

## Verifying Kottwitz' conjecture by computer

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In these notes I will discuss how I used a computer to verify the main conjecture of Kottwitz (1997) for the groups  $E_6$ ,  $E_7$ ,  $E_8$ , and the subsidiary one for  $F_4$ . At the end I will include tables of the relevant computer output. I begin by recalling briefly what is to be computed.

### The main conjecture

Suppose  $W$  to be a finite Weyl group. An **involution** in  $W$  is any element of order at most two. If  $\sigma$  is an involution, let  $W_\sigma$  be the centralizer of  $\sigma$ . A root  $\lambda$  is called **imaginary** if  $\sigma\lambda = -\lambda$  (as opposed to **real** if  $\sigma\lambda = \lambda$ ). Let  $I_\sigma$  be the set of all imaginary roots of  $\sigma$ . Any element commuting with  $\sigma$  permutes  $I_\sigma$ . Therefore if

$$P_\sigma = \prod_{\lambda > 0, \lambda \in I_\sigma} \lambda$$

then for any  $w$  in  $W_\sigma$

$$wP_\sigma = \text{sgn}_\sigma(w)P_\sigma$$

where  $\text{sgn}_\sigma = \pm 1$  is a multiplicative homomorphism from  $W_\sigma$  to  $\{\pm 1\}$ . It can be calculated explicitly as  $(-1)^{\ell_\sigma(w)}$  where

$$\ell_\sigma(w) = \#\{\lambda \in I_\sigma \mid \lambda > 0, w^{-1}\lambda < 0\} = I_\sigma \cap \Lambda_w$$

if

$$\Lambda_w = \{\lambda > 0 \mid w^{-1}\lambda < 0\}.$$

Kottwitz' conjecture concerns the multiplicity of  $\text{sgn}_\sigma$  in the restriction to  $W_\sigma$  of irreducible representations of  $W$ . In other words we must compute

$$m(\sigma, E) = \langle \text{sgn}_\sigma, E \mid W_\sigma \rangle = \frac{1}{\#W_\sigma} \sum_{W_\sigma} \text{sgn}_\sigma(w) \chi_E(w)$$

for the irreducible representations  $E$  of  $W$ .

Some cases are simple. If  $\sigma = 1$  then it has no imaginary roots and  $\text{sgn}_\sigma$  is the trivial character of  $W$ . In this case  $m(\sigma, E) = 0$  unless  $E$  is equal to the trivial character. If the longest element  $w_\ell$  of  $W$  happens to be  $-1$  and  $\sigma = w_\ell$  then all roots are imaginary and  $\text{sgn}_\sigma$  is then the usual sign-character  $\text{sgn}_W$  of  $W$ . Again in this case  $m(\sigma, E) = 0$  unless  $E = \text{sgn}_W$  itself. Along these lines, it can be seen more generally that if  $-1$  lies in  $W$  then  $m(\sigma, E) = m(\sigma_*, E_*)$  whenever  $\sigma_* = -\sigma$ ,  $E_* = E \cdot \text{sgn}_W$ .

Kottwitz' conjecture asserts that the sum

$$m(E) = \sum_{\sigma} m(\sigma, E)$$

(where the sum is over representatives of all conjugacy classes of involutions) is equal to the Lustzig-Fourier transform of a function  $\varphi_0$  which I do not define here. (But I shall recall later exactly what the computation has to agree with.) For classical groups as well as  $G_2$  and  $F_4$ , Kottwitz was able to verify his conjecture by hand. This leaves the exceptional groups  $E_6$  (51, 840 elements),  $E_7$  (2, 903, 040 elements), and  $E_8$  (696, 729, 600 elements), for which it was apparently necessary to do the computations by machine.

At first it looked as though it would be a great deal of work just getting the known character and conjugacy class information into computer-readable form, but luckily — and just in time — Meinolf Geck made available to us some recently developed programs for use with the well known algebra package **GAP**, which were able to produce exactly the information we needed. (These files are part of a larger project called **CHEVIE** involving Geck and several of his colleagues, and are available for public use. For information, see the reference to **CHEVIE** at the end of this note.)

Once we had the character and conjugacy class data that we needed in a form which a program could read, it was not too difficult to write a program that could calculate the multiplicities. It was something of a challenge to construct an efficient program, since the groups  $E_6$ ,  $E_7$ , and  $E_8$  are so large. The basic technique in all cases was the same — for each one of the involution classes  $\{\sigma\}$  in  $W$  the whole group was scanned to find elements in the centralizer  $W_\sigma$  (the order of the centralizer is known, and is one of the data produced by **GAP**). For each  $w$  in  $W_\sigma$  the parity of  $I_\sigma \cap \Lambda_w$  was calculated as well as its conjugacy class in  $W$ , from which the terms  $m(\sigma, E)$  could be calculated, again from the **GAP** data.

The details of the calculation were important, if the program were to be fast. The fastest known way to scan the group, as far as I can see, is that described in Casselman (1994), and involves building and then traversing the automaton describing the **ShortLex** language of strings of simple generators for elements in  $W$  which are of minimal length and lexicographically least. The fastest way to perform multiplication in the group is to use the ideas of du Cloux (1995), based in turn on ideas of Deodhar, representing an element of  $W$  as a sequence of elements of Weyl group cosets (with respect to smaller Weyl groups), using canonical representatives of these cosets. Here the basic calculation is to multiply an element of the group, represented as a sequence of coset representatives, by a simple generator  $s_i$ . The group  $E_6$  is in fact small enough so that in fact the whole table of  $ws_i$  can be stored in an array. It would have been possible to use tables of representatives already constructed by du Cloux, but in fact the program built these tables on the fly, using the multiplication algorithm described in Casselman (1994), which in turn is based on ideas of Brink-Howlett (1993).

There seems to be no uniform efficient way to handle the conjugacy class problem for these groups. Here again, the group  $E_6$  is small enough that the most efficient and dependable thing to do was simply to build a list, in the obvious way, of classes for every element in the group. For  $E_8$ , Geck suggested using the fact that the map from conjugacy classes of  $E_8$  to those in the group of permutations of its 240 roots is an embedding. In other words, a conjugacy class in  $E_8$  is distinguished by its representation in terms of cycles in  $\mathfrak{S}_{240}$ . There are simple and efficient algorithms for finding the cycle representation of a permutation, but nonetheless to do this several million times for permutations of 240 items is necessarily a slow business.

For  $E_7$  there are 126 roots. The map from conjugacy classes into conjugacy classes of  $\mathfrak{S}_{126}$  fails to be an injection, but the map into conjugacy classes in  $E_8$  is injective. Those classes in  $E_7$  which coincide in  $\mathfrak{S}_{126}$  were distinguished in this way.

Thus in traversing the group there were essentially three things to do: (1) tell whether the element  $w$  commutes with the current involution  $\sigma$ ; (2) calculate  $\text{sgn}_\sigma(w)$ ; (3) find the conjugacy class of  $w$ . The data needed for these calculations is easily updated in going from  $w$  to  $ws$  with  $ws > w$ , which is the way in which the automaton is traversed.

The amount of memory needed to run the program was negligible, but the amount of time required was substantial. I used several machines to develop the program with the small  $A_n$  and  $F_4$ , to compare with Kottwitz' calculations, and then to deal with the cases  $E_6$  and  $E_7$ , which are still relatively small. They took a few seconds and a few minutes, respectively. The group  $E_8$  is large by almost any standard, however. It contains almost a billion elements, and for each one of these a large number of calculations had to be made. The final run for  $E_8$  took about 36 hours on a SPARC 20.

The tables of  $m(\sigma, E)$  will be exhibited at the end of this note. In order to understand exactly how they imply Kottwitz' conjecture, I present here some relevant data mentioned in Kottwitz' article, but in a tabular form so that immediate comparison is straightforward. According to Lusztig (1984), the irreducible representations  $E$  of  $W$  are partitioned into families  $\mathcal{F}$ . To each family is associated a finite group  $\mathcal{G}$ , and to  $\mathcal{G}$  is associated a finite set  $\mathcal{M}(\mathcal{G})$  of conjugacy classes of pairs  $(g, \rho)$ , where  $g$  is an element of  $\mathcal{G}$  and  $\rho$  an irreducible representation of the centralizer  $\mathcal{G}_g$ . For this theory, refer to Lusztig (1984), Chapter 13 of Carter (1985), and Lusztig (1987). Each family maps injectively into a subset of  $\mathcal{M}(\mathcal{G})$ , but this map is not surjective. In the tables below, the image  $(g, \rho)$  of each  $E$  is indicated. Kottwitz' conjecture concerning the  $m(E)$  is that they agree with the Lusztig-Fourier transform of a certain function  $\varphi_0$  on  $\mathcal{M}(\mathcal{G})$  defined in §1.10 of Kottwitz' article. Kottwitz has listed the relevant values of  $\hat{\varphi}_0$  in his article; it will be simpler for the reader to verify his conjecture from my calculations if he has these in a tabular form.

**Table of  $\hat{\varphi}_0$**

$\mathcal{G}$	$(g, \rho)$	$\widehat{\varphi}_0(g, \rho)$
$\mathfrak{S}_1$	$(1, 1)$	1
$\mathfrak{S}_2$	$(1, 1)$	2
	$(g_2, 1)$	0
	$(1, \varepsilon)$	0
$\mathfrak{S}_3$	$(1, 1)$	2
	$(g_2, 1)$	0
	$(1, r)$	1
	$(g_3, 1)$	1
	$(1, \varepsilon)$	0
$\mathfrak{S}_4$	$(1, 1)$	3
	$(1, \lambda^1)$	1
	$(1, \lambda^2)$	0
	$(1, \sigma)$	2
	$(g_2, 1)$	0
	$(g_2, \varepsilon'')$	0
	$(g'_2, 1)$	1
	$(g'_2, \varepsilon'')$	0
	$(g'_2, \varepsilon')$	0
	$(g_3, 1)$	1
	$(g_4, 1)$	0
	$\mathfrak{S}_5$	$(1, 1)$
$(g_3, 1)$		2
$(g'_2, 1)$		1
$(1, \nu)$		2
$(1, \lambda^1)$		2
$(g_5, 1)$		1
$(g_3, \varepsilon)$		0
$(1, \nu')$		1
$(g'_2, \varepsilon'')$		0
$(1, \lambda^2)$		0
$(g'_2, \varepsilon')$		0
$(1, \lambda^3)$		0
$(g_2, 1)$		0
$(g_4, 1)$		0
$(g_6, 1)$		0
$(g_2, r)$		0
$(g_2, \varepsilon)$	0	

Throughout the tables, Carter's name conventions for both conjugacy classes and representations of  $W$  are followed (see Carter (1972) and Carter (1985)). Carter's naming scheme for representations refers to a representation  $\varphi_{n,d}$  where  $n$  is its dimension and  $d$  is the lowest degree it appears in the canonical representation of  $W$  on  $S(V)$ , the symmetric algebra of the root space  $V$ . That for conjugacy classes takes advantage of the fact that most conjugacy classes in a Weyl group are Coxeter elements in some Weyl subgroup. The roots, hence the numbering of the elementary reflections indexed in the reduced expressions, are numbered as in Bourbaki (1972). The characters and other conjugacy class data were provided by **GAP**.

As Kottwitz mentions, it was the computer results for  $E_7$  (which appeared before those for  $E_8$ ) which forced him to deal with the six exceptional representations (two of  $E_7$ , four of  $E_8$ ) specially. As far as I know, it is only in the paper Opdam (1995) that any theoretical explanation of some of the phenomena attached to these occurs in the literature.

I include here also the tables for the exceptional groups  $G_2$  and  $F_4$ , which will perhaps allow the reader to orient himself in reading these tables, even though they are treated by Kottwitz directly.

### The subsidiary conjecture

Kottwitz' second conjecture is that if  $\sigma$  is an involution, then the number of involutions in any right cell  $[\Gamma]$  and conjugate to  $\sigma$  is equal to

$$m(\sigma, [\Gamma]) = \sum n_i m(\sigma, E_i)$$

if the canonical  $W$ -representation  $[\Gamma]$  associated to  $[\Gamma]$  has the irreducible decomposition

$$[\Gamma] = \sum n_i E_i .$$

He demonstrates this in a number of cases, but leaves it open for several others. Again, using a computer program, we verified this to be true for  $F_4$ . This program was more interesting than the one used previously, since it involved computing explicitly all the  $W$ -graphs associated to the right cells of  $W$ . (It was also more interesting because I implemented it one year after the other, and this time completely in **Java**.) Explicitly, we have the following table of data, which in combination with the table of the  $m(\sigma, E)$  for  $E$  an irreducible representation of  $F_4$  implies the result:

Number of equivalent cells	Involution classes in cell	Decomposition
1	1	$\varphi_{1,0}$
2	$A_1, \tilde{A}_1$	$\varphi''_{2,4} + \varphi_{4,1}$
2	$A_1, \tilde{A}_1$	$\varphi'_{2,4} + \varphi_{4,1}$
9	$A_1 \times \tilde{A}_1$	$\varphi_{9,2}$
8	$\tilde{A}_1$	$\varphi''_{8,3}$
8	$A_1$	$\varphi'_{8,3}$
8	$A_1^3$	$\varphi''_{8,9}$
8	$A_1^2 \times \tilde{A}_1$	$\varphi'_{8,9}$
9	$A_1 \times \tilde{A}_1$	$\varphi_{9,10}$
1	$[5] A_1 \times \tilde{A}_1, [2] A_1^2$	$\varphi''_{1,12} + 2\varphi''_{9,6} + \varphi''_{6,6} + \varphi_{12,4} + \varphi''_{4,7} + \varphi_{16,5}$
3	$[4] A_1 \times \tilde{A}_1, A_1^2$	$\varphi''_{9,6} + \varphi'_{6,6} + \varphi_{12,4} + \varphi''_{4,7} + \varphi_{16,5}$
4	$[2] A_1^2, [5] A_1 \times \tilde{A}_1$	$\varphi_{4,8} + \varphi''_{9,6} + \varphi'_{9,6} + \varphi''_{6,6} + \varphi_{12,4} + 2\varphi_{16,5}$
3	$[4] A_1 \times \tilde{A}_1, A_1^2$	$\varphi'_{9,6} + \varphi'_{6,6} + \varphi_{12,4} + \varphi'_{4,7} + \varphi_{16,5}$
2	$A_1^3, A_1^2 \times \tilde{A}_1$	$\varphi''_{2,16} + \varphi_{4,13}$
1	$[2] A_1^2, [5] A_1 \times \tilde{A}_1$	$\varphi'_{1,12} + 2\varphi'_{9,6} + \varphi''_{6,6} + \varphi_{12,4} + \varphi'_{4,7} + \varphi_{16,5}$
2	$A_1^3, A_1^2 \times \tilde{A}_1$	$\varphi'_{2,16} + \varphi_{4,13}$
1	$A_1^4$	$\varphi_{1,24}$

By the time this paper appears, the case  $E_6$  ought to be decided. Doing this in the obvious way involves an enormous amount of memory to record the Bruhat order, but this problem can be alleviated again by suggestions of du Cloux, following Deodhar.

## References

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4. W. A. Casselman, 'Automata to perform basic calculations in Coxeter groups', in *Representations of groups*, CMS Conference Proceedings (published by the AMS) **16** (1995), pp. 35–58.
5. Fokko du Cloux, 'A transducer approach to Coxeter groups', preprint, Institut de Mathématiques et Informatique, Université Lyon - I, 1995.
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<http://www.iwr.uni-heidelberg.de//iwr/compalg/chevie/chevieann.html>

7. R. Kottwitz, 'Involutions in Weyl groups', preprint, Department of Mathematics, University of Chicago, 1997.
8. G. Lusztig, *Characters of reductive groups over a finite field*, *Annals of Math. Studies* **107**, Princeton, 1984.
9. G. Lusztig, 'Leading coefficients of character values of Hecke algebras', *Proc. Symp. Pure Math.* **47** (1987), pp. 235–262.
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**The tables**

**The group  $G_2$**

It has 12 elements.

**Conjugacy class data:**

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
$\tilde{A}_1$	[1]	3
$A_1$	[2]	3
$A_1 \times \tilde{A}_1$	[121212]	1

**Multiplicities:**

$\mathcal{G}$	$E$	$(g, \rho)$	$m(\sigma, E)$				
			1	$\tilde{A}_1$	$A_1$	$A_1 \times \tilde{A}_1$	
$\mathfrak{S}_1$	$\varphi_{1,0}$	(1, 1)	1	0	0	0	1
	$\varphi_{1,6}$	(1, 6)	0	0	0	1	1
$\mathfrak{S}_3$	$\varphi_{2,1}$	(1, 1)	0	1	1	0	2
	$\varphi'_{1,3}$	(1, r)	0	0	1	0	1
	$\varphi''_{1,3}$	(g <sub>3</sub> , 1)	0	1	0	0	1
	$\varphi_{2,2}$	(g <sub>2</sub> , 1)	0	0	0	0	0

**The group  $F_4$**

It has 1152 elements.

**Conjugacy class data:**

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
$A_1^4$	[121321323432132343213234]	1
$A_1^2$	[2323]	18
$A_1$	[1]	12
$A_1^3$	[232343234]	12
$\tilde{A}_1$	[3]	12
$A_1^2 \times \tilde{A}_1$	[121321323]	12
$A_1 \times \tilde{A}_1$	[13]	72

**Multiplicities:**

The group  $F_4$  is unusual, in that Kondo's names are still commonly used, and in particular in Lusztig (1984). They are therefore given here, just after those of Carter.

$\mathcal{G}$	$E$	$(\text{Kondo}) (g, \rho)$	$m(\sigma, E)$								
			1	$A_1^4$	$A_1^2$	$A_1$	$A_1^3$	$\tilde{A}_1$	$A_1^2 \times \tilde{A}_1$	$A_1 \times \tilde{A}_1$	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$	1 <sub>1</sub>	1	0	0	0	0	0	0	0	1
	$\varphi_{1,24}$	1 <sub>4</sub>	0	1	0	0	0	0	0	0	1
	$\varphi_{9,10}$	9 <sub>4</sub>	0	0	0	0	0	0	0	1	1
	$\varphi''_{8,3}$	8 <sub>1</sub>	0	0	0	0	0	1	0	0	1

	$\varphi'_{8,3}$	$8_3$		0	0	0	1	0	0	0	0	1
	$\varphi''_{8,9}$	$8_4$		0	0	0	0	1	0	0	0	1
	$\varphi'_{8,9}$	$8_2$		0	0	0	0	0	0	1	0	1
	$\varphi_{9,2}$	$9_1$		0	0	0	0	0	0	0	1	1
$\mathfrak{S}_2$	$\varphi_{4,1}$	$4_2$	$(1, 1)$	0	0	0	1	0	1	0	0	2
	$\varphi'_{2,4}$	$2_1$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0
	$\varphi_{2,4}$	$2_3$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{4,13}$	$4_5$	$(1, 1)$	0	0	0	0	1	0	1	0	2
	$\varphi''_{2,16}$	$2_4$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0
	$\varphi'_{2,16}$	$2_2$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_4$	$\varphi_{12,4}$	$12_1$	$(1, 1)$	0	0	1	0	0	0	0	2	3
	$\varphi''_{9,6}$	$9_2$	$(g'_2, 1)$	0	0	0	0	0	0	0	1	1
	$\varphi'_{9,6}$	$9_3$	$(1, \lambda^1)$	0	0	0	0	0	0	0	1	1
	$\varphi''_{1,12}$	$1_2$	$(g'_2, \varepsilon')$	0	0	0	0	0	0	0	0	0
	$\varphi'_{1,12}$	$1_3$	$(1, \lambda^2)$	0	0	0	0	0	0	0	0	0
	$\varphi_{4,8}$	$4_1$	$(g'_2, \varepsilon'')$	0	0	0	0	0	0	0	0	0
	$\varphi''_{4,7}$	$4_3$	$(g_4, 1)$	0	0	0	0	0	0	0	0	0
	$\varphi'_{4,7}$	$4_4$	$(g_2, \varepsilon'')$	0	0	0	0	0	0	0	0	0
	$\varphi'_{6,6}$	$6_1$	$(g_3, 1)$	0	0	0	0	0	0	0	1	1
	$\varphi''_{6,6}$	$6_2$	$(1, \sigma)$	0	0	1	0	0	0	0	1	2
	$\varphi_{16,5}$	$16_1$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0

**The group  $E_6$**

It has 51, 840 elements.

**Conjugacy class data:**

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
$A_1^4$	[343243543245]	45
$A_1^2$	[14]	270
$A_1$	[1]	36
$A_1^3$	[146]	540

**Multiplicities:**

$\mathcal{G}$	$E$	$(g, \rho)$	$m(\sigma, E)$					$m(E)$
			1	$A_1^4$	$A_1^2$	$A_1$	$A_1^3$	
$\mathfrak{S}_1$	$\varphi_{1,0}$		1	0	0	0	0	1
	$\varphi_{1,36}$		0	1	0	0	0	1
	$\varphi_{6,1}$		0	0	0	1	0	1
	$\varphi_{6,25}$		0	0	0	0	1	1
	$\varphi_{20,2}$		0	0	1	0	0	1
	$\varphi_{20,20}$		0	1	0	0	0	1
	$\varphi_{24,6}$		0	0	1	0	0	1
	$\varphi_{24,12}$		0	1	0	0	0	1
	$\varphi_{60,5}$		0	0	0	0	1	1
	$\varphi_{60,11}$		0	0	0	0	1	1
	$\varphi_{64,4}$		0	0	1	0	0	1
	$\varphi_{64,13}$		0	0	0	0	1	1
	$\varphi_{81,6}$		0	0	1	0	0	1
	$\varphi_{81,10}$		0	0	1	0	0	1
$\mathfrak{S}_2$	$\varphi_{30,15}$	$(1, 1)$	0	0	0	0	2	2
	$\varphi_{15,17}$	$(1, \varepsilon)$	0	0	0	0	0	0
	$\varphi_{15,16}$	$(g_2, 1)$	0	0	0	0	0	0

$\mathfrak{S}_2$	$\varphi_{30,3}$	(1, 1)	0	0	0	1	1	2
	$\varphi_{15,5}$	(1, $\varepsilon$ )	0	0	0	0	0	0
	$\varphi_{15,4}$	( $g_2$ , 1)	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{80,7}$	(1, 1)	0	0	0	0	2	2
	$\varphi_{10,9}$	( $g_3$ , 1)	0	0	0	0	1	1
	$\varphi_{20,10}$	(1, $\varepsilon$ )	0	0	0	0	0	0
	$\varphi_{60,8}$	( $g_2$ , 1)	0	0	0	0	0	0
	$\varphi_{90,8}$	(1, $r$ )	0	0	0	0	1	1

**The group  $E_7$**

It has 2, 903, 040 elements.

**Conjugacy class data:**

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
$A_1^6$	[767567456724567345672456345243]	63
$(A_1'')^4$	[545245345243]	315
$A_1^2$	[75]	945
$(A_1')^4$	[7523]	3780
$A_1^7$	[767567456724567345672456345243134567245634524313456724563452431]	1
$A_1$	[7]	63
$(A_1')^3$	[752]	315
$A_1^5$	[7545245345243]	945
$(A_1'')^3$	[753]	3780

**Multiplicities:**

The exceptional classes are marked *Exc*.

$\mathcal{G}$	$E$	$(g, \rho)$	$m(\sigma, E)$	1	$A_1^6$	$(A_1'')^4$	$A_1^2$	$(A_1')^4$	$A_1^7$	$A_1$	$(A_1')^3$	$A_1^5$	$(A_1'')^3$	$m(E)$
$\mathfrak{S}_1$	$\varphi_{1,0}$		1	0	0	0	0	0	0	0	0	0	0	1
	$\varphi_{1,63}$		0	0	0	0	0	1	0	0	0	0	0	1
	$\varphi_{7,46}$		0	1	0	0	0	0	0	0	0	0	0	1
	$\varphi_{7,1}$		0	0	0	0	0	0	1	0	0	0	0	1
	$\varphi_{21,36}$		0	0	1	0	0	0	0	0	0	0	0	1
	$\varphi_{21,3}$		0	0	0	0	0	0	0	1	0	0	0	1
	$\varphi_{27,2}$		0	0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{27,37}$		0	0	0	0	0	0	0	0	1	0	0	1
	$\varphi_{105,6}$		0	0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{105,21}$		0	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{105,12}$		0	0	1	0	0	0	0	0	0	0	0	1
	$\varphi_{105,15}$		0	0	0	0	0	0	0	1	0	0	0	1
	$\varphi_{168,6}$		0	0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{168,21}$		0	0	0	0	0	0	0	0	1	0	0	1
	$\varphi_{189,22}$		0	0	0	0	1	0	0	0	0	0	0	1
	$\varphi_{189,5}$		0	0	0	0	0	0	0	0	0	0	1	1
	$\varphi_{189,20}$		0	0	1	0	0	0	0	0	0	0	0	1
	$\varphi_{189,7}$		0	0	0	0	0	0	0	1	0	0	0	1
	$\varphi_{210,6}$		0	0	0	1	0	0	0	0	0	0	0	1
	$\varphi_{210,21}$		0	0	0	0	0	0	0	0	1	0	0	1
$\varphi_{210,10}$		0	0	0	0	1	0	0	0	0	0	0	1	
$\varphi_{210,13}$		0	0	0	0	0	0	0	0	0	1	0	1	
$\varphi_{378,14}$		0	0	0	0	1	0	0	0	0	0	0	1	
$\varphi_{378,9}$		0	0	0	0	0	0	0	0	0	1	0	1	
$\mathfrak{S}_2$	$\varphi_{56,3}$	(1, 1)	0	0	0	0	0	0	1	0	0	1	2	

	$\varphi_{35,4}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{21,6}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{56,30}$	$(1, 1)$	0	1	0	0	1	0	0	0	0	2
	$\varphi_{35,31}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{21,33}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{120,4}$	$(1, 1)$	0	0	0	1	1	0	0	0	0	2
	$\varphi_{15,7}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{105,5}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{120,25}$	$(1, 1)$	0	0	0	0	0	0	0	1	1	2
	$\varphi_{15,28}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{105,26}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{405,8}$	$(1, 1)$	0	0	0	0	2	0	0	0	0	2
	$\varphi_{216,9}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{189,10}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{405,15}$	$(1, 1)$	0	0	0	0	0	0	0	0	2	2
	$\varphi_{216,16}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{189,17}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{420,10}$	$(1, 1)$	0	0	0	1	1	0	0	0	0	2
	$\varphi_{84,12}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{336,11}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_2$	$\varphi_{420,13}$	$(1, 1)$	0	0	0	0	0	0	0	1	1	2
	$\varphi_{84,15}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{336,14}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
<i>Exc</i>	$\varphi_{512,11}$	$(1, 1)$	0	0	0	0	0	0	0	0	1	1
	$\varphi_{512,12}$	$(1, \varepsilon)$	0	0	0	0	1	0	0	0	0	1
$\mathfrak{S}_3$	$\varphi_{315,7}$	$(1, 1)$	0	0	0	0	0	0	0	0	2	2
	$\varphi_{280,8}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{280,9}$	$(1, r)$	0	0	0	0	0	0	0	0	1	1
	$\varphi_{70,9}$	$(g_3, 1)$	0	0	0	0	0	0	0	0	1	1
	$\varphi_{35,13}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{S}_3$	$\varphi_{315,16}$	$(1, 1)$	0	0	0	0	2	0	0	0	0	2
	$\varphi_{280,17}$	$(g_2, 1)$	0	0	0	0	0	0	0	0	0	0
	$\varphi_{280,18}$	$(1, r)$	0	0	0	0	1	0	0	0	0	1
	$\varphi_{70,18}$	$(g_3, 1)$	0	0	0	0	1	0	0	0	0	1
	$\varphi_{35,22}$	$(1, \varepsilon)$	0	0	0	0	0	0	0	0	0	0

**The group  $E_8$**

It has 696, 729, 600 elements.

**Conjugacy class data:**

Carter's name	Representative reduced word expression	Conjugacy class size
1	$\emptyset$	1
$A_1^8$	[878678567845678145678345678145673456145341134567814567345614... 534113456781456734561453411345678145634514313456714563451431]	1
$(A_1^4)^4$	[545245345243]	3150
$A_1^2$	[61]	3780
$A_1^6$	[767567456724567345672456345243]	3780
$(A_1'')^4$	[7523]	113400
$A_1$	[3]	120





