 1-densities on $\mathbb{P}^1(\mathbb{R})$. With a suitable choice of measures, the integral formula

$$\int_{N(\Gamma\cap P)\backslash G} f(x)\, dx = \int_{P\backslash G} \overline{f}(x)\, dx$$

is valid, where

$$\overline{f}(x) = \int_A \delta_P^{-1}(a) f(ax)\, da$$

lies in I_1.

The product of an element of I_s and one in I_{1-s} lies in I_1, and may then be integrated. The space I_{1-s} is therefore the contragredient of I_s. If $\Re(s) = 1/2$ so that $s = 1/2 + it$, then $1 - s = 1/2 - it = \overline{s}$; the representation of $(\mathfrak{g}, K)$ on I_s is therefore unitary.

Let

$$\mathcal{I} = \text{ the space of } K\text{-finite functions on } K \cap P\backslash K \,.$$

Since $G = PK$, restriction to K is a K-covariant isomorphism of I_s with $\mathcal{I}$. Thus as vector spaces and as representations of K, all the I_s may be identified with each other. It therefore makes sense to say that they form a holomorphic family, or that the representation of $\mathfrak{g}$ varies holomorphically with s. Restriction to K can be used to define a norm on the I_s. For f in I_s with the decomposition $f = \sum f_\chi$ into K-components, define

$$\|f\|^2 = \int_{K\cap P\backslash K} |f(k)|^2\, dk = \sum_\chi \|f_\chi\|^2 \,.$$

For $\Re(s) = 1/2$, $\|f\|$ is the same as the norm induced by the identity of I_{1-s} with the contragredient of I_s, the G-invariant Hilbert space norm on I_s.

Fourier analysis decomposes functions on $N(\Gamma\cap P)\backslash G$ into its components in the spaces I_s. As with classical analysis, there are several variants.

◇ *A Paley-Wiener theorem.* Suppose φ to be a smooth K-finite function on $N(\Gamma \cap P)\backslash G$ which is rapidly decreasing at infinity on $N(\Gamma \cap P)\backslash G$ in both directions, in the sense that for any integer m whatsoever (positive or negative) $R_X\varphi = O(\delta^m)$ for all X in $U(\mathfrak{g})$. Then for any s in $\mathbb{C}$ we can define an element $\widehat{\varphi}_s$ in I_s by the condition

$$\langle \widehat{\varphi}_s, \psi \rangle = \int_{N(\Gamma\cap P)\backslash G} \varphi(x)\psi(x)\,dx$$

for each ψ in I_{1-s}. More explicitly, we can write the integral as

$$\int_{N(\Gamma\cap P)\backslash G} \varphi(x)\psi(x)\,dx = \int_{P\backslash G}\int_A \delta_P^{-1}(a)\varphi(ax)\psi(ax)\,da\,dx$$

so that

$$\widehat{\varphi}_s(g) = \int_0^\infty \delta(a)^{-s}\varphi(ag)\,da\ .$$

The function $\widehat{\varphi}_s$ will determine a section of I over all of $\mathbb{C}$, rapidly decreasing at $\pm i\infty$. We can recover φ from the functions $\widehat{\varphi}_s$ by the formula

$$\varphi(g) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma} \widehat{\varphi}_s(g)\,ds$$

for any real number σ.

If φ and ψ are two such functions on $N(\Gamma\cap P)\backslash G$ then their inner product can be calculated from their Fourier transforms by the formula

$$\int_{N(\Gamma\cap P)\backslash G} \varphi(g)\psi(g)\,dg = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \langle \widehat{\varphi}_s, \widehat{\psi}_{1-s}\rangle\,ds\ .$$

The map taking φ to $\widehat{\varphi}$ is an isomorphism of $\mathcal{S}(N(\Gamma\cap P)\backslash G)$ with that of all holomorphic sections Φ_s of I_s over all of $\mathbb{C}$ satisfying the condition that for all $m > 0$ we have

$$\|\Phi_{\sigma+it}\| = O\left(\frac{1}{1+|t|^m}\right)$$

uniformly on horizontally bounded vertical strips.

◇ *The Laplace transform.* Suppose φ to be a smooth function on $N(\Gamma\cap P)\backslash G$, such that each right derivative $R_X\varphi$ is bounded overall and rapidly decreasing at ∞ (but not necessarily at 0). These conditions are satisfied, for example, by the constant terms of functions in $\mathcal{S}(\Gamma\backslash G)$. Then for any s in $\mathbb{C}$ with $\Re(s) < 1/2$ the integral

$$\widehat{\varphi}_s(g) = \int_0^\infty \delta(a)^{-s}\varphi(ag)\,da$$

converges and defines a function in I_s. In other words, we now have a holomorphic section of I_s over the region $\Re(s) < 1/2$, which can reasonably be called the **Laplace transform** of φ. When only one K-component is involved, this amounts to the usual Laplace transform on the multiplicative group $\mathbb{R}^{\mathrm{pos}}$. Standard arguments from the theory of the Laplace transform on the multiplicative group of positive reals then imply that the function φ can be recovered from $\widehat{\varphi}$:

$$\varphi(g) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{\varphi}_s(g)\,ds$$

for all $\sigma < 1/2$. The integral over each line makes sense because under the assumptions on φ the magnitude of $\widehat{\varphi}(s)$ decreases rapidly at $\pm i\infty$. In particular, if $\widehat{\varphi}(s)$ vanishes identically, then $\varphi = 0$. This is a consequence of our assumption that $\Gamma\cap P$ contains ± 1—without this assumption we would have to take into account characters of A not necessarily trivial on ± 1.

In particular:

- *If f lies in $\mathcal{S}(\Gamma\backslash G)$ and the Laplace transform of f_P vanishes, then so does f_P.*

◇ *Square-integrable functions.* The map taking f in from $\mathcal{S}(N(\Gamma \cap P)\backslash G)$ to $\widehat{f}$ extends to an isomorphism of $L^2(N(\Gamma \cap P)\backslash G)$ (square-integrable half-densities) with the space $L^2(1/2 + i\mathbb{R}, \mathcal{I})$ of all square-integrable functions Φ on $1/2 + i\mathbb{R}$ with values in $\mathcal{I}$, i.e. those such that

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \|\Phi(s)\|^2 \, ds < \infty \, .$$

6. Eisenstein series

Suppose Φ to be an element of I_s with $\Re(s) > 1$. Then the **Eisenstein series**

$$E(\Phi) = \sum_{\Gamma \cap P\backslash\Gamma} \Phi(\gamma g)$$

will converge absolutely to a function of uniform moderate growth—in fact, an automorphic form—on $\Gamma\backslash G$.

Let ι_s be the identification of $\mathcal{I}$ with I_s, extending φ on $K \cap P\backslash K$ to $\varphi_s = \iota_s\varphi$ on $N(\Gamma \cap P)\backslash G$ where

$$\varphi_s(pk) = \delta^s(p)\varphi(k) \, .$$

Then the composite

$$E_s(\varphi) = E(\varphi_s)$$

will vary holomorphically for s with $\Re(s) > 1$.

The map

$$E_s \colon \mathcal{I} \to A_{\mathrm{umg}}(\Gamma\backslash G)$$

continues meromorphically to all of $\mathbb{C}$, defining where it is holomorphic a $(\mathfrak{g}, K)$-covariant map from I_s to $\mathcal{A}(\Gamma\backslash G)$. It is holomorphic in the region $\Re(s) \geq 1/2$ except for a simple pole at $s = 1$ and possibly a few more simple poles on the line segment $(1/2, 1)$.

The constant term of $E(\varphi_s)$ is for generic s a sum

$$\varphi_s + \tau(\varphi_s)$$

where τ is a covariant $(\mathfrak{g}, K)$ map from I_s to I_{1-s}. Let τ_s be the composite

$$\tau_s \colon \varphi \to \varphi_s \to \tau(\varphi_s)|K \, .$$

It is meromorphic in s. For $\Re(s) > 1/2$ and $s \notin [1/2, 1]$, the Eisenstein series $E(\varphi_s)$ is determined uniquely by the conditions that (1) near ∞ it is the sum of φ_s and something square-integrable; (2) it is an eigenfunction of the Casimir operator in $U(\mathfrak{g})$. As a result of uniqueness, the Eisenstein series satisfies a **functional equation**

$$E_s(\varphi) = E_{1-s}(\tau_s\varphi) \, .$$

In any event, the operator τ_s satisfies the condition $\tau_s\tau_{1-s} = 1$, and is a unitary operator when $\Re(s) = 1/2$. When $\varphi \equiv 1$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, as I have already mentioned, $\tau_s(\varphi)$ is related to the Riemann ζ function. In this case, the functional equation for the Eisenstein series is implied by—but does not imply—that for $\xi(s)$. Poles of E_s in the region $\Re(s) < 1/2$ will in this case arise from zeroes of $\zeta(s)$.

It is not important in this context to know exactly what happens to the left of the critical line $\Re(s) = 1/2$. This is just as well, because this is uncharted—and perhaps unchartable—territory.

The truncation $\Lambda^T E$ of any Eisenstein series E will be square-integrable. There is a relatively simple formula, called the **Maass-Selberg formula**, for the inner product of two of these. For generic values of s and t we have a formal rule

$$\langle \Lambda^T \Phi_s, \Lambda^T \Psi_t \rangle = -\int_{N(\Gamma\cap P)\backslash G_T} \langle \varphi_s, \psi_t \rangle \, dx$$

where Φ_s lies in the image of E_s, etc. and φ_s is its constant term. The integral is defined by analytic continuation and, if necessary, l'Hôpital's rule. If we take Ψ to be the conjugate of Φ, we get a formula for $\|\Lambda^T \Phi_s\|$.

If φ is K-invariant, there is always a pole of $E(\varphi_s)$ at $s=1$, and its residue is a constant function whose value is related to the volume of $\Gamma\backslash G$. For other K-eigenfunctions there will be no poles at $s=1$. These phenomena occur because the trivial representation of $(\mathfrak{g}, K)$ is a quotient of I_1 and embeds into I_0. Since I_s is irreducible for $1/2 < s < 1$, poles in $(1/2, 1)$ will occur simultaneously for all K-components of I_s.

7. The cuspidal decomposition

A function F on $\Gamma\backslash G$ is said to be **cuspidal** if its constant term vanishes identically. If F lies in $A_{\mathrm{umg}}(\Gamma\backslash G)$ and it is cuspidal then it will lie in $\mathcal{S}(\Gamma\backslash G)$. Define $\mathcal{S}_{\mathrm{cusp}}$ to be the subspace of cuspidal functions in $\mathcal{S}(\Gamma\backslash G)$.

If φ lies in $\mathcal{S}(N(\Gamma\cap P)\backslash G)$ then the Eisenstein series

$$E_\varphi(g) = \sum_{\Gamma\cap P\backslash\Gamma} \varphi(\gamma g)$$

will converge to a function in $\mathcal{S}(\Gamma\backslash G)$, and the map from $\mathcal{S}(N(\Gamma\cap P)\backslash G)$ to $\mathcal{S}(\Gamma\backslash G)$ is continuous. Define $\mathcal{S}_{\mathrm{Eis}}$ to be the closure in $\mathcal{S}(\Gamma\backslash G)$ of the image of $\mathcal{S}(N(\Gamma\cap P)\backslash G)$.

Proposition. *The Schwartz space $\mathcal{S}(\Gamma\backslash G)$ is the direct sum of its two subspaces $\mathcal{S}_{\mathrm{cusp}}$ and $\mathcal{S}_{\mathrm{Eis}}$.*

As a preliminary:

Lemma. *The space $L^{2,\infty}(\Gamma\backslash G)$ is contained in A_{umg}.*

I recall that the space $L^{2,\infty}$ is that of all functions Φ on $\Gamma\backslash G$ such that the distributional derivatives $R_X F$ ($X \in U(\mathfrak{g})$) are all square-integrable. It is to be shown that every Φ in $L^{2,\infty}$ is a smooth function on $\Gamma\backslash G$ and that for some single $m > 0$ independent of Φ we have

$$R_X \Phi(g) = O(\delta^m(g))$$

on G_T, for all $X \in U(\mathfrak{g})$.

A much more general result is proven in [Casselman:1989] (Proposition 1.16 and remarks afterwards), but circumstances here allow a simpler argument.

Proof of the Lemma. According to the Decomposition Theorem (see §1.2 of [Cartier:1974]) we can express the Dirac δ at 1 as

$$\delta_1 = \sum \xi_i * f_i$$

where ξ_i are in $U(\mathfrak{g})$, the f_i in $C_c^k(G)$, and k is arbitrarily high. As a consequence, every Φ in $L^{2,\infty}$ can be expressed as a sum of vectors $R_f F$, where F lies in $L^{2,\infty}$ and f in $C_c^k(G)$. Furthermore, if

$$\Phi = \sum R_{f_i} F_i$$

then

$$R_X \Phi = \sum R_{X f_i} F_i \; .$$

It therefore suffices to prove that for some $m > 0$, all f in $C_c(G)$, and all F in $L^{2,\infty}$ the convolution $R_f F$ is continuous on $\Gamma\backslash G$ and satisfies

$$R_f F(g) = O(\delta^m(g))$$

on G_T. On a fundamental domain of Γ, the function $\delta(g)$ and the norm $\|g\|$ are asymptotically equivalent, hence it is sufficient to verify

$$R_f F(g) = O(\|g\|^m)\ .$$

Formally we can write

$$\begin{aligned} R_f F(g) &= \int_G F(gx) f(x)\, dx \\ &= \int_{\Gamma\backslash G} F(y) \sum_\Gamma f(g^{-1}\gamma y)\, dy \\ &= \langle F, \Theta_{L_g f}\rangle \\ &\le \|F\|\, \|\Theta_{L_g f}\| \end{aligned}$$

where Θ is the map taking f in $C_c(G)$ to

$$\Theta_f(y) = \sum_\Gamma f(\gamma y)\ .$$

There are only a finite number of non-zero terms in this series, which therefore converges to a continuous function of compact support on $\Gamma\backslash G$, so the formal calculation at least makes sense.

Since

$$\|\Theta_f\| \le \mathrm{vol}(\Gamma\backslash G)^{1/2} \sup_{\Gamma\backslash G} \left|\Theta_f(x)\right|$$

we must find a bound on the values of Θ_f, and then see how the bound for $\Theta_{L_g f}$ changes with g.

Choose a compact open subgroup U such that

$$\Gamma \cap U^{-1} \cdot U = \{1\}$$

and let

$$\|U\| = \max_{u\in U} \|u\|\ .$$

Then for u in U, γ in Γ, x in G

$$\begin{aligned} \|u\gamma x\| &\le \|u\|\, \|\gamma x\| \\ &\le \|U\|\, \|\gamma x\| \\ \frac{1}{\|\gamma x\|} &\le \frac{\|U\|}{\|u\gamma x\|}\ . \end{aligned}$$

and

$$\sum_\Gamma \frac{1}{\|\gamma x\|^{1+\epsilon}} \le \frac{\|U\|^{1+\epsilon}}{\mathrm{meas}(U)} \int_{U\gamma x} \frac{1}{\|y\|^{1+\epsilon}}\, dy\ .$$

If $C_{f,\epsilon} = \max \|x\|^{1+\epsilon}\,|f(x)|$ then

$$\begin{aligned}\left|\Theta_f(x)\right| &\le \sum_\Gamma \left|f(\gamma x)\right| \\ &\le \sum_\Gamma \frac{C_{f,\epsilon}}{\|\gamma x\|^{1+\epsilon}} \\ &= C_{f,\epsilon} \sum_\Gamma \frac{1}{\|\gamma x\|^{1+\epsilon}} \\ &\le C_{f,\epsilon} \frac{\|U\|^{1+\epsilon}}{\mathrm{meas}(U)} \sum_\Gamma \int_{U\gamma x} \frac{1}{\|y\|^{1+\epsilon}}\,dy \\ &\le C_{f,\epsilon} \frac{\|U\|^{1+\epsilon}}{\mathrm{meas}(U)} \int_G \frac{1}{\|y\|^{1+\epsilon}}\,dy\end{aligned}$$

(since the $U\gamma x$ are disjoint) and

$$\left|\Theta_{L_g f}(x)\right| \le K\|g\|^{1+\epsilon} C_{f,\epsilon}$$

for a constant $K > 0$ depending only on ϵ. Everything we want to know follows from this. ▫

Proof of the Proposition. Let L^2_{cusp} be the subspace of functions in $L^2(\Gamma\backslash G)$ whose constant terms vanish, and L^2_{Eis} its orthogonal complement. Any f in $\mathcal{S}$ can be expressed as a sum of two corresponding components

$$f = f_{\mathrm{cusp}} + f_{\mathrm{Eis}}$$

where *a priori* each component is known only to lie in L^2. But the first component lies in $L^{2,\infty} \subseteq A_{\mathrm{umg}}$ and has constant term equal to 0, so lies itself in $\mathcal{S}$. Therefore the second does, too. This proves that

$$\mathcal{S} = \mathcal{S}_{\mathrm{cusp}} \oplus \left(\mathcal{S} \cap L^2_{\mathrm{Eis}}\right) .$$

It remains to be shown that the second component here is the closure of the functions $E(\varphi)$ with φ in $\mathcal{S}(N(\Gamma \cap P)\backslash G)$.

For this, because of the Hahn-Banach theorem, it suffices to show that if Φ is a tempered distribution which is equal to 0 on both $\mathcal{S}_{\mathrm{cusp}}$ and all the $E(\varphi)$, then it is 0. On the one hand, the constant term of Φ vanishes, and therefore so does that of every $R_f\Phi$, which since it lies in A_{umg} must also lie in $\mathcal{S}_{\mathrm{cusp}}$. But on the other hand, the orthogonal complement of $\mathcal{S}_{\mathrm{cusp}}$ is G-stable, so all these $R_f\Phi$ also lie in this complement. But since they themselves are cuspidal, they must vanish, too. However, Φ is the weak limit of $R_f\Phi$ if f converges weakly to the Dirac distribution δ_1. Therefore Φ itself vanishes. ▫

An analogous result for groups of arbitrary rank, essentially a reformulation of a result due to Langlands, is proven in [Casselman:1989].

8. Definition of the Fourier-Eisenstein transform

Suppose f to be in $\mathcal{S}(\Gamma\backslash G)$. For $s \in \mathbb{C}$ where the Eisenstein series map E_{1-s} is holomorphic, define its Fourier-Eisenstein transform $\widehat{f}(s)$ to be the unique element of I_s such that

$$\langle f, E_{1-s}(\varphi)\rangle = \langle \widehat{f}(s), \iota_s\varphi\rangle$$

for every φ in $\mathcal{I}$. The section $F = \widehat{f}$ of $\mathcal{I}$ is meromorphic in s and has poles where E_{1-s} does. It clearly satisfies this condition:

(PW1) $F(1-s) = \tau_s F(s)$

The next step is to investigate more carefully the singularities of $F(s)$. They will only occur at the poles of E_{1-s}. In the region $\Re(s) < 1/2$, which is all we will care about, they are simple. What can we say about its residues in that region?

(PW2) *The function $F(s)$ has simple poles on $[0, 1/2)$ where E_{1-s} does, and the residue $F^{\#}(s)$ at such a pole lies in the image of the residue of τ_{1-s}.*

Proof. For $\Re(s) < 0$ we a simple rearrangemnt of a converging series shows that

$$\langle f, E_{1-s}(\varphi)\rangle_{\Gamma\backslash G} = \langle f_P, \varphi_{1-s}\rangle_{N(\Gamma\cap P)\backslash G}$$

so that $\widehat{f} = 0$ if $f_P = 0$. The kernel of this transform is therefore precisely the subspace $\mathcal{S}_{\mathrm{cusp}}$ of 'The cuspidal decomposition', and the transform is completely determined by its restriction to $\mathcal{S}_{\mathrm{Eis}}$. The space $\mathcal{S}_{\mathrm{Eis}}$ is the closure of the image of the functions E_f for f in $\mathcal{S}(\Gamma\backslash G)$. Any particular K-constituent in I_s is finite-dimensional, so the image of all of $\mathcal{S}(\Gamma\backslash G)$ in I_s under the Fourier-Eisenstein transform is the same as the image of the functions E_f for f in $\mathcal{S}_{N(\Gamma\cap P)\backslash G}$.

If f lies in $\mathcal{S}_{N(\Gamma\cap P)\backslash G}$ we can express it as

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{f}(s)\, ds$$

for any real σ. If we choose $\sigma > 1$ this gives us

$$E_f = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E(\widehat{f}(s))\, ds$$

and then

$$\begin{aligned}
[E_f]_P &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E(\widehat{f}(s))_P\, ds \\
&= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{f}(s) + \tau_s \widehat{f}(s)\, ds \\
&= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{f}(s)\, ds + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \tau_s \widehat{f}(s)\, ds \\
&= \frac{1}{2\pi i}\int_{1-\sigma-i\infty}^{1-\sigma+i\infty} \widehat{f}(s) + \tau_{1-s}\widehat{f}(1-s)\, ds\ .
\end{aligned}$$

In the last step we move the contour of one integral and make a substritution of $1-s$ for s in the other. This implies that *the Fourier-Eisenstein transform of E_f is* $\widehat{f}(s) + \tau_{1-s}\widehat{f}(1-s)$. If we take residues of this expression at a pole, we obtain (PW2).

Keep in mind that since $\tau_s \tau_{1-s} = 1$, on this image E_s is well defined and $\tau_s = 0$. Hence the constant term of $E(F_s)$ will just be F_s itself.

9. The Plancherel theorem

Suppose φ_s to be a smooth function of compact support on the critical line $\Re(s) = 1/2$ with values in $\mathcal{I}$. Define the Eisenstein series E_φ to be

$$E_\varphi = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} E(\varphi_s)\, ds\ .$$

It will be a smooth function on $\Gamma\backslash G$. The **Plancherel Formula** for $\Gamma\backslash G$ asserts that it will be in $L^2(\Gamma\backslash G$ and that its L^2-norm will be given by the equation

$$\frac{1}{2}\, \|E_\varphi\|^2 = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \|\varphi_s\|^2\, ds\ .$$

As a consequence, the map $\varphi \mapsto E_\varphi$ extends to one from $L^2(1/2+i\mathbb{R})$ to $L^2(\Gamma\backslash G)$. The principal consequence of the Plancherel Theorem for our purposes is this:

(PW3) *For f in $\mathcal{S}(\Gamma\backslash\mathcal{H})$ the function $\widehat{f}(s)$ is square-integrable on $1/2+i\mathbb{R}$ in the sense that*

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \|\widehat{f}(s)\|^2\, ds < \infty$$

Proof. For φ of compact support

$$\begin{aligned}
\langle f, E_\varphi\rangle &= \frac{1}{2\pi i} \int_{\Re(s)=1/2} \langle f, E(\varphi_s)\rangle\, ds \\
&= \frac{1}{2\pi i} \int_{\Re(s)=1/2} \langle \widehat{f}_{1-s}, \varphi_{1-s}\rangle\, ds \\
&\le \|f\|\, \|E_\varphi\| \\
&= \frac{1}{2}\, \|f\|\, \|\varphi\|
\end{aligned}$$

so $\widehat{f}(s)$ extends to a continuous functional on $L^2(1/2+i\mathbb{R})$, and must itself lie in $L^2(1/2+i\mathbb{R})$ by Radon-Nikodym.

10. Spectral considerations

The Casimir operator is self-adjoint on any one K-component of $L^2(\Gamma\backslash\mathcal{H})$. A standard argument about self-adjoint operators implies that

$$\|\mathfrak{C} - \lambda\|^{-1} \le |\Im(\lambda)|^{-1}$$

and here

$$\Im(s(s-1)) = 2\sigma t, \quad s = 1/2 + \sigma + it$$
$$\|\mathfrak{C} - s(s-1)\|^{-1} \le \frac{1}{2|\sigma t|}\ .$$

The construction of Eisenstein series in, for example, [Colin de Verdière:1981] shows then that

$$\|\Lambda^T E(\varphi_s)\| = O\left(\frac{1}{2\sigma|t|}\right)\ .$$

Since we can write

$$E(\varphi_s) = \Lambda^T E(\varphi_s) + C^T(\varphi_s)$$

we have

$$\langle f, E(\varphi_{1-s})\rangle = \langle f, \Lambda^T E(\varphi_{1-s})\rangle + \langle f, C^T E(\varphi_{1-s})\rangle$$
$$|\langle f, E(\varphi_{1-s})\rangle| \le \|f\|\, \|\Lambda^T E(\varphi_{1-s})\| + |\langle f, C^T E(\varphi_{1-s})\rangle|\ .$$

The second term involves an easy calculation on $\mathbb{R}^{\mathrm{pos}}$, and since the same reasoning applies to all $\mathfrak{C}^n f$ we deduce

(PW4) *In any sub-region of $\Re(s) < 1/2, |\Re(s)| > \tau$ bounded to the left*

$$\|F(s)\| = O\left(\frac{1}{\sigma|t|^m}\right)$$

for all $m > 0$, where $s = 1/2 - \sigma + it$.

11. The Paley-Wiener theorem

If f lies in $\mathcal{S}$ then so does every $\mathfrak{C}^m f$. Define $PW(\Gamma\backslash G)$ to be the space of all meromorphic functions $F(s)$ with values in $\mathcal{I}$ such that every $\Phi(s) = \Delta(s)^m F(s)$ satisfies (PW1)–(PW4). These translate to the following conditions on $F(s)$ itself:

- $F(1-s) = \tau_s F(s)$
- *$F(s)$ has only simple poles on $[0, 1/2)$ in the region $\Re(s) \le 0$, located among the poles of E_{1-s}. The residue $F^{\#}(s)$ at s lies in the image of τ_{1-s}.*
- *The restriction of any $s^m F(s)$ to $(1/2 + i\mathbb{R})$ is square-integrable.*
- *In any region $s = 1/2 - \sigma + it$ with σ bounded, t bounded away from 0, we have*

$$\|F(s)\| = O\left(\frac{1}{\sigma|t|^m}\right) .$$

  for all $m > 0$.

For F in $PW(\Gamma\backslash G)$, let $F^{\#}(s)$ be its residue at any s in $[0, 1/2)$. Define

$$\mathcal{E}(F) = -\sum E_s(F^{\#}(s)) + \frac{1}{2}\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} E_s(F(s))\, ds .$$

Theorem. *(1) The map $\mathcal{E}$ has image in $\mathcal{S}(\Gamma\backslash G)$. (2) If $F = \widehat{f}$ then $\mathcal{E}(F)$ has the same constant term as f.*

Proof. It comes to showing that the constant term of $\mathcal{E}(F)$ is

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\, ds$$

for $\sigma \ll 0$. The crucial point, as before, is that we are allowed to move contours by the results of the last section.

12. Cusp forms

We now have a map from $\mathcal{S}(\Gamma\backslash G)$ to a space of meromorphic sections of $\mathcal{I}$ satisfying certain conditions, with an inverse map back from the space of such sections to $\mathcal{S}(\Gamma\backslash G)$. The kernel of this map is precisely the subspace of functions in $\mathcal{S}(\Gamma\backslash G)$ whose constant term vanishes identically. This is the subspace of **cusp forms**. We therefore have an explicit version of the direct sum decomposition

$$\mathcal{S}(\Gamma\backslash G) = \mathcal{S}_{\mathrm{cusp}} \oplus \mathcal{S}_{\mathrm{Eis}} .$$

The space of cusp forms is itself a direct sum of irreducible G-representations, each with finite multiplicity. If π is one of these components, then the map $f \mapsto \langle f, v\rangle$ $(v \in V_\pi)$ induces a map from $\mathcal{S}(\Gamma\backslash G)$ to the dual of a cuspidal representation π. The cuspidal component of $\mathcal{S}(\Gamma\backslash G)$ is a kind of Schwartz discrete sum of irreducible unitary representations of G. In order to say more we must know about the asymptotic distribution of cusp forms. But that is another story.

13. A calculus exercise

In the next section we shall need this result:

Lemma. *Suppose $f(x)$ to be a function in $C^{r+1}(0,\rho]$, such that for some κ_{r+1}*

$$\left|f^{(r+1)}(x)\right| \le \frac{\kappa_{r+1}}{x^r}$$

for all $0 < x \le \rho$. Then

$$f_0 = \lim_{x\to 0} f(x)$$

exists, and

$$|f_0| \le A\,\kappa_{r+1} + \sum_{0\le k\le r} \frac{\rho^k}{k!}|f^{(k)}(\rho)|$$

for some positive coefficient A independent of f.

In effect, the function $f(x)$ extends to a continuous function on all of $[0,\rho]$.

As an illustration of the Lemma, let $\ell(x) = x\log x - x$. We have on the one hand

$$\begin{aligned}
\ell(x) &= x\log x - x \\
\ell'(x) &= \log x \\
\ell''(x) &= \frac{1}{x} \\
\ell'''(t) &= -\frac{1}{x^2} \\
\ell^{(p)}(x) &= (-1)^p\,\frac{(p-2)!}{x^{p-1}} \\
\ell^{(r+1)}(x) &= (-1)^{r+1}\,\frac{(r-1)!}{x^r}\ ,
\end{aligned}$$

and on the other $\lim_{x\to 0}\ell(x) = 0$. The function $\ell(x)$ will play a role in the proof of the Lemma.

Proof. It is an exercise in elementary calculus. The cases $r = 0$, $r \ge 1$ are treated differently. Begin by recalling the elementary criterion of Cauchy: *If $f(x)$ is continuous in $(0,\rho]$ then $\lim_{x\to 0} f(x)$ exists if and only if for every $\varepsilon > 0$ we can find $\delta > 0$ such that $|f(y) - f(z)| < \varepsilon$ whenever $0 < y, z < \delta$.*

(1) *The case $r = 0$.* By assumption, f is C^1 on $(0,\rho]$ and f' is bounded by κ_1 on that interval. For any y, z in $(0,\rho]$.

$$f(z) - f(y) = \int_y^z f'(x)\,dx, \quad |f(z) - f(y)| \le \kappa_1\,|z - y|\ .$$

Therefore Cauchy's criterion is satisfied, and the limit $f_0 = \lim_{x\to 0} f(x)$ exists. Furthermore

$$\begin{aligned}
f_0 &= -f(\rho) + \int_0^\rho f'(x)\,dx \\
|f_0| &\le |f(\rho)| + \kappa_1\rho\ .
\end{aligned}$$

(2) *The case $r > 0$.* For any y in $(0,\rho]$ we can write

$$\begin{aligned}
f(\rho) - f(y) &= \int_y^\rho f'(x_1)\,dx_1 \\
f(y) &= -\int_y^\rho f'(x_1)\,dx_1 + f(\rho)\ .
\end{aligned}$$

We extend this by repeating the same process with $f'(x_1)$ etc. to get

$$\begin{aligned}
f'(x_1) &= -\int_{x_1}^{\rho} f''(x_2)\,dx_2 + f'(\rho) \\
f(y) &= -\int_{y}^{\rho} f'(x_1)\,dx_1 + f(\rho) \\
&= -\int_{y}^{\rho} \left(-\int_{x_1}^{\rho} f''(x_2)\,dx_2 + f'(\rho) \right) dx_1 + f(\rho) \\
&= \int_{y}^{\rho} \int_{x_1}^{\rho} f''(x_2)\,dx_2\,dx_1 + (y-\rho) f'(\rho) + f(\rho) \\
&= -\int_{y}^{\rho} \int_{x_1}^{\rho} \int_{x_2}^{\rho} f'''(x_3)\,dx_3\,dx_2\,dx_1 + \frac{(y-\rho)^2}{2} f''(\rho) + (y-\rho) f'(\rho) + f(\rho) \\
&= \quad \ldots \\
&= (-1)^p \int_{y}^{\rho} \ldots \int_{x_{p-1}}^{\rho} f^{(p)}(x_p)\,dx_p \ldots dx_1 \\
&\qquad + \frac{(y-\rho)^{p-1}}{(p-1)!} f^{(p-1)}(\rho) + \frac{(y-\rho)^{p-2}}{(p-2)!} f^{(p-2)}(\rho) + \cdots + f(\rho) \; .
\end{aligned}$$

This is the familiar calculation leading to Taylor series at ρ. If we apply this also to z in $(0,\rho]$ and set $p = r+1$ we get by subtraction

$$\begin{aligned}
f(y) - f(z) = (-1)^{r+1} \int_{y}^{z} \ldots \int_{x_r}^{\rho} & f^{(r+1)}(x_{r+1})\,dx_{r+1} \ldots dx_1 \\
& + \left[(y-\rho)^r - (z-\rho)^r \right] \frac{f^{(r)}(\rho)}{r!} \\
& + \left[(y-\rho)^{r-1} - (z-\rho)^{r-1} \right] \frac{f^{(r-1)}(\rho)}{(r-1)!} \\
& + \cdots + [y-z]\, f'(\rho)
\end{aligned}$$

In order to apply Cauchy's criterion, we must show how to bound

$$\left| \int_{y}^{z} \ldots \int_{x_r}^{\rho} f^{(r+1)}(x_{r+1})\,dx_{r+1} \ldots dx_1 \right| \le \int_{y}^{z} \ldots \int_{x_r}^{\rho} \frac{\kappa_{r+1}}{x_{r+1}^r}\,dx_{r+1} \ldots dx_1 \; .$$

We do not have to do a new calculation to find an explicit formula for the iterated integral

$$K_{y,z,r} = \int_{y}^{z} \ldots \int_{x_r}^{\rho} \frac{1}{x_{r+1}^r}\,dx_{r+1} \ldots dx_1 \; .$$

If we set $f = \ell$ above we get

$$\begin{aligned}
\ell(y) &= y \log y - y \\
&= \int_{y}^{\rho} \ldots \int_{x_r}^{\rho} \frac{(r-1)!}{x_{r+1}^r}\,dx_{r+1} \ldots dx_1 \\
&\qquad + \frac{(y-\rho)^r}{r!} \ell^{(r)}(\rho) + \frac{(y-\rho)^{r-1}}{(r-1)!} \ell^{(r-1)}(\rho) + \cdots + \ell(\rho) \; ,
\end{aligned}$$

so that

$$\begin{aligned} K_{y,\rho,r} &= \int_y^\rho \dots \int_{x_r}^\rho \frac{(r-1)!}{x_{r+1}^r}\, dx_{r+1} \dots dx_1 \\ &= \ell(y) - \frac{(y-\rho)}{r!}^r \ell^{(r)}(\rho) - \frac{(y-\rho)}{(r-1)!}^{r-1} \ell^{(r-1)}(\rho) - \dots - \ell(\rho)\ . \\ K_{y,z,r} &= K_{z,\rho,r} - K_{y,\rho,r} \end{aligned}$$

Since $\ell(x)$ is continuous on $[0,\rho]$ we may now apply Cauchy's criterion in the other direction to see that the limit f_0 exists. Furthermore, the bound on $f^{(r+1)}$ together with the equation for $f(y)-f(z)$ enable us to to see that

$$|f_0| \le \kappa_{r+1}\, |K_{0,\rho,r+1}| + \sum_0^r \frac{\rho}{k!}^k \left|f^{(k)}(\rho)\right|\ .$$

This concludes the proof of the Lemma.

14. Moving contours

In the proof of Paley-Wiener theorems for the Schwartz space of arithmetic quotients, it is necessary to allow a change of contour of integration which is not obviously justifiable. This is a consequence of the following very general result. In this paper I require only the special case $n=1$, but it is only slightly more difficult to deal with the general case, which will be needed for Paley-Wiener theorems for groups of higher rank.

For the next result, for $\varepsilon > 0$ let

$$\Sigma_\varepsilon = \{ s \in \mathbb{C}^n \mid 0 < \Re(s_i) < \varepsilon \}$$

and

$$\overline{\Sigma}_\varepsilon = \{ s \in \mathbb{C}^n \mid 0 \le \Re(s_i) < \varepsilon \}\ .$$

Theorem. *Suppose $\Phi(s)$ to be holomorphic in Σ_ε. Suppose that in addition that for some positive integers m and $r \ge 0$ it satisfy an inequality*

$$\Phi(\sigma + it) = O\left(\frac{1 + \|t\|^m}{\prod \sigma_i^r} \right)\ .$$

Thus for a fixed s in Σ_ε the function $t \mapsto \Phi(s+it)$ is of moderate growth and therefore defines by integration a tempered distribution Φ_s. For every s in the region $\overline{\Sigma}_\varepsilon$ the weak limit

$$\Phi_s = \lim_{x \in \Sigma_\varepsilon, x \to s} \Phi_x$$

exists as a tempered distribution. If the tempered distribution φ_0 is the inverse Fourier transform of Φ_0, then for every s in $\overline{\Sigma}_\varepsilon$ the product distribution $\varphi_s = e^{-\langle s, \bullet \rangle}\varphi_0$ is tempered and has Fourier transform Φ_s.

Proof. It is a straightforward modification of that of a similar result to be found on p. 25 in volume II of the series on methods of mathematical physics by by Mike Reed and Barry Simon (which also contains an implicit version of the Lemma in the previous section).

Suppose for the moment that $s=0$, and choose λ a real point in Σ_ε. Suppose $\Psi(t)$ to be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. For each x in $(0,1]$ let

$$f_\lambda(x) = \int_{\mathbb{R}^n} \Phi(x\lambda + it)\Psi(t)\, dt$$

i.e. integration against Ψ on the space $\Re(s) = x\lambda$. Then

$$\begin{aligned}
f'_\lambda(x) &= \int_{\mathbb{R}^n} \frac{d}{dx}\Phi(x\lambda + it)\Psi(t)\, dt \\
&= \int_{\mathbb{R}^n} \sum \lambda_k \left[\frac{\partial\, \Phi}{\partial s_k}\right](x\lambda + it)\Psi(t)\, dt \\
&= \int_{\mathbb{R}^n} \sum \lambda_k \frac{1}{i}\frac{\partial}{\partial t_k}\big[t \mapsto \Phi(x\lambda + it)\big]\Psi(t)\, dt \\
&= i\int_{\mathbb{R}^n} \Phi(x\lambda + it)\sum \lambda_k \frac{\partial\, \Psi(t)}{\partial t_k}\, dt \quad \text{(integration by parts)} \\
&= \int_{\mathbb{R}^n} \Phi(x\lambda + it) D_\lambda \Psi(t)\, dt
\end{aligned}$$

where

$$D_\lambda = i\sum \lambda_k \frac{\partial}{\partial t_k}\ .$$

Therefore for all p

$$f^{(p)}_\lambda(x) = \int_{\mathbb{R}^n} \Phi(x\lambda + it) D^p_\lambda \Psi(t)\, dt\ .$$

The assumptions on Φ and Ψ ensure that for all large integers k and suitable C_{m+k}

$$\begin{aligned}
\big|\Phi(x\lambda + it) D^p_\lambda \Psi(t)\big| &\le C\,\frac{1 + \|t\|^m}{x^{nr}\prod \lambda^r_k}\,\frac{C_{m+k}}{1 + \|t\|^{m+k}} \\
\big|f^{(p)}_\lambda(x)\big| &\le \frac{1}{x^{nr}}\,\frac{CC_{m+k}}{\prod \lambda^r_k}\int_{\mathbb{R}^n} \frac{1 + \|t\|^m}{1 + \|t\|^{m+k}}\, dt\ .
\end{aligned}$$

The Lemma can therefore be applied to $f_\lambda(x)$ to see that $f_\lambda(0)$ exists and depends continuously on the norms of Ψ, therefore defining in limit the tempered distribution

$$\langle \Phi_{0,\lambda}, \Psi\rangle = \lim_{x\to 0}\langle \Phi_{x\lambda}, \Psi\rangle$$

where

$$\langle \Phi_\sigma, \Psi\rangle = \int_{\mathbb{R}^n} \Phi(\sigma + it)\Psi(t)\, dt\ .$$

Define $\varphi_{0,\lambda}$ to be the inverse Fourier transform of $\Phi_{0,\lambda}$, a tempered distribution on $\mathbb{R}^n$. It remains to be shown that the product φ_σ of $e^{-\sigma x}$ and $\varphi_{0,\lambda}$ is also tempered for σ in Σ_ε, and that Φ_σ is the Fourier transform of φ_σ. This will prove among other things that $\Phi_{0,\lambda}$ doesn't actually depend on the choice of λ.

Choose a function ψ in $C^\infty_c(\mathbb{R}^n)$. Its Fourier transform

$$\Psi(s) = \int_{\mathbb{R}^n} \psi(x) e^{-\langle s,x\rangle}\, dx$$

will be entire, satisfying inequalities

$$|\Psi(s)| = O\left(\frac{1}{1 + \|\Im(s)\|^m}\right)$$

for every $m > 0$, uniformly on vertical strips $\|\Re(s)\| < C$.

Then for every σ in $\mathbb{R}^n$ with $\sigma_i > 0$ the product $e^{-\langle\sigma,x\rangle}\psi(x)$ will also be of compact support with Fourier transform

$$\Psi_\sigma(s) = \int_{-\infty}^{\infty} e^{-\langle\sigma,x\rangle} e^{-\langle s,x\rangle} f(x)\, dx = \Psi(\sigma + s)$$

Recall that if φ is a tempered distribution on $\mathbb{R}^n$ and and ψ in $\mathcal{S}(\mathbb{R}^n)$ with Fourier transforms Φ and Ψ then (expressing it formally)

$$\langle \varphi, \psi \rangle = \left(\frac{1}{2\pi i} \right)^n \int_{(i\mathbb{R})^n} \Phi(s)\Psi(-s)\, ds \; .$$

Thus

$$\begin{aligned} \langle \varphi_\sigma, \psi(x) \rangle &= \langle \varphi_0, e^{-\langle \sigma, x \rangle} \psi(x) \rangle \\ &= \lim_{x \to 0} \left(\frac{1}{2\pi i} \right)^n \int_{(i\mathbb{R})^n} \Phi(x\lambda + it)\Psi(\sigma - it)\, dt \end{aligned}$$

We change of contour of integration from $\sigma + (i\mathbb{R})^n$ to $(i\mathbb{R})^n$, which is permissible by our assumptions. The calculation continues

$$\begin{aligned} \langle \varphi_\sigma, \psi(x) \rangle &= \lim_{x \to 0} \left(\frac{1}{2\pi i} \right)^n \int_{(i\mathbb{R})^n} \Phi(x\lambda + \sigma + iu)\Psi(-iu)\, du \\ &= \left(\frac{1}{2\pi i} \right)^n \int_{(i\mathbb{R})^n} \Phi(\sigma + iu)\Psi(-iu)\, du \; . \end{aligned}$$

This result implies that the limit of Φ_s as s approaches 0 does not depend on the way in which the limit is taken, since $\Phi_0 = e^{\langle s,x \rangle}\Phi_s$ for all s in Σ_ε.

Dealing with an arbitrary s in $\overline{\Sigma}_\varepsilon$ is straightforward, since $e^{-\langle s, \bullet \rangle}\varphi_0$ is clearly tempered.

This concludes the proof of the Theorem.

Corollary. *Suppose $\Phi(s)$ to be holomorphic in the region Σ_ε, having as continuous limit as $\Re(s) \to 0$ a function in $L^2((i\mathbb{R})^n)$. Assume that for some integer $r > 0$ it satisfies an inequality*

$$|\Phi(\sigma + it)| \le \frac{C_m}{(1 + \|t\|^m) \prod \sigma_i^r}$$

for all $m > 0$ in the region Σ_ε. Then

$$\lim_{T \to \infty} \left(\frac{1}{2\pi i} \right)^n \int_{\|s\| \le T} \Phi(s) e^{\langle s,x \rangle}\, ds = \left(\frac{1}{2\pi i} \right)^n \int_{\Re(s) = \sigma} \Phi(s) e^{\langle s,x \rangle}\, ds$$

for any σ in Σ_ε.

The limit here is to be the limit in the L^2 norm of the functions

$$\varphi_T(x) = \left(\frac{1}{2\pi i} \right)^n \int_{\|s\| \le T} \Phi(s) e^{\langle s,x \rangle}\, ds$$

Formally, this is just a change of contours, but a direct argument allowing this does not seem possible. Instead, apply the Theorem to the function $\Phi(s)$, using the hypotheses to compute its inverse Fourier transform in two ways.

To apply the results of this section to the principal results of this paper, a change from additive to multiplicative coordinates is necessary. Thus $e^{\langle s,x \rangle}$ is replaced by $x^s = \prod x_k^{s_k}$.

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