# Truncation exercises

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One of the most basic tools in the theory of automorphic forms is the truncation operator, first defined in [Langlands: 1966] but elaborated in later work of Jim Arthur (particularly [Arthur: 1978] and [Arthur: 1980]). In spite of its fundamental nature, there are many mysteries about it. In this article I'll not dispel many of these, but I'll at least motivate its definition by relating it to recent work on truncation operators in Euclidean space.

The results in this paper have been presented at conferences in Dubrovnik and Hang Zhou, and I wish to thank the organizers of those conferences for the excuse to do the research explained in this paper. I also wish to thank Peter Trapa for pointing out to me the previous literature.

## Part I. Arithmetic quotients

I begin by recalling a classical case, and then Arthur's definitions.

### 1. The upper half plane

Let's look first at the simplest case. Let

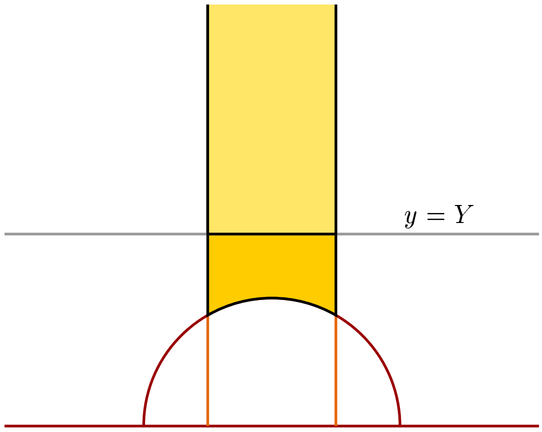
$$\begin{aligned} G &= \mathrm{SL}_2(\mathbb{R}) \\ \Gamma &= \mathrm{SL}_2(\mathbb{Z}) \\ \mathcal{H} &= \{x+iy \,|\, y>0\} \\ P &= \text{ upper triangular matrices in } G \\ N &= \text{ unipotent matrices in } P\ . \end{aligned}$$

Elements in the group $G$ act on $\mathcal{H}$ by fractional linear transformations. Elements of $N$ act by translations $z \mapsto z+x$, and elements of $\Gamma \cap P$ act by integral horizontal translations $z \mapsto z+n$. A fundamental domain for $\Gamma \cap P$ is the region $|x| \le 1/2$ (where $z = x+iy$), and one for $\Gamma$ is the region with the additional condition $|z| \ge 1$. For $Y \ge 0$ let

$$\mathcal{H}_Y = \{x+iy \,|\, y>Y\}\ .$$

The significant fact for us is:

**Proposition.** *For $Y \ge 1$ the canonical map from $\Gamma \cap P\backslash\mathcal{H}_Y$ to $\Gamma\backslash\mathcal{H}$ is an embedding. The complement of its image is compact.*




The partition of $\Gamma\backslash\mathcal{H}$

Recall that the **constant term** of any function $F$ on $\Gamma\backslash\mathcal{H}$ is the function

$$F_P(y) = \int_0^1 F(x+iy)\,dx \ .$$

It is invariant under both $\Gamma\cap P$ and $N$, hence a function on the quotient $(\Gamma\cap P)N\backslash\mathcal{H} \cong (0,\infty)$. On this quotient define the operator

$$C^Y f(y) = f(y)\cdot \mathfrak{char}_{(Y,\infty)}$$

where $\mathfrak{char}$ is the characteristic function. The natural domain of definition for the operator $C^Y$ is the subspace of $N$-invariant functions on $\Gamma\cap P\backslash\mathcal{H}$, but by the Proposition it may be defined equally well on functions on $\Gamma\backslash\mathcal{H}$. Equivalently, we may define

$$C^Y F(z) = \sum_{\Gamma\cap P\backslash\Gamma} C^Y F_P(\gamma z) \ ,$$

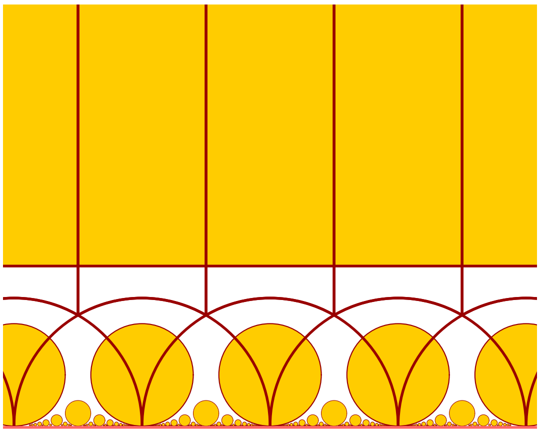
which makes $\Gamma$-invariance manifest. There is yet a third equivalent definition that is in many ways more convenient. If $F(z)$ is a function on $(\Gamma\cap P)N_P\backslash\mathcal{H}$ then $L_\gamma F(z) = F(\gamma^{-1}z)$ is one on $(\Gamma\cap Q)N_Q\backslash\mathcal{H}$ where $Q = \gamma P\gamma^{-1}$. If $F$ is $\Gamma$-invariant then $L_\gamma F_P = F_Q$, where

$$F_Q(z) = \int_{\Gamma\cap N_Q\backslash N_Q} f(xz)\,dx \ .$$

Define $C^Y_Q$ to be $L_\gamma C^Y_P$. The third formula is

$$C^Y F = \sum_Q C^Y_Q F_Q$$

where the sum is over all rational parabolic subgroups $Q$.



Truncation is a sum of local truncations at cusps

The **truncation** on $\Gamma\backslash\mathcal{H}$ of $F$ at $Y$ is the difference

$$\Lambda^Y F = F - C^Y F \ .$$

In other words, truncating $F$ chops away its constant term near all cusps.

The decomposition

$$F = \Lambda^Y F + C^Y F$$

is orthogonal. The most important, if somewhat subtle, property of truncation is that if $F$ satisfies a mild growth condition (one of **uniform moderate growth**) then $\Lambda^Y F$ is rapidly decreasing at infinity.

In general, if $f$ is any function on $(0,\infty)$ then the associated **Eisenstein series** is, at least formally, the sum

$$\sum_{\Gamma\cap P\backslash\Gamma} f\big(\mathrm{IM}(\gamma z)\big) \ .$$

If $f(y) = y^{1/2+s/2}$ with $\mathrm{RE}(s) > 1$ then this series converges, and is the same as Maass' series

$$E_s(z) = \frac{1}{2}\sum_{\gcd(c,d)=1} \frac{y^{1/2+s/2}}{|cz+d|^{1+s}} \ .$$

The function $E_s(z)$ is an eigenfunction of the non-Euclidean Laplacian

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

since $y^{1/2+s/2}$ is one, and $\Delta$ commutes with $\Gamma$. It can be meromorphically extended in $s$ to all of $\mathbb{C}$. Its constant term is

$$y^{1/2+s/2} + c(s)y^{1/2-s/2} \quad \text{with } c(s) = \frac{\xi(s)}{\xi(s+1)} \ ,$$

where $\xi(s)$ is Riemann's function $\pi^{-s/2}\Gamma(s/2)\zeta(s)$. It satisfies the functional equation

$$E_s = c(s)E_{-s} \ ,$$

which is compatible with, but not equivalent to, the functional equation for $\xi(s)$.

Since $\Lambda^Y E_s$ is rapidly decreasing at infinity, it is in particular square-integrable. The inner product of two Eisenstein series can be evaluated by the **Maass-Selberg formula**

$$\langle \Lambda^Y E_s, \Lambda^Y E_t \rangle = \int_0^Y (y^{1/2+s/2} + c(s) y^{1/2-s/2})(y^{1/2+t/2} + c(t) y^{1/2-t/2})\, dy/y^2$$

which is evaluated according to the formal rule

$$\int_0^Y y^r \, dy/y = Y^r/r$$

even when the integral doesn't converge—or in other words by analytic continuation. As explained in more detail in [Casselman: 1993], this expression can be justified by using a suggestion of Hadamard that allows one to make sense of and prove the orthogonality relation

$$\langle E_s, E_t \rangle = 0 \ .$$

One noteworthy feature of this truncation is that it is essentially a local operator in the neighbourhood of infinity on the compactification of $\Gamma\backslash\mathcal{H}$ obtained by adding one point at each cusp.

## 2. Arthur's partition

Now suppose $G$ to be the group of real points on a rational reductive group with $G(\mathbb{C})$ connected. Fix a maximal compact $K$ and let $X = X_G$ be the symmetric space of $G$, which may be identified with $G/K$. For each rational parabolic subgroup $P$ let

$$\begin{aligned} N_P &= \text{ its unipotent radical} \\ M_P &= P/N_P \\ &= M \text{ for the moment} \\ K_M &= \text{ the image of } K \cap P \text{ in } M (\text{isomorphic to } K \cap P) \\ A_P &= \text{ the maximal split torus in the center of } M_P \ . \end{aligned}$$

Thus $M/K_M$ may be identified with the symmetric space $X_M$ of $M$. Since $G = PK$ we may also identify $X$ with $P/K \cap P$ and $N_P\backslash X$ with $X_M$. The quotient $A_P/K \cap A_P$ is isomorphic to the connected component $|A_P|$ of $A_P$.

For $\mathrm{SL}_2$ the group $M_P = A_P$ is isomorphic to the multiplicative group of real numbers, and truncation at $Y \geq 1$ is straightforward. But for groups of higher rank there are a few complications. As with $\mathrm{SL}_2$, truncation will subtract from a function various parts of constant terms with respect to rational parabolic subgroups, but how this is to be done is not straightforward. In this section I'll explain a partition, essentially due to Arthur, of $X$ into pieces parametrized by rational parabolic subgroups, and in the next I'll say something about the more sophisticated notion of truncation.

Suppose $P$ and $Q$ to be rational parabolic subgroups of $G$ with $gPg^{-1} = Q$ for an element $g$ of $G(\mathbb{Q})$. Then $gN_Pg^{-1} = N_Q$, inducing an isomorphism of the quotients $M_P$ and $M_Q$ as well, taking $A_P$ to $A_Q$. Since a parabolic subgroup is its own normalizer, the isomorphism of $M_P$ with $M_Q$ is unique up to conjugation at either end, so the isomorphism of $A_P$ with $A_Q$ is independent of the choice of $g$. Since all minimal rational parabolic subgroups are conjugate, there exists a canonical isomorphism of $A_P$ and $A_Q$ if $P$ and $Q$ are both minimal. The transpose of this isomorphism takes $\mathbb{Q}$-roots to $\mathbb{Q}$-roots, positive roots to positive roots. We obtain in this way a **canonical rational split torus** $A$ and a based root system $(\Sigma, \Delta)$ associated to it. Let $\mathfrak{a}$ be the real Lie algebra of $A$, which is isomorphic to the connected component $|A|$ through the exponential map. The set of roots $\Sigma$ may be considered a subset of the dual $\mathfrak{a}^\vee$ of $\mathfrak{a}$, and $\Delta$ is a subset of $\Sigma$. Associated to $\Sigma$ are the coroots $\Sigma^\vee$ in $\mathfrak{a}$ itself. The vector space $\mathfrak{a}$ may be assigned a Euclidean structure preserved by the Weyl group.

For every subset $\Theta \subset \Delta$ let

$$\mathfrak{a}_\Theta = \{v \in \mathfrak{a} \mid \langle \alpha, v \rangle = 0 \text{ for } \alpha \in \Theta\} \ .$$

Thus $\mathfrak{a} = \mathfrak{a}_\emptyset$ and $\mathfrak{a}_\Delta$ is canonically isomorphic to the Lie algebra of the maximal split torus in the centre of $G$. Let

$$\mathfrak{a}_\Theta^\Delta = \mathfrak{a}_\Theta / \mathfrak{a}_\Delta \ .$$

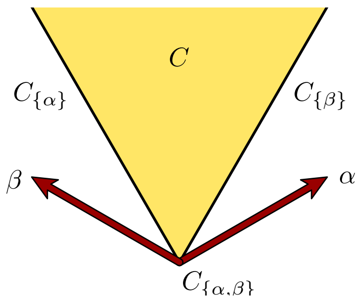
A fundamental domain for the Weyl group $W$ is the closed cone

$$C = \{v \in \mathfrak{a} \mid \langle \alpha, v \rangle \geq 0 \text{ for all } \alpha \in \Delta\} \ .$$

The faces of $C$ are the cones

$$C_\Theta = \mathfrak{a}_\Theta \cap C = \{v \in \mathfrak{a} \mid \langle \alpha, v \rangle = 0 \text{ for all } \alpha \in \Theta, \langle \alpha, v \rangle \geq 0 \text{ for all } \alpha \in \Delta - \Theta\}$$

for $\Theta \subseteq \Delta$. Thus $C = C_\emptyset$.




The dominant cone and its faces

Each $C_\Theta$ is stable under translation by $C_\Delta = \mathfrak{a}_\Delta$. Let $C_\Theta^\Delta$ be the quotient, a subset of $\mathfrak{a}_\Theta^\Delta$.

If $P$ is an arbitrary minimal rational parabolic subgroup and $Q$ any rational parabolic subgroup, then $Q$ is rationally conjugate to a unique parabolic subgroup containing $P$. This gives rise to a canonical isomorphism of $A_Q$ with a unique $A_\Theta$ and of $\mathfrak{a}_Q$ with $\mathfrak{a}_\Theta$. The subset $\Theta = \Theta(Q)$ is well determined. The Lie algebra $\mathfrak{m}_P$ of $M_P$ possesses the Lie algebra $\mathfrak{a}_P$ as a uniquely determined summand. There then exists a canonical map from $M_P$ to $\mathfrak{a}_P$, then to $\mathfrak{a}_{\Theta(P)}$. This in turn induces a canonical map

$$(\Gamma \cap P) N_P \backslash P \longrightarrow \mathfrak{a}_\Theta \quad \big(\Theta = \Theta(P)\big)$$

inducing in turn

$$(\Gamma \cap P) N_P \backslash X = (\Gamma \cap P) N_P \backslash P / (K \cap P) \longrightarrow \mathfrak{a}_\Theta$$

and a related one onto the quotient:

$$\varepsilon_P \colon \ (\Gamma \cap P) N_P \backslash X = (\Gamma \cap P) N_P \backslash P / (K \cap P) \longrightarrow \mathfrak{a}_\Theta^\Delta \ .$$

These last two maps depend on the choice of maximal compact subgroup $K$. Following Arthur, let $\mathfrak{X}_P^G \subset X$ be the inverse image of $C_\Theta^\Delta$ with respect to $\varepsilon_P$, and $\tau_P^G$ its characteristic function. In addition, for each $T$ in $\mathfrak{a}_\Theta^\Delta$ let $\mathfrak{X}_{P,T}^G$ be the inverse image of $C_\Theta^\Delta$ translated by $T$ and $\tau_{P,T}^G$ its characteristic function. If $P^{++}(T)$ is the inverse image of $T\,C_\Theta^\Delta$ and $x_K$ is the point of $X$ fixed by $K$ then then $\mathfrak{X}_P^G$ is also $P^{++}(T) x_K$.

Before I describe Arthur's partition, let me look again at the case of $\mathrm{SL}_2$. We start off with the domain $\mathcal{H}_Y$, and then construct a $\Gamma$-invariant partition of all of $\mathcal{H}$ by transforming $\mathcal{H}_Y$. That is to say, if $Q = \gamma P \gamma^{-1}$ then

$$\mathcal{H}_Q = \gamma \mathcal{H}_Y \ .$$

Each $\mathcal{H}_Q$ is invariant with respect to $\Gamma \cap Q$ as well as $N_Q$. This suffices to partition all of $\Gamma\backslash\mathcal{H}$ because all the rational parabolic subgroups are $\Gamma$-conjugate—if the quotient $\Gamma\backslash\mathcal{H}$ had several cusps we'd have to do something more complicated. One other complication is that we don't have explicit coordinates in the general case and must use instead identifications with group quotients.

In general:

**Proposition.** *If $P$ is a rational parabolic subgroup and $\gamma = k_\gamma p_\gamma$ lies in $\Gamma$ then*

$$\gamma\, \mathfrak{X}^G_{P,T} = \mathfrak{X}^G_{Q,TT_\gamma}$$

*where $Q = \gamma P \gamma^{-1}$ and $T_\gamma$ is the image of $p_\gamma$ in $\mathfrak{a}^\Delta_\Theta$.*

The proof is just a calculation. We have

$$\begin{aligned} Q^{++}(T)x_K &= \gamma P^{++}(T)\gamma^{-1}x_K \\ &= \gamma P^{++}(T)p_\gamma^{-1}x_K \\ &= \gamma P^{++}(T)p_\gamma^{-1}k_\gamma^{-1}x_K \\ &= \gamma P^{++}(TT_\gamma^{-1})x_K \ . \end{aligned}$$

For a general group $G$, the number of $\Gamma$-conjugacy classes of rational parabolic subgroups is finite. Let $\mathcal{P}$ be a set representatives for the maximal proper ones. For $P$ in $\mathcal{P}$ with $\Theta = \Theta(P)$, the space $\mathfrak{a}^\Delta_\Theta$ is one-dimensional. For each $P$ in $\mathcal{P}$ choose $T_P$ in $\mathfrak{a}^\Delta_\Theta$ subject to the condition that $\langle \alpha_\Theta, T_P\rangle \gg 1$ where $\alpha_\Theta$ is the unique element of $\Delta - \Theta(P)$. For any other maximal proper parabolic subgroup $Q = \gamma P \gamma^{-1}$ with $P$ in $\mathcal{P}$ let $T_Q = T_P T_\gamma$. This guarantees that

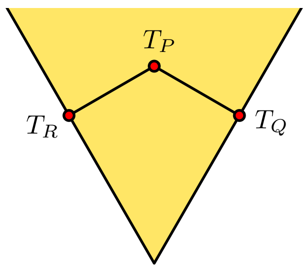
$$C^G_{Q,T_Q} = \gamma C^G_{P,T_P} \ .$$

according to the previous Proposition.

If $P$ is now any rational parabolic subgroup, it can be expressed as the intersection of maximal proper ones:

$$P = \bigcap_{Q \in \mathfrak{max}_P} Q \ .$$

For each maximal proper rational parabolic subgroup $Q$, the group $\mathfrak{a}^\Delta_{\Theta(Q)}$ is a line, and for the groups that occur in $\mathfrak{max}_P$ these are linearly independent. By means of orthogonal projection the $T_Q$ correspond to hyperplanes in $\mathfrak{a}^\Delta_{\Theta(P)}$, which meet in a single point $T_P$.




Defining $T_P$

It is now true for any conjugate parabolic subgroups $P$ and $Q = \gamma P \gamma^{-1}$ that

$$C^G_{Q,T_Q} = \gamma C^G_{P,T_P} \ .$$

Finally, we can define Arthur's partition of $X$ into pieces $X^G_{P,T}$ parametrized by rational parabolic subgroups $P$ given a choice of $T = (T_P)$. There are two ways to do this.

The first is by induction on the semi-simple rank of $G$, that is to say the dimension of $\mathfrak{a}^{\Delta}_{\emptyset}$. If it is 0 then $\Gamma\backslash X$ modulo $A_\Delta$ is compact, and the only rational parabolic subgroup is $G$ itself.

Now we make the induction assumption that the partition is known for all the Levi subgroups $M_P$. For each rational $P$ the quotient $N_P\backslash X$ may be identified with $X^G_{M_P}$. The parameters $T$ determine a family of parameters for $M_P$ as well. Define $X_{M_P,T}$ to be the inverse image of $X^{M_P}_{M_P,T}$ in $X^G$. Define for $P \neq G$

$$X^G_{P,T} = X_{M_P,T} \cap C^G_{P,T} \ ,$$

and then let $X^G_{G,T}$ be what's left over. Arthur's Theorem, proved by him for adelic groups and rediscovered since by many others, is that for $T$ large enough these sets partition $X$.

The second way is equivalent, but not easily seen to be so. Define $X^G_P$ as above, but define $X^G_G$ to be the complement of the regions $C^G_{P,T}$ as $P$ varies over all maximal rational parabolic subgroups. This is essentially Arthur's definition in [Arthur: 1978]. His proof does not readily suggest what to do for arithmetic groups. The nicest exposition in this case, very readable if a little long-winded, is that of [Saper: 1997]. A relatively simple and direct derivation can also be carried out following Satake's construction of arithmetic compactifications.

One thing to keep in mind is that the regions $X^G_{P,T}$ are neighbourhoods of well defined regions in rational boundary components of certain Satake compactifications.

## 3. Arthur's truncation

The constant term with respect to the rational parabolic subgroup $P$ of a function $F$ on $\Gamma\backslash G$ is the function

$$F_P(x) = \int_{\Gamma\cap N_P\backslash N_P} F(xn)\, dn \ ,$$

a function on the quotient $\Gamma \cap P\backslash G$ invariant under $N_P$, hence also on $N_P(\Gamma \cap P)\backslash G$.

To give Arthur's definition of truncations, we need to define another collection of regions on root spaces and on $X$, one somewhat similar to that of the regions $C^G_P$. On $\mathfrak{a}^{\Delta}_{\Theta}$ define $\widehat{C}^{\Delta}_{\Theta}$ to be the obtuse cone spanned by the projections of the roots in $\Delta$, and correspondingly define $\widehat{\mathfrak{X}}^G_P$ and $\widehat{\mathfrak{X}}^G_{P,T}$. Let $\widehat{\tau}^G_{P,T}$ be the characteristic function of $\widehat{\mathfrak{X}}^G_{P,T}$.

Arthur's definition of truncation, one related to an earlier definition due to Langlands, is

$$\Lambda^T_G F = \sum_P (-1)^{\mathrm{corank}\, P} \widehat{\tau}_{P,T}\, F_P$$

where the sum is over all rational parabolic subgroups $P$. Here the **corank** of $P$ is $\dim A_P - \dim A_G$.

This definition agrees with the previous definition for $\mathrm{SL}_2$. But for groups of higher rank it looks more puzzling the more you think about it. Nonetheless it is in some sense, and certainly for Arthur's purposes, exactly the right definition. Again, under a mild growth condition the truncation of $F$ is rapidly decreasing at infinity, and for Eisenstein series induced from cusp forms there is an extension of the Maass-Selberg formulas. It is related to the definition of the partition defined in the previous section because of this result, which says that this truncation of $F$ is one component in an orthogonal decomposition of $F$:

**Proposition.** *For any $F$ on $\Gamma\backslash G$ we have an orthogonal decomposition*

$$F = \sum_{\mathcal{C}} \sum_{Q\in\mathcal{C}} \tau_{Q,T}\, \Lambda^T_{M_Q} F_Q \ ,$$

*where the (finite) outer sum is over $\Gamma$-conjugacy classes of rational parabolic subgroups, and the inner over groups in the class.*

The decomposition is well known, but the orthogonality seems to be is slightly stronger than what is stated in the literature, although it is implicit in Arthur's work.

There are a number of puzzling features of Arthur's definition. They can best be summed up by saying that except for groups of rank one truncation doesn't *seem* to be local on any of the natural compactifications. The situation looks confusing already for rank two groups, for which the constant term chopped away for maximal proper parabolic subgroups has 'wings' that are not evidently local. I don't know how to account for this in any satisfactory way.

# Part II. Euclidean spaces

The rest of this paper is devoted to motivating the definition through relating it to analogous notions in Euclidean spaces.

## 4. A simple model for the theory of Eisenstein series

There are two aspects to dealing with Arthur's truncation operators, combinatorial and analytical. In Arthur's work the separation is not always so clear, but there is a ridiculously simple model for automorphic forms in which the analytical problems disappear, leaving only combinatorial ones. In the most pleasant of all worlds solving the combinatorial problems for this model could then be applied directly to Arthur's work, but alas! this doesn't seem to be quite true. Nonetheless the model is useful.

In this model, the group $G$ becomes the semi-direct product of the torus $A = A_\emptyset$ and the Weyl group $W$ of a root system $\Sigma$, and $\Gamma$ just becomes $W$. The group $K$ is the semi-direct product of $W$ and the torsion in $A$ which is $K \cap A$, so the symmetric space is $X = A/K \cap A$. Intuitively, this is what you get by shrinking all unipotent elements in a reductive group to 1.

In showing how this drastically reduced group can still be interesting, I'll write $A/K \cap A$ additively as the real vector space $V = V_\emptyset$, just as I did earlier. I'll also use the definitions I introduced there.

The parabolic groups in this scheme are the semi-direct products of $W_*$ and $V$, where $W_*$ is one of the Weyl groups generated by a basic subset of reflections. Every one of them is conjugate to a unique $W_\Theta$, the subgroup of elements of $W$ fixing vectors in $V_\Theta$.

Constant terms and Eisenstein series are very simple. Any constant term of a function on $W\backslash V$ is just itself, since all unipotent groups are trivial. And if $\Phi$ is invariant under $W_P$ then the associated Eisenstein series is

$$E(\varphi \,|\, P, G)(v) = \sum_{W_P\backslash W} \varphi(wv) \ .$$

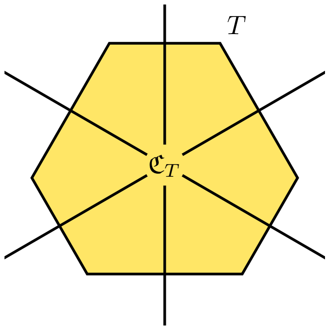
If $s$ is a character of $V$ then the Eisenstein series associated to it is

$$E_s = \sum_W w^{-1}s \ .$$

In other words, this is a universe where all $c(s) = 1$. Another simplification is that there are no cusp forms. One can see directly that everything is indeed an Eisenstein series, but the intuitive reason for this is that since all unipotent groups are trivial, any cusp form automatically vanishes identically.

As I said, analytic problems vanish. Combinatorial ones—interesting ones—remain.

Given a regular point $T$ in $V$, there is a natural definition of truncation associated to it. If $\mathfrak{C} = \mathfrak{C}_T$ is the convex hull of the $W$-orbit of $T$, the truncation operator $\Lambda^T$ is simply multiplication by its characteristic function $\chi_\mathfrak{C}$.




Truncation is restriction to the convex hull of $W \cdot T$

Arthur's definition in this context is at first sight astonishing. The parabolic subgroups correspond precisely to faces $F$ of $\mathfrak{C}$. If $F$ is a face of $\mathfrak{C}$ of codimension one, its **exterior** $E_F^{\mathfrak{C}}$ is the open half-space on is side away from $\mathfrak{C}$. To other faces is assigned an exterior in this way:

$$E_F^{\mathfrak{C}} = \begin{cases} V & \text{if } F = \mathfrak{C} \\ \bigcap_{F_*} E_{F_*}^{\mathfrak{C}} & \text{if } F \text{ is the intersection of faces } F_* \text{ of codimension one} \end{cases}$$

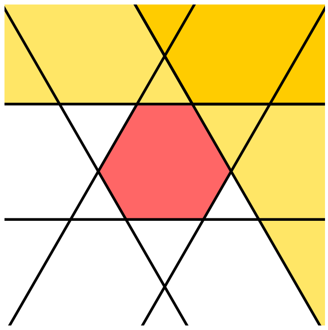
Let $\mathcal{E}_F^{\mathfrak{C}}$ be the characteristic function of $E_F^{\mathfrak{C}}$.

Arthur's definition of truncation suggests this result:

**Theorem.** *We have*

$$\chi_{\mathfrak{C}} = \sum_P (-1)^{\operatorname{codim} F} \mathcal{E}_F^{\mathfrak{C}} .$$

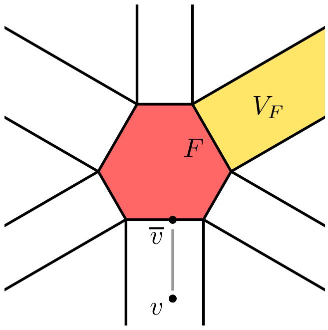
It is not obvious.



Why does Arthur's truncation agree with the natural one?

As I'll show in the next few sections, this is a special case of a much more general result about convex polyhedra, more particularly one apparently first observed by Brion and Vergne around 1995. It is implicit in Arthur's much older work, where the convex polyhedra are hulls of Weyl orbits.

The orthogonal decomposition is natural here, too. The convex set $\mathfrak{C}$ determines a partition of $V$ according to the criterion of nearest points—to $v$ in $V$ is associated $\overline{v}$ of $\mathfrak{C}$ that is nearer to $v$ than any other point of $\mathfrak{C}$.




The partition according to nearest face

We thus get a partition of $V$—the set $V_F$ is that of all points $v$ for which $\overline{v}$ lies in $F$. This determines determines an orthogonal decomposition of the $W$-invariant functions on $V$ analogous to Arthur's.

If $\varphi$ is any function on $V$ and $\Phi$ a $W$-invariant one, then the inner product

$$\langle \Lambda^T \Phi, E_\varphi \rangle$$

is finite. It can be expressed in a curious fashion.

Let $\mathfrak{S} = \mathfrak{S}_\Delta$ be the obtuse cone spanned by $V_\Delta$ and $-\Delta$. The faces of $\mathfrak{S}$ are the sets $\mathfrak{S}_\Theta$ with $\Theta \subseteq \Delta$, the interiors of the faces of $\mathfrak{S}$ spanned by $V_\Delta$ and $-\Theta$. For any one of these faces $F$ let $E_F^{\mathfrak{S}}$ be its exterior (defined along lines similar to the definition of the exteriors of faces of $\mathfrak{C}$). Define the function

$$\mathsf{sgn}_W = \sum_\Theta (-1)^{|\Delta|-|\Theta|} \frac{1}{|W_\Theta|}\ \mathcal{E}_\Theta^{\mathfrak{S}}\ ,$$

where $W_\Theta$ is the Weyl group generated by reflections in the roots of $\Theta$. This odd-looking function is motivated very simply, because if $T$ lies in the positive Weyl chamber and $\mathsf{sgn}_{W,T}$ is the shift by $T$, then an easy calculation shows that

$$E_\emptyset^\Delta(\mathsf{sgn}_{W,T}) = \sum_F (-1)^{\operatorname{codim} F} \mathcal{E}_F^{\mathfrak{C}} = \mathfrak{char}_{\mathfrak{C}}\ .$$

Let $E = E_{W_\emptyset}^{W_\Delta}$. From this it is immediate that:

**Proposition.** *For $f, \varphi$ in $L^2(V)$*

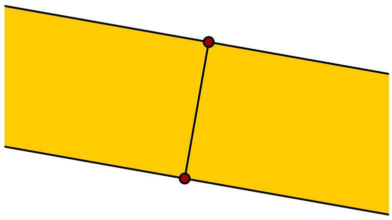
$$\langle \Lambda^T E_f, E_\varphi \rangle_{W\backslash V} = \langle\, \mathsf{sgn}_{W,T} \cdot \textit{ constant term of } E_f,\ \textit{constant term of } E_\varphi \rangle_V\ .$$

This formula turns out to remain valid even in the case of arithmetic quotients, when formulated correctly, and is a crucial step in verifying the Plancherel measure for arithmetic quotients without the contour motions of Langlands' classic arguments. This argument is presented in [Casselman: 1999] for $\mathrm{SL}_2(\mathbb{Z})$.

In the rest of this paper, I'll cease to restrict myself to $W$-invariant functions on a root space, and instead consider arbitrary convex polyhedra in Euclidean space. It's not clear to me to what extent the more general theory will be of interest for investigating automorphic forms, but it seems likely to me that it will be.

### 5. Euclidean truncation

Suppose $C$ to be any convex polyhedron in the Euclidean space $V$. I'll assume it to possess a finite number of faces, but how to get along without that assumption looks like an interesting question. Assume further that $C$ has a non-empty interior. If it does not, replace $C$ by the product of $C$ and the linear subspace of all vectors perpendicular to its affine support.

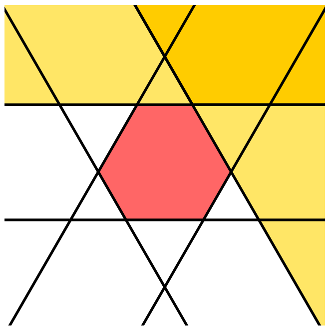


The polyhedron extended to have an interior

By no means do I assume $C$ to be bounded.

For any face $F$ of $C$ of codimension one, define its **exterior** $E_F^C$ to be the open half-space with $F$ in its boundary that does not contain $C$. As earlier, define $E_F^C$ more generally by the specification

$$E_F^{\mathfrak{C}} = \begin{cases} V & \text{if } F = \mathfrak{C} \\ \bigcap_{F_*} E_{F_*}^{\mathfrak{C}} & \text{if } F \text{ is the intersection of faces } F_* \text{ of codimension one} \end{cases}$$



The exterior of some faces

Let $\mathcal{E}_F^{\mathfrak{C}}$ be the characteristic function of $E_F^{\mathfrak{C}}$.

The following result seems to have been observed first by M. N. Ishida in the case of homogeneous cones and by M. Brion and M. Vergne in the case of bounded polyhedra. It seems not to have been previously announced for arbitrary polyhedra.

**Theorem T.** *We have*

$$\mathfrak{char}_C = \sum_{F \preceq C} (-1)^{\operatorname{codim} F} \mathcal{E}_F^C$$

*where the sum is over all faces $F$ of $C$.*

The proof goes in steps—first come bounded polyhedra, next cones, then a local version, and finally the general case.

Here is an equivalent formulation:

*For any $P$ in $V$*

$$\sum_{F \preceq C \,|\, P \in E_F^C} (-1)^{\operatorname{codim} F} = \begin{cases} 1 & P \in C \\ 0 & \textit{otherwise.} \end{cases}$$

The proof will occupy the next several sections. There are a few cases that are straightforward, and one implicit in work of Arthur whose argument I'll also present.

**Simplicial cones.** A simplicial cone is a coordinate octant

$$C = \{v \mid x_i \le 0 \text{ for } 1 \le i \le n\}$$

Its faces are the degenerate cones

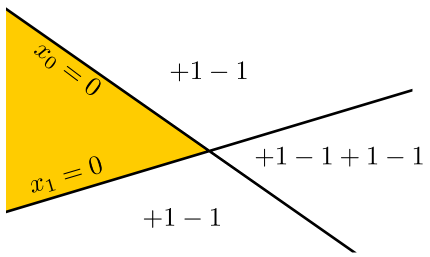
$$C_\Theta^\Delta = \{x \mid x_i = 0 \text{ for } i \in \Theta, x_i \le 0 \text{ for } i \notin \Theta\}$$

with $\Theta \subseteq \Delta = \{1, \dots, n\}$. The exterior of $C_\Theta^\Delta$ is the set

$$E_\Theta^\Delta = \{x \mid x_i > 0 \text{ for } i \in \Theta\} \; .$$

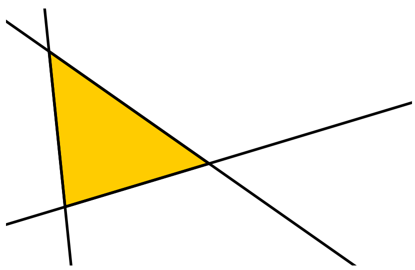
For $x$ in $V$ let $\Theta(x) = \{i \mid x_i > 0\}$. According to the binomial theorem the sum in the theorem is

$$\sum_{\Theta \subseteq \Theta(x)} (-1)^{|\Theta|} = \begin{cases} 1 & \text{if } \Theta(x) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$




For simplicial cones the proof is straightforward

**Simplices.** A simplex are created by slicing a simplicial cone transversely, with any one of its vertices possibly functioning as the vertex of the cone. On the vertex side of the slice, the configurations defined by simplex and cone are the same.



The case of a simplex reduces to that of a simplicial cone

**Weyl hulls.** Suppose $T$ to be a point in the interior of the positive Weyl chamber in a root system, and $\mathfrak{C}$ the convex hull of the Weyl group orbit of $T$. Arthur's proof of one of the basic formulas involving truncation (Lemma 1.1 of [Arthur: 1980]) is valid also for Theorem T in this case. His argument is very elementary, but depends strongly on the structure of Weyl groups.

To pick up with the general argument—the expression

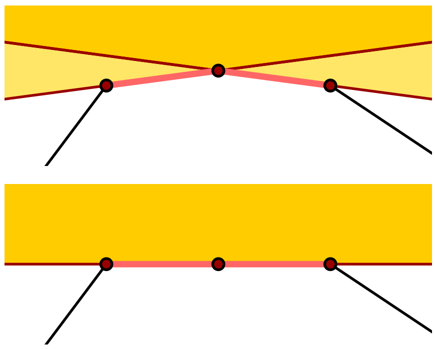
$$\sum_{F \preceq C \,|\, P \in E_F^C} (-1)^{\operatorname{codim} F}$$

suggests that cohomological **Euler-Poincaré characteristics** are going to play a role. We'll need to know that

$$\sum_{F \preceq C} (-1)^{\mathrm{codim} F} = \begin{cases} (-1)^{\dim C} & \text{if } C \text{ is a closed, bounded, convex polyhedron} \\ 0 & \text{if } C \text{ is a closed cone} \\ 1 & \text{if } C \text{ is open.} \end{cases}$$

The first two are equivalent, since a suitable slice through a cone is a bounded polyhedron. The last also follows from the first by subtracting from the first formula the faces on the boundary, which make up a sphere.

Two faces are allowed to have the same affine support. In effect, the following two configuratioms are almost the same, and the second should be considered legitimate.

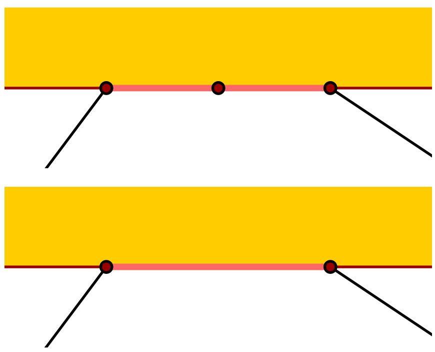


Degenerate faces are limits of non-degenerate ones . . .

In effect, the Theorem is really about a cell decomposition of $C$ compatible with its convex structure. If a number of faces $F_*$ partition an open geometric face $F^\circ$, their total contribution is just

$$\sum_{F_* \subseteq F^\circ} (-1)^{\mathrm{codim} F_*} E^C_{F_*} = E^C_F$$

since all $E^C_{F_*} = E^C_F$ here and the Euler-Poincaré characteristic of the open face is $1$. The following two configurations are therefore completely equivalent.



. . . but it doesn't matter

This gives me the feeling that some deeper topological phenomenon is at the heart of the matter, but I don't know how to make this precise.

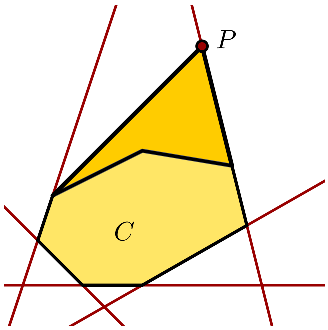
## 6. Bounded convex polyhedra

In this section I'll follow Brion and Vergne very closely in proving the Theorem for bounded polyhedra. All that's new here are the remarks above about possible facial degeneracy, a difficulty they ignore (without serious harm).

We are given a bounded polyhedron $C$ and want to prove that for any point $P$

$$\sum_{F \preceq C \,|\, P \in E_F^C} (-1)^{\mathrm{codim} F} = \begin{cases} 1 & P \in C \\ 0 & \text{otherwise} \end{cases}$$

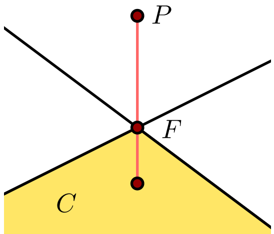
If $P$ is a point of $C$ this is immediate, since the only exterior $P$ belongs to is that of $C$ itself. If $P$ is not in $C$, let $H$ be the convex hull of $C$ and $P$.




Let $H$ be the hull of $P$ and $C$

Then $H$ is the union of all segments $[P, Q]$ with $Q$ in $C$, and $H^\circ$ is the union of all $[Q, P)$ with $Q$ in $C^\circ$.

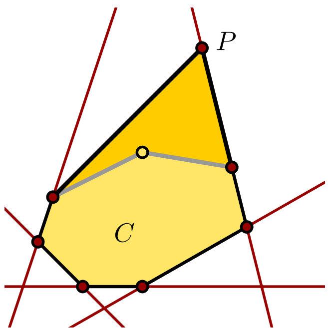
**Proposition.** *If $F$ is a face of $C$ then $F^\circ \subset H^\circ$ if and only if $P \in E_F^C$.*




Symmetry near a face

This is because near $F^\circ$ the polyhedron $C$ looks like the tangent cone to $C$ at $F$, and for a cone there is symmetry between interior and exterior.

**Corollary.** *A face $F$ of $C$ is one of the cells in the boundary of $H$ if and only if $P \notin E_F^C$.*




Which faces of $C$ are also faces of $H$?

Now we can argue as follows:

$$\sum_{F\preceq H}(-1)^{\mathrm{codim} F} = (-1)^{\dim C} \ \text{ because } P \text{ is in } H$$

$$\sum_{F\preceq H, P\in F}(-1)^{\mathrm{codim} F} = 0 \ \text{ because } H \text{ is a cone near } P \text{ with E. P. equal to } 0$$

$$\sum_{F\preceq H, P\notin F}(-1)^{\mathrm{codim} F} = (-1)^{\dim C} \ \text{ by subtraction}$$

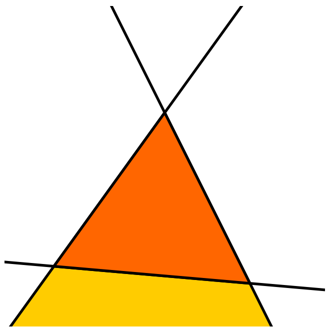
$$\sum_{F\preceq C, P\notin E_F^C}(-1)^{\mathrm{codim} F} = (-1)^{\dim C} \ \text{ by the Lemma}$$

$$\sum_{F\preceq C}(-1)^{\mathrm{codim} F} = (-1)^{\dim C} \ \text{ because } C \text{ is a bounded polyhedron}$$

$$\sum_{P\in E_F^C}(-1)^{\mathrm{codim} F} = 0 \ \text{ by subtraction. Q.E.D.}$$

## 7. Cones

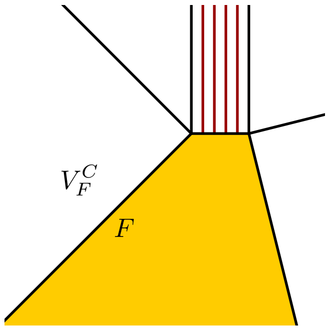
The formula for cones reduces to that for bounded convex sets by taking slices, as has already been remarked in the simplicial case.



Above the slice, the two configurations are the same

## 8. A local version

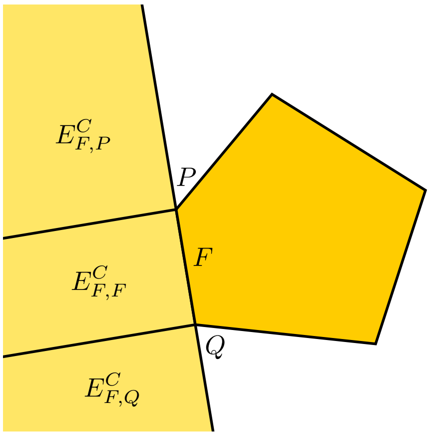
For each face $F$ of $C$, let $\widehat{V}_F^C$ be the set of points in $V$ for which the point of $C$ nearest to it lies in $F$. Then let $V_C^F$ be the points of $\widehat{V}_F^C$ that are not in some $\widehat{V}_{F_*}^C$ for a larger $F_*$. Thus $V_C^C$ is all of $C$ and $V_F^C$ for vertices is the interior of the points for which $F$ is nearest. These $V_F^C$ partition $V$.




$V_F^C$ possesses a product structure $F \times T_F^C$

For each couple of faces $F_* \preceq F$ let

$$E^C_{F,F_*} = E^C_F \cap V^F_{F_*} \,.$$




The nearest face partition of $E^C_F$

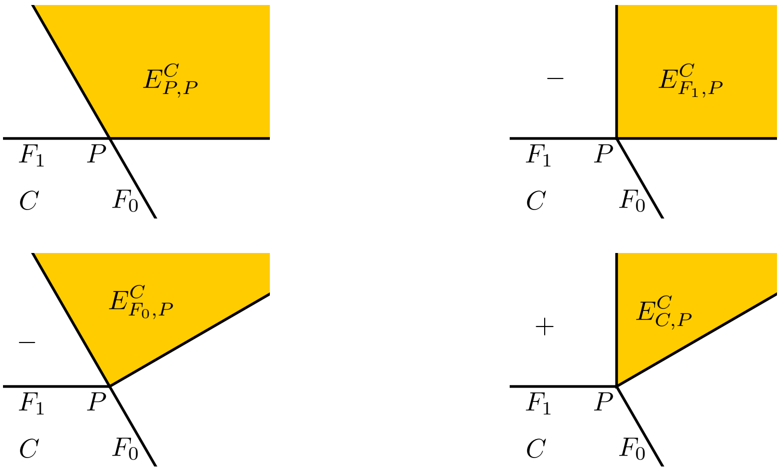
Thus a point $v$ of $E^C_F$ lies in $E^C_{F,F_*}$ if and only if the point of $F$ closest to it lies in $F_*$.

**Theorem L.** *For each face $F_*$ of $C$*

$$\sum_{F \mid F_* \preceq F} (-1)^{\operatorname{codim} F}\, \mathcal{E}^C_{F,F_*} = \begin{cases} 0 & F_* \neq C \\ \chi_C & F_* = C \end{cases}$$

This is one variation of Langlands' combinatorial lemma.

Theorem L in two dimensions is covered by these images . . .




. . . whose secret is given away by these:


Theorem T for cones implies Theorem L. Theorem L asserts that

$$\sum_{F \mid F_* \preceq F} (-1)^{\operatorname{codim} F} \, \mathcal{E}^C_{F,F_*} = \begin{cases} 0 & F_* \neq C \\ \chi_C & F_* = C \end{cases}$$

In this, $C$ may be replaced by its tangent cone at $F_*$. At any face but a vertex, the tangent cone at that face has a simple product structure, and induction proves the claim. The formula for the full cone can be rearranged to give it for the vertex.

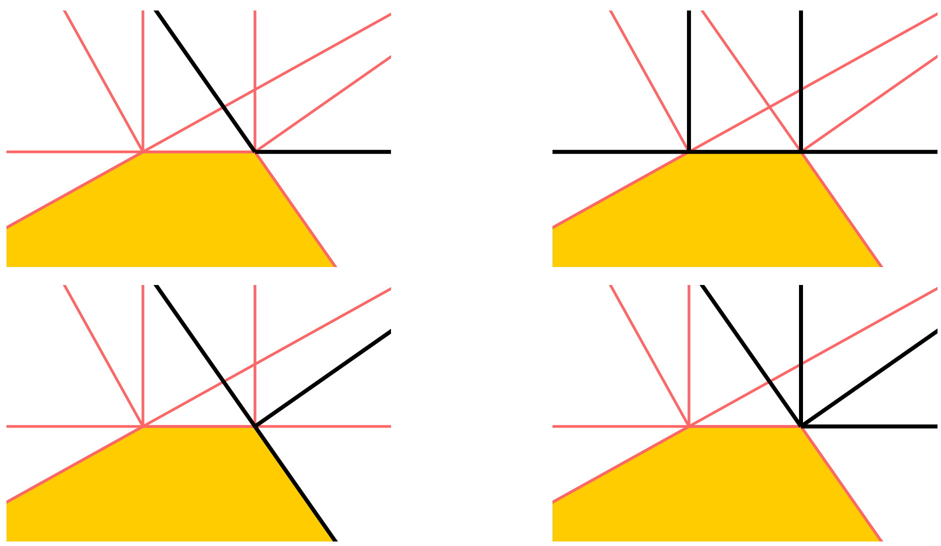
### 9. The general case

Theorem C now follows from Theorem L by introducing the partition

$$\mathcal{E}^C_F = \sum_{F_* \preceq F} \mathcal{E}^C_{F,F_*}$$

and then rearranging the sum:

$$\begin{aligned} \sum_F (-1)^{\operatorname{codim} F} \, \mathcal{E}^C_F &= \sum_{F, F_* \mid F_* \preceq F} (-1)^{\operatorname{codim} F} \, \mathcal{E}^C_{F,F_*} \\ &= \sum_{F_*} \sum_{F \mid F_* \preceq F} (-1)^{\operatorname{codim} F} \, \mathcal{E}^C_{F,F_*} \\ &= \chi_C \, . \end{aligned}$$



#### 10. Langlands' combinatorial lemma

If $F$ is a face of $C$, let $T_F^C$ be the translation of $V_F^C$ by the support of $F$, and $\tau_F^C$ its characteristic function. Thus $V_F^C = T_F^C \times F^\circ$. When $C$ is an obtuse simplicial cone the following result is essentially the same as the original combinatorial lemma of Langlands.

**Theorem.** *For any face $F$ of $C$*

$$\sum_{F \preceq F_* \preceq C} (-1)^{\mathrm{codim} F} \tau_{F_*}^C \mathcal{E}_F^{F_*} = \begin{cases} 1 & \text{if } F = C \\ 0 & \text{otherwise} \end{cases}$$

The case $F = C$ is trivial. The proof for other $F$ uses the partition of $V$ into the $V_F^C$, and goes by induction.

The original applied to simplicial cones and was announced by Langlands without proof in his 1965 Boulder talk on Eisenstein series, and a result equivalent to this one is contained in the appendices to a recent paper by Goresky et al.

#### 11. The analogue of the Maass-Selberg formula

The Maass-Selberg formula is an explicit formula for the inner product of two Eisenstein series. Its analogue here is a formula for the Fourier transform of the characteristic function of a bounded convex set with $C^\circ \neq \emptyset$, observed by Brion & Vergne:

$$\widehat{\chi}_C(s) = \sum_P (-1)^{\mathrm{codim} F} \widehat{\mathcal{E}}_P^C(s)$$

where the right hand sum is over the vertices of $C$, and the expression is taken to be the analytic continuation of the obvious integral. It is not difficult to deduce, among other things by reversing interiors and exteriors, from this the following result, apparently first observed by Brion and Vergne:

**Proposition.** *The Fourier transform of the characteristic function of a bounded convex polyhedron is the sum of the formal Fourier transforms of the tangent cones at its vertices.*

This seems to be useful only when the vertices are simplicial, in which case these Fourier transforms can be evaluated easily. There ought to be an elegant formula for the Fourier transform of a cone in terms of elementary information in the general case, but I am not aware of one.

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