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Bruhat filtrations and Whittaker vectors for real groups

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Abstract. In this paper we continue earlier work of Bruhat, Jacquet, and Shalika on Whittaker models for smooth representations of real groups. We prove exactness in a certain category of such smooth representations, and also give a new proof of holomorphicity as a function of the inducing parameter of principal series. This makes the theory of Whittaker models for real groups almost as simple to formulate as that of p-adic groups.

Contents

1. Introduction

Suppose G to be a real reductive Lie group of Harish-Chandra class [**15**], P a minimal parabolic subgroup of G and N the unipotent radical of P. Let K be a maximal compact subgroup of G. Denote by $\mathfrak{g}_0$, $\mathfrak{n}_0$ and $\mathfrak{k}_0$ the Lie algebras of G, N and K respectively. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{q}_0$ be the Cartan decomposition of $\mathfrak{g}_0$. Denote by $\mathfrak{a}_0$ a maximal abelian Lie subalgebra of $\mathfrak{q}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$ and $\mathfrak{a}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{n}_0$ and $\mathfrak{a}_0$ respectively. Let R be the (restricted) root

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system in $\mathfrak{a}^*$ of the pair $(\mathfrak{g}, \mathfrak{a})$. Then $\mathfrak{n}$ is the direct sum of root subspaces $\mathfrak{g}_\alpha$ for roots α in a set R^+ of positive roots in R. Denote by B the corresponding set of simple roots in R^+. Let ψ be a unitary character of N, and η its differential. Then η is completely determined by its restrictions to the root subspaces $\mathfrak{g}_\alpha$ corresponding to the roots $\alpha \in B$. The character ψ is said to be *non-degenerate* if all of these restrictions are non-trivial. Denote by $\mathbb{C}_\eta$ the one-dimensional representation of N with action given by ψ. If (π, V) is a representation of $\mathfrak{g}$, a *Whittaker functional* on π is an $\mathfrak{n}$-covariant map from V to $\mathbb{C}_\eta$. If π is the restriction to $\mathfrak{g}$ of a smooth representation of G, then a Whittaker functional corresponds to an N-covariant map to $\mathbb{C}_\eta$ as well, therefore a G-covariant map from V into the space of smooth functions f on G such that

$$f(ng) = \psi(n)f(g)$$

for all $g \in G$, $n \in N$. Functions in the image are called Whittaker functions, because in simple cases they essentially coincide with confluent hypergeometric functions treated extensively in Whittaker's classic text. They, or rather their analogues for certain finite groups, were apparently introduced into representation theory by I. M. Gelfand and colleagues. The most important role they play is probably in constructing archimedean factors of L-functions associated to certain representations of adèle groups, and in particular to automorphic forms. There have been many interesting papers about them, notably Jacquet's thesis [**18**], an early paper by Shalika [**25**], a paper of Kostant [**21**], and the thesis of Tze-Ming To [**26**]. There remain so far unanswered, however, a few interesting technical but important questions which we propose to answer here. We will also offer new proofs of known results, and along the way explain a fundamental result in the representation theory which as so far been only weakly exploited, what we call the *Bruhat filtration* of the smooth principal series. This is a refinement of much earlier work of Bruhat [**6**].

The main results of this paper depend strongly on the Bruhat filtration, and include (1) a proof of the exactness of what we call the Whittaker functor, (2) uniqueness of Whittaker functionals for suitable smooth irreducible representations of quasisplit groups, and (3) holomorphicity of Whittaker functionals for the smooth principal series and other analytic families of induced representations. Many results in this paper have appeared before in various work of others, but not in as useful or as general a form as we would like (see Matumoto [**22**], Wallach [**29**]). The Bruhat filtration can be used to give more elegant proofs of many other results in the representation theory of G. It might serve, in fact, as the cornerstone of a reasonably elegant exposition of the subject from its beginnings. We hope to return to these ideas in a subsequent paper.

It is well known that Harish-Chandra was much concerned with these matters right up to his untimely death, and we therefore think it is not inappropriate to dedicate this paper to his memory. We would like to thank Hervé Jacquet, who pointed out to us a long time ago that there is some truth to the claim that nearly all of analysis reduces to integration by parts. One of us (D.M.) would also like to thank the Department of Mathematics, Harvard University, for their support during the period when the final draft of the paper was written.

1.1. The simplest case. The main part of this paper is tough and abstract, and we want to explain the themes here in the simplest case, when the arguments can be laid out in very classical terms. Let G be the group $\mathrm{SL}(2, \mathbb{R})$ of unimodular two-by-two real matrices, P the subgroup of upper triangular matrices and N the unipotent radical of P.

Further, let

$$\psi\colon N \to \mathbb{C}, \qquad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mapsto e^{\eta x}$$

be a unitary character of N, with η purely imaginary. At first we shall make no assumption on η, but eventually we shall assume $\eta \neq 0$.

For $s \in \mathbb{C}$ we define define the smooth principal series representation

$$I_s = \{f \in C^\infty(G) \mid f(nag) = |y|^{s+1} f(g)\}$$

for $g \in G$, $n \in N$, and

$$a = \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} .$$

The group G is the disjoint union of two subsets PwN and P, where

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

The space I_s is that of smooth sections of a suitable real analytic line bundle on $\mathbb{P}^1(\mathbb{R})$. It contains the subspace $I_{s,w}$ of all sections vanishing of infinite order at ∞, and has as quotient the space $I_{s,1}$ of formal sections (Taylor series) of the analytic line bundle at ∞. For example, if $s = -1$ the space I_s is just $C^\infty(\mathbb{P}^1(\mathbb{R}))$, which is non-canonically the same as the space of smooth functions on the circle, $I_{s,w}$ is the subspace of functions vanishing of infinite order at ∞, and the quotient may be identified with $\mathbb{C}[[\frac{1}{x}]]$. The complement of ∞ on $\mathbb{P}^1(\mathbb{R})$ is a single orbit under N, so that functions in $I_{s,w}$ may be identified with functions on N. It is elementary in this case that this allows us to identify $I_{s,w}$ as an N-module with $\mathcal{S}(N)$, the Schwartz space of the additive group. This is essentially a special case of the well known observation of Laurent Schwartz that the Schwartz space of $\mathbb{R}^n$ may be identified with the space of all smooth functions on the unit n-dimensional sphere S^n vanishing of infinite order at the north pole, *via* stereographic projection. The exact sequence

$$0 \to I_{s,w} \to I_s \to I_{s,1} \to 0$$

is what we call the Bruhat filtration of I_s. The last map is surjective because of the well known result of Émile Borel. Note that these spaces are all stable under N and $\mathfrak{g}$. The dual of the space $I_{s,1}$, a space of distributions with support at ∞, may be identified with a certain Verma module over $\mathfrak{g}$.

For a function $f \in I_{s,w}$, the integral

$$\Omega_{\psi,s}(f) = \int_N \psi^{-1}(n) f(wn)\, dn$$

converges absolutely, and defines a ψ-covariant functional on $I_{s,w}$ holomorphic in s. The basic question we are going to investigate here is this: to what extent can this functional be extended covariantly to one on all of I_s? It turns out that the integral defining $\Omega_{\psi,s}$ is convergent for s in a certain half-plane of $\mathbb{C}$, and that it possesses a meromorphic continuation for all s, holomorphic in the case $\eta \neq 0$. It is the holomorphicity that we are really concerned with here.

These results can be formulated in entirely classical terms.

1.2. Some elementary functional analysis. In this section we shall investigate some problems which have no obvious relevance to representation theory, but in the next we shall explain the connection.

Suppose f to be in $\mathcal{S}(\mathbb{R})$, $s \in \mathbb{C}$, $\eta \in i\mathbb{R}$. The integral

$$\int_{-\infty}^{\infty} e^{-\frac{\eta}{x}} |x|^s f(x) dx$$

converges for $\mathrm{Re}(s) > -1$ and defines in this region a tempered distribution on $\mathbb{R}$. As long as f vanishes of infinite order at 0 this integral is defined and analytic for all values of s. Does it have for all s a natural extension to all of $\mathcal{S}(\mathbb{R})$?

If $\eta = 0$ the answer is negative, but the situation is perfectly well understood. Integration by parts gives us

$$\int_{-\infty}^{\infty} |x|^s f(x)\, dx = -\frac{1}{s+1} \int_{-\infty}^{\infty} |x|^s x f'(x)\, dx$$

and this allows us to continue the distribution at least into the region where $\mathrm{Re}(s) > -2$. If we multiply by $s+1$ and take the limit as $s \to -1$ we get

$$-\int_{-\infty}^{\infty} \mathrm{sgn}(x) f'(x)\, dx = \int_{-\infty}^{0} f'(x)\, dx - \int_{0}^{\infty} f'(x)\, dx = 2f(0)\ .$$

In other words, integration by parts allows to extend the distribution meromorphically to the region $\mathrm{Re}(s) > -2$, with a residue at $s = -1$ equal to $2\delta_0$. Of course we can again integrate by parts and eventually we see that we have a meromorphic map from $\mathbb{C}$ to the space of tempered distributions on $\mathbb{R}$, with simple poles at $-1, -3, -5, \dots$ and residues that are proportional to derivatives of δ_0.

Now suppose that $\eta \neq 0$. Since η is purely imaginary, the function $e^{-\frac{\eta}{x}}$ oscillates wildly near 0, and it is reasonable to expect that the integral

$$\int_{-\infty}^{\infty} e^{-\frac{\eta}{x}} |x|^s f(x)\, dx$$

can be holomorphically extended. A different kind of integration by parts will do the trick. The function

$$\varphi_s(x) = |x|^s e^{-\frac{\eta}{x}}$$

satisfies the differential equation

$$x^2 \varphi_s'(x) - (sx + \eta)\varphi_s(x) = 0$$

Therefore the integral becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi_s(x) f(x)\, dx &= \frac{1}{\eta} \int_{-\infty}^{\infty} x^2 \varphi_s'(x) f(x)\, dx - \frac{1}{\eta} \int_{-\infty}^{\infty} s x \varphi_s(x) f(x)\, dx \\
&= -\frac{1}{\eta} \int_{-\infty}^{\infty} x^2 \varphi_s(x) f'(x)\, dx - \frac{1}{\eta} \int_{-\infty}^{\infty} (s+2) x \varphi_s(x) f(x)\, dx \\
&= \frac{1}{\eta} \int_{-\infty}^{\infty} \varphi_s(x) \left(-x^2 f'(x) - (s+2) x f(x)\right)\, dx.
\end{aligned}$$

This converges for $\mathrm{Re}(s) > -2$, since the function $x \longmapsto -x^2 f'(x) - (s+2) x f(x)$ vanishes at 0. As before, we can repeat this procedure, eventually obtaining a distribution holomorphic as a function of s.

What is relevant for the rest of this paper is that the second argument reduces in essence to this observation: *The differential operator*

$$f(x) \longmapsto -x^2 f'(x) - (s+2)xf(x)$$

induces a topologically nilpotent map of the space of formal power series $\mathbb{C}[[x]]$ *into itself.*

1.3. Relevance to representation theory. Return to the notation of the earlier section on $\mathrm{SL}(2,\mathbb{R})$. The group G is the disjoint union of PwN and P, and the overlapping union of open subsets PwN and $P\bar{N}$, where $\bar{N}$ is the subgroup of lower unipotent matrices. A function in I_s may therefore be expressed as a sum of functions f_w and f_1, where f_w has compact support modulo P on PwN and f_1 has compact support modulo P on $P\bar{N}$. In considering how to extend the functional $\Omega_{\psi,s}$, we may assume $f = f_1$.

Therefore suppose now that f has support on $P\bar{N}$. How do we evaluate $\Omega_{\psi,s}(f)$?

We have

$$\Omega_{\psi,s}(f) = \int_N \psi^{-1}(n) f(wn)\, dn$$

and also $f(p\bar{n}) = |c|^{s+1} f(\bar{n})$ if $p \in P$, $\bar{n} \in \bar{N}$, where the restriction of f to $\bar{N}$ may be identified with a function $\varphi(y)$ of compact support. Here

$$p = \begin{bmatrix} c & x \\ 0 & c^{-1} \end{bmatrix}, \qquad \bar{n} = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$

We can write

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} = \begin{bmatrix} 1 & -x^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{-1} & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x^{-1} & 1 \end{bmatrix}$$

and therefore the integral becomes

$$\int_{-\infty}^{\infty} |x|^{-s-1} \varphi(x^{-1}) e^{-\eta x}\, dx = -\int_{-\infty}^{\infty} |z|^{s-1} e^{-\frac{\eta}{z}} \varphi(z)\, dz$$

which of course leads us back to the calculation in the previous section.

This is admittedly elementary, but not entirely satisfactory. The explicit calculation does not really tell us what is going on in terms of representations of $\mathrm{SL}(2,\mathbb{R})$. So we shall now reformulate it. Consider the exact sequence

$$0 \to I_{s,w} \to I_s \to I_{s,1} \to 0$$

as a sequence of modules over $\mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of N. The integral defining the functional $\Omega_{\psi,s}$ induces, as we have seen, an $\mathfrak{n}$-covariant map from $I_{s,w}$ to $\mathbb{C}_\eta$, or equivalently an $\mathfrak{n}$-covariant map from $\mathbb{C}_{-\eta}$ to the strong dual ${I_{s,w}}'$ of $I_{s,w}$, or again equivalently an $\mathfrak{n}$-invariant element of ${I_{s,w}}' \otimes \mathbb{C}_\eta$. We can now, however, look at the long Lie algebra cohomology sequence derived from the exact sequence above:

$$\begin{aligned} 0 \to H^0(\mathfrak{n}, {I_{s,1}}' \otimes \mathbb{C}_\eta) \to H^0(\mathfrak{n}, {I_s}' \otimes \mathbb{C}_\eta) \to H^0(\mathfrak{n}, {I_{s,w}}' \otimes \mathbb{C}_\eta) \\ \to H^1(\mathfrak{n}, {I_{s,1}}' \otimes \mathbb{C}_\eta) \to H^1(\mathfrak{n}, {I_s}' \otimes \mathbb{C}_\eta) \to H^1(\mathfrak{n}, {I_{s,w}}' \otimes \mathbb{C}_\eta) \to 0. \end{aligned}$$

We have already remarked that ${I_{s,1}}'$ is a Verma module, which means that $\mathfrak{n}$ acts locally nilpotently on it, and this implies in turn that each nonzero element of $\mathfrak{n}$

acts bijectively on ${I_{s,1}}' \otimes \mathbb{C}_\eta$, as long as ψ is not trivial. Therefore the $\mathfrak{n}$-cohomology of ${I_{s,1}}'$ vanishes completely, and the map

$$H^0(\mathfrak{n}, {I_s}' \otimes \mathbb{C}_\eta) \to H^0(\mathfrak{n}, {I_{s,w}}' \otimes \mathbb{C}_\eta)$$

is an isomorphism. This guarantees the canonical extension of $\Omega_{\psi,s}$ from $I_{s,w}$ to all of I_s. Holomorphicity in s follows, among other reasons, from a slightly more technical argument about various topological spaces of holomorphic functions, as will be shown in the main part of this paper.

How does this argument relate to the simple one in the previous section? The Lie algebra $\mathfrak{n}$ of N is spanned by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The action of the corresponding differential operator on functions in I_s restricted to $\bar{N}$ is

$$-y^2 \frac{d}{dy} - (s+1)y$$

if we identify $\bar{N}$ with $\mathbb{R}$ as before. The Verma module ${I_{s,1}}'$ corresponds to the module of distributions with the support at 0. Therefore, the local nilpotency we mentioned before is just the statement dual to the observation from the end of last subsection.

1.4. What happens for more general groups. In other words, the simple argument involving integration by parts can be interpreted, essentially, in terms of cohomology. This is not necessary for $\mathrm{SL}(2, \mathbb{R})$, as we have seen, but already extremely convenient for arbitrary groups of rank one. For groups of higher rank the Bruhat filtration is indexed by elements of the Weyl group W of R, with closure among closed N-stable subsets mirrored by a well known combinatorial ordering of W. The filtration itself is more complicated in that the various modules in the associated graded module are in some sense a mixture of Schwartz spaces and Verma modules. A weak description of this associated graded module was already known to Bruhat, of course, but using well known results originating with Whitney's general theory of smooth functions on singular varieties allows a stronger statement. For these groups, it is impossible even to contemplate carrying out the calculations necessary to reduce everything to integration by parts. Instead, we introduce a rather complicated formal machine utilizing Lie algebra cohomology and functional analysis. Various simple but technical results allow one to see that the $\mathfrak{n}$-cohomology groups associated to non-degenerate characters of N of all of the quotients of the associated graded module except the bottom one vanish completely, and that only H^0 of the bottom one is non-trivial.

If G is linear quasisplit then every smooth principal series has essentially a unique Whittaker functional (in the literature, commonly called a Whittaker vector). Further cohomological arguments allow us to conclude that every irreducible smooth Harish-Chandra representation of G (in the sense of [**9**]) has at most one. This is an important fact, on which depends the local functional equation of Whittaker functionals. For p-adic groups, this is an elementary result and has been known for a long time. Shalika's results were perhaps the first to suggest the strong statement we prove here. But for a long time the situation for real groups has been confused by the intriguing fact that uniqueness is true only for a certain category of smooth representations of G. For algebraic Harish-Chandra modules, the dimension of the space of Whittaker functionals is equal to the cardinality of the Weyl group, a result first pointed out in Kostant's remarkable paper [**21**]. Already in the book by Jacquet and Langlands [**19**] uniqueness was tied to the asymptotic growth at ∞ of certain N-covariant functions (the classical Whittaker functions, in fact), and the impressive thesis of To [**26**] analyzed growth conditions of Whittaker functions for groups of higher rank. We believe, however that the context in which we prove uniqueness here is the most natural one. It might also be of historical interest for us to mention that the construction of the useful category of smooth representations found in Casselman [**9**] and Wallach [**28**] was initiated by investigations about uniqueness of Whittaker functionals.

1.5. Plan of the rest of the paper. The body of the paper is organized as follows. In the second section we discuss the definition of Schwartz spaces and spaces of tempered distributions in the context of vector bundles on compact analytic manifolds. We prove some technical results about these spaces which are simple consequences of deep results originating in the work of Whitney and Łojasiewicz. In our exposition these results are hidden in the formalism of subanalytic sets. In the next section we study some natural filtrations on spaces of smooth sections of analytic vector bundles attached to finite stratifications of the base spaces by closed subanalytic sets. These filtrations together with the corresponding filtrations on the spaces of distributions are of fundamental importance in our paper. Up to that point, our discussion is quite general and devoid of any representation theoretic context. The example critical for our representation theoretic applications is that of a homogeneous vector bundle on a compact homogeneous manifold X for a Lie group G and the orbit stratification corresponding to a Lie subgroup $N \subset G$ acting on X with finitely many orbits. It is discussed at the end of the third section.

In the fourth section we specialize this example even further to a reductive Lie group G acting on its real flag manifold X. The group N is the nilpotent radical of a minimal parabolic subgroup. The corresponding filtration of the space of sections of a homogeneous vector bundle is the Bruhat filtration. In the fifth section we compute Lie algebra (co)homology of twisted regular representations of a nilpotent Lie group N on Schwartz spaces of functions on homogeneous spaces of N and their strong duals. The calculation is based on a simple Fourier transform argument in the abelian case, and an inductive argument using the Hochschild-Serre spectral sequence in general. The main results of the paper are proved in the sixth and seventh section where we study the existence of Whittaker functionals for smooth principal series representations of reductive Lie groups and their construction as holomorphic continuation of Jacquet integrals (which are a generalization of the integral $\Omega_{\psi,s}$ from §1.3). Using the Bruhat filtration and some standard cohomological arguments, the existence follows from the results on Lie algebra cohomology in the fifth section. In the eight section we reformulate our main results for arbitrary smooth representations. Finally, in the ninth section we specialize our results to linear quasisplit groups.

Throughout the paper we freely use a number of results from the theory of nuclear spaces. These results are notoriously hard to find in the literature so we decided to include flowcharts of their proofs in two appendices at the end of the paper.

2. Schwartz spaces and tempered distributions

Let X be a compact analytic manifold. We denote by $C^\infty(X)$ the algebra of complex valued smooth functions on X and by $\mathrm{Diff}(X)$ the algebra of differential operators with smooth coefficients on X. We equip $C^\infty(X)$ with the topology of uniform convergence of functions and their derivatives on X (more precisely, this is the locally convex topology given by seminorms $p_D : f \longmapsto \max_{x\in X} |(Df)(x)|$ for $D \in \operatorname{Diff}(X)$). It is well known that $C^\infty(X)$ with this topology is a nuclear Fréchet space (or an NF-space, for short)[1]. The strong dual $C^\infty(X)'$ of $C^\infty(X)$ is the space of distributions on X. It is a dual nuclear Fréchet space (or a DNF-space, for short). Clearly, differential operators act as continuous transformations on $C^\infty(X)$ and $C^\infty(X)$ has a natural structure of a left $\operatorname{Diff}(X)$-module. For any $D \in \operatorname{Diff}(X)$, let D^t be the adjoint endomorphism of $C^\infty(X)'$. Clearly, D^t is a continuous linear map on $C^\infty(X)'$ and $(D,T) \longmapsto D^tT$ for $D \in \operatorname{Diff}(X)$ and $T \in C^\infty(X)'$, defines a structure of a right $\operatorname{Diff}(X)$-module on $C^\infty(X)'$.

More generally, let E be an analytic vector bundle on X. Denote by $p : E \to X$ the natural projection. Let $\operatorname{Diff}(E)$ be the algebra of differential operators with smooth coefficients acting on sections of E. For $m \in \mathbb{Z}$, denote by $\operatorname{Diff}_m(E)$ the subspace of all differential operators of order $\leq m$. Let E^* be the dual vector bundle. Let $C^\infty(E)$ denote the space of all smooth sections of the vector bundle E. Then, using the natural pairing $C^\infty(E) \times C^\infty(E^*) \to C^\infty(X)$, we can define seminorms $p_{D,f} : s \longmapsto \max_{x\in X} |f(x)((Ds)(x))|$ on $C^\infty(E)$, for any $D \in \operatorname{Diff}(E)$ and $f \in C^\infty(E^*)$. They define the locally convex topology of uniform convergence of sections and their derivatives on X. With this topology $C^\infty(E)$ is an NF-space. As above, it also has the structure of a left $\operatorname{Diff}(E)$-module. Let $C^\infty(E)'$ be its strong dual. By abuse of language, we call it the space of *E-distributions* on X. It is a DNF-space and a right $\operatorname{Diff}(E)$-module.

Let T be an E-distribution on X. Then there exist a finite family $\{(D_j, f_j); 1 \leq j \leq k\}$, of $D_j \in \operatorname{Diff}(E)$ and sections f_j of E^* such that $|T(s)| \leq \sum_{j=1}^k p_{D_j,f_j}(s)$ for any $s \in C^\infty(E)$. If the orders of differential operators D_j are $\leq p$, we say that the *order* of T is $\leq p$. Clearly, this defines an exhaustive increasing filtration of $C^\infty(E)'$. If $D \in \operatorname{Diff}_s(E)$ and T is of order $\leq p$, D^tT is of order $\leq s+p$. Therefore, $C^\infty(E)'$ is a filtered $\operatorname{Diff}(E)$-module.

Let U be an open subset in X. For any compact set $K \subset U$, $D \in \operatorname{Diff}(E)$ and a smooth section f of E^*, define a seminorm $p_{K,D,f} : s \longmapsto \max_{x\in K} |f(x)((Ds)(x))|$ on the space $C^\infty(U,E)$ of smooth sections of E on U. Let $C_0^\infty(U,E)$ be the subspace of all compactly supported smooth sections of E on U. Also, let $C_K^\infty(U,E)$ be its subspace consisting of all sections with support in K. Clearly, as a linear space, $C_0^\infty(U,E) = \varinjlim_{K\subset U} C_K^\infty(U,E)$. We endow $C_K^\infty(U,E)$ with the locally convex topology given by the family of seminorms $p_{K,D,f}$ where $D \in \operatorname{Diff}(E)$ and f are smooth sections of E^*. Moreover, we endow $C_0^\infty(U,E)$ with the topology of the direct limit of locally convex spaces. The strong dual $C_0^\infty(U,E)'$ of $C_0^\infty(U,E)$ is the space of E-distributions on U. If V is an open subset of U, we have an obvious continuous inclusion of $C_0^\infty(V,E) \longrightarrow C_0^\infty(U,E)$. The adjoint of this inclusion is the restriction map $\operatorname{res}_{U,V} : C_0^\infty(U,E)' \longrightarrow C_0^\infty(V,E)'$. Clearly, this defines a sheaf of E-distributions on X.

Let T be an E-distribution on X. The *support* $\operatorname{supp}(T)$ of T is the complement of the largest open set U in X such that $T|_U = \operatorname{res}_{X,U}(T) = 0$.

The following result is a simple generalization of a well known fact.

LEMMA 2.1. *Let T be an E-distribution of order $\leq m$ on X. Then $T(s) = 0$ for any smooth section s of E such that Ds vanishes on* $\operatorname{supp}(T)$ *for all $D \in \operatorname{Diff}_m(E)$.*

[1] For our terminology and notation concerning locally convex spaces see Appendix A.

Proof. Let U be an open set in X such that $E|U$ is trivial and U can be covered by one coordinate chart. If the support of T is contained in U, the result is [**24**, Ch. III, Th. XXVIII].

As we already remarked, if T is of order $\leq p$ and ϕ a smooth function on X, the distribution ϕT is of order $\leq p$. Therefore, by a partition of unity argument, we can reduce the general situation to the above special case. □

Fix an open set U in X, and denote by Z its complement. Following [**20**, §3], we call the elements of the image of $\operatorname{res}_U = \operatorname{res}_{X,U} : C^\infty(E)' \longrightarrow C_0^\infty(U, E)'$ *tempered E-distributions* on U with respect to X. The kernel $\ker \operatorname{res}_U$ of res_U consists of all E-distributions on X with the support in Z. Therefore, $\ker \operatorname{res}_U$ is a closed linear subspace of $C^\infty(E)'$ and we have the short exact sequence

$$0 \longrightarrow \ker \operatorname{res}_U \longrightarrow C^\infty(E)' \longrightarrow \operatorname{im} \operatorname{res}_U \longrightarrow 0$$

of right $\operatorname{Diff}(E)$-modules. We equip the kernel $\ker \operatorname{res}_U$ with the induced topology, and the image $\operatorname{im} \operatorname{res}_U$ with the quotient topology. Then $\ker \operatorname{res}_U$ and $\operatorname{im} \operatorname{res}_U$ are DNF-spaces. By dualizing, we have the exact sequence

$$0 \longrightarrow (\operatorname{im} \operatorname{res}_U)' \longrightarrow C^\infty(E) \longrightarrow (\ker \operatorname{res}_U)' \longrightarrow 0,$$

where $(\operatorname{im} \operatorname{res}_U)' = (\ker \operatorname{res}_U)^\perp$, i.e., this is the space of all smooth global sections of E on which the distributions supported in Z vanish.

We say that a smooth section $s \in C^\infty(E)$ *vanishes at x with all of its derivatives of order $\leq m$* if $(Ds)(x) = 0$ for all differential operators $D \in \operatorname{Diff}_m(E)$. Also, we say that a smooth section $s \in C^\infty(E)$ *vanishes at x with all of its derivatives* if $(Ds)(x) = 0$ for all differential operators $D \in \operatorname{Diff}(E)$.

Let $x \in X$. As in [**24**, Ch. III, Th. XXXV], one can easily check that any E-distribution supported in $\{x\}$ has the form $s \longmapsto f((Ds)(x))$ for some differential operator $D \in \operatorname{Diff}(E)$ and a linear form f on the fiber E_x of E at x. Therefore, a smooth section of E vanishes at x with all of its derivatives if and only if all E-distributions with support in $\{x\}$ vanish on s.

If $x \in Z$, all distributions supported in $\{x\}$ vanish on sections in $(\operatorname{im} \operatorname{res}_U)'$. Therefore, this space is contained in the subspace of $C^\infty(E)$ consisting of all sections vanishing with all of their derivatives on Z. The converse follows from 2.1.

Lemma 2.2. *The linear subspace of all global sections in $C^\infty(E)$ vanishing with all of their derivatives on Z is equal to* $(\operatorname{im} \operatorname{res}_U)'$.

We denote this space by $\mathcal{S}(U, E)$ and call it the *(relative) Schwartz space of sections of E on U with respect to X*. It is equipped with the natural topology induced by $C^\infty(E)$. Hence, $\mathcal{S}(U, E)$ has a natural structure of an NF-space. Its strong dual $\mathcal{S}(U, E)'$ is the space $\operatorname{im} \operatorname{res}_U$ of all tempered E-distributions on U.

Example 2.3. Let $X = S^n$ be an n-dimensional sphere, p a point in S^n and $\pi : S^n \longrightarrow \mathbb{R}^n$ the stereographical projection of $U = S^n - \{p\}$ onto $\mathbb{R}^n$ with respect to the pole p. The closed subspace of the space $C^\infty(S^n)$ consisting of all functions vanishing with all of their derivatives at the pole p can be identified, via the diffeomorphism π, with the "classical" Schwartz space $\mathcal{S}(\mathbb{R}^n)$ [**24**, Ch. VII, Th. II]. Also, this identifies the space of all restrictions to U of the distributions on the sphere S^n with the space $\mathcal{S}(\mathbb{R}^n)'$ of "classical" tempered distributions on $\mathbb{R}^n$. The kernel of the restriction map is the space of all distributions supported in p. As it is well known [**24**, Ch. III, Th. XXXV], this is the space of all derivatives of the

Dirac δ-distribution in p. Therefore, it is isomorphic to the space of polynomials $\mathbb{C}[\partial_1, \partial_2, \dots, \partial_n]$ in partial derivatives, equipped with the topology of the inductive limit of finite-dimensional spaces of polynomials of degree $\leq p$, $p \in \mathbb{Z}_+$. Its strong dual is the space of formal power series $\mathbb{C}[[X_1, X_2, \dots, X_n]]$ with its natural topology. Therefore, the above short exact sequence corresponds to the exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow C^\infty(S^n) \xrightarrow{\mathrm{TS}} \mathbb{C}[[X_1, X_2, \dots, X_n]] \longrightarrow 0,$$

where TS is the Taylor series map which attaches to a smooth function on the sphere S^n its Taylor series at the point p. The surjectivity of this map is the classic theorem of Émile Borel.

As remarked in [**20**, 3.2], temperedness is a a local property. More precisely an E-distribution T on an open set U is tempered if and only if for any point $x \in X$ there exists a neighborhood V of x and a tempered E-distribution S on V such that $T|_{U\cap V} = S|_{U\cap V}$. This remark, combined with local triviality of E, implies the following variant of the "gluing" result of [**20**, 3.6].

LEMMA 2.4. *Let U be a subanalytic open set in X, and $(U_i; 1 \leq i \leq n)$ a covering of U by subanalytic open sets. Then, an E-distribution T on U is tempered if and only if its restrictions $T|_{U_i}$ are tempered for all $1 \leq i \leq n$.*

PROOF. Clearly, if T is tempered, $T|_{U_i}$ are also tempered for $1 \leq i \leq n$.

Assume that all $T|_{U_i}$, $1 \leq i \leq n$, are tempered. Since any point x in X has a fundamental system of subanalytic open neighborhoods, we can pick a subanalytic open neighborhood V such that $E|V$ is trivial vector bundle over V. Now, $T|_{U_i\cap V}$ are tempered for $1 \leq i \leq n$. By [**20**, 3.6], $T|_{U\cap V}$ is a restriction of an E-distribution on V. By the above remark, it follows that T is tempered. □

Let Y be a submanifold[2] of X. Then there exists an open subset U in X such that Y is closed in U. Let U' be another such open set contained in U. The kernel of the restriction map $\mathrm{res}_{U,U'}$ consists of distributions on U supported in the closed subset $U - U'$ of U. Therefore, this map induces a bijection of the space of E-distributions on U with support in Y onto the space of E-distributions on U' supported in Y. The inverse of this map is "the extension by zero". Since the restriction of a tempered distribution on U is obviously tempered distribution on U', we see that the restriction map $\mathrm{res}_{U,U'}$ induces an injection of the space of tempered distributions on U with support in Y into the space of all tempered distributions on U' with support in Y.

Assume that Y is in addition a subanalytic set in X. Let U be an open subset of X containing Y which is subanalytic in X. We say that U is a *subanalytic neighborhood* of Z in X if Z is a closed subset of U.

LEMMA 2.5. *Let Y be a submanifold of X which is also a subanalytic set in X. Then Y has a subanalytic neighborhood in X.*

PROOF. Since the closure $\bar{Y}$ of Y in X is compact, there exists a finite covering $(U_i; 1 \leq i \leq n)$ of Y consisting of subanalytic open sets in X. Since $\partial Y = \bar{Y} - Y$ is a closed subanalytic subset in X, we also see that $U = (\bigcup_{i=1}^n U_i) - \partial Y$ is an open subanalytic set in X. Clearly, Y is a closed subset of U. □

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[2]We follow [**2**] in our terminology. In particular, our submanifolds are locally closed.

Subanalytic neighborhoods of Y have the following property.

LEMMA 2.6. *Let Y be a submanifold of X which is also a subanalytic set in X. Let U and U' be two subanalytic neighborhoods of Y such that $U' \subset U$. Then the restriction* $\mathrm{res}_{U,U'}$ *induces an isomorphism of the space of tempered E-distributions on U with support in Y onto the space of all tempered E-distributions on U' with support in Y.*

PROOF. By the preceding discussion, it is enough to show that the extension by zero of a tempered distribution on U' with support in Z is tempered on U. Since U and Y are subanalytic in X, their difference $U - Y$ is an open subanalytic set in X. Therefore, the assertion follows from 2.4. □

If U and U' are two arbitrary subanalytic neighborhoods of Y in X, their intersection $U \cap U'$ is also a subanalytic neighborhood of Y in X. The restriction maps $\mathrm{res}_{U,U\cap U'}$ and $\mathrm{res}_{U',U\cap U'}$ induce isomorphisms of the spaces of tempered E-distributions on U, resp. U', with support in Y into the space of all tempered E-distributions on $U \cap U'$ with support in Y. It follows that the spaces of tempered E-distributions with support in Y on any subanalytic neighborhood of Y are naturally isomorphic. Therefore, by abuse of language, we call these distributions the *tempered E-distributions supported in Y*. We denote by $\mathcal{T}(Y, E)$ the space of all tempered E-distributions supported in Y. Clearly, $\mathcal{T}(Y, E)$, as a closed subspace of $\mathcal{S}(U, E)'$ for an subanalytic neighborhood U of Y, has a natural structure of a DNF-space. This structure doesn't depend on the choice of U by the open mapping theorem. Also, $\mathcal{T}(Y, E)$ is a right $\mathrm{Diff}(E)$-module.

Now we want to define a natural exhaustive filtration on $\mathcal{T}(Y, E)$. First, fix a subanalytic neighborhood U of Y and consider $\mathcal{T}(Y, E)$ as a subspace of $\mathcal{S}(U, E)'$. For $p \in \mathbb{Z}$, let $\mathcal{M}_p$ denote the closed subspace of $\mathcal{S}(U, E)$ consisting of all sections s which vanish along Y with their derivatives of order $\leq p$. Clearly, $(\mathcal{M}_p; p \in \mathbb{Z})$ is a decreasing filtration of $\mathcal{S}(U, E)$. We put $\mathrm{F}_p\,\mathcal{T}(Y, E) = \mathcal{M}_p^{\perp}$, i.e., $\mathrm{F}_p\,\mathcal{T}(Y, E)$ consists of all tempered E-distributions supported in Y which vanish on the sections in $\mathcal{M}_p$. Then $\mathrm{F}\,\mathcal{T}(Y, E)$ is an increasing filtration of $\mathcal{T}(Y, E)$ by closed subspaces.

First, we remark that this filtration is exhaustive.

LEMMA 2.7. *The filtration $\mathrm{F}\,\mathcal{T}(Y, E)$ of $\mathcal{T}(Y, E)$ is exhaustive.*

PROOF. Fix a subanalytic neighborhood U of Y. Let $Z = X - U$. Let T be a distribution in $\mathcal{T}(Y, E)$. Then it is a restriction of an E-distribution $\bar{T}$ on X supported in $Y \cup Z$. Assume that the order of $\bar{T}$ is $\leq p$. Then $T(s) = 0$ for all smooth sections s of $\mathcal{S}(U, E)$ such that Ds vanishes on Y for all $D \in \mathrm{Diff}_p(E)$, i.e., for all $s \in \mathcal{M}_p$. Therefore, T is in $\mathrm{F}_p\,\mathcal{T}(Y, E)$. □

If we consider $\mathcal{S}(U, E)'$ as a filtered right $\mathrm{Diff}(E)$-module with respect to the filtration induced by the order filtration, the above result means that the natural inclusion of $\mathcal{T}(Y, E)$ into $\mathcal{S}(U, E)'$ is compatible with the filtrations.

Since $\mathrm{F}_p\,\mathcal{T}(Y, E)$ are closed subspaces of $\mathcal{T}(Y, E)$, they are DNF-spaces with the induced topology. Therefore, by 2.7 and the open mapping theorem, we have the following result.

PROPOSITION 2.8. *As a locally convex space,* $\mathcal{T}(Y, E) = \varinjlim_{p\to\infty} \mathrm{F}_p\,\mathcal{T}(Y, E)$.

Let $\mathcal{M}_{0p}$ be the subspace of $\mathcal{M}_p$ consisting of sections with compact support contained in U.

Lemma 2.9. *We have $\mathcal{M}_{0p}^{\perp} = \mathcal{M}_p^{\perp}$ or any $p \in \mathbb{Z}$.*

Proof. Fix $p \in \mathbb{Z}$. Obviously, $\mathcal{M}_{0p} \subset \mathcal{M}_p$ and we have $\mathcal{M}_p^{\perp} \subset \mathcal{M}_{0p}^{\perp}$. Therefore, it is enough to show that $\mathcal{M}_{0p}^{\perp} \subset \mathcal{M}_p^{\perp}$.

Let T be a distribution $\mathcal{T}(Y,E)$ which vanishes on $\mathcal{M}_{0p}$. Then there exists a distribution $\bar{T}$ on X such that T is the restriction of $\bar{T}$ to U. The support of $\bar{T}$ is contained in $Y \cup Z$. Assume first that the support of $\bar{T}$ is "small", i.e., there exists an open set V in X such that $\operatorname{supp}(\bar{T}) \subset V$, $E|V$ is trivial and V can be covered by a coordinate chart. If $\operatorname{supp}(\bar{T}) \cap Z = \emptyset$, the support of T in U is compact and it is clear that T is perpendicular to $\mathcal{M}_p$.

Therefore, we can assume that $\operatorname{supp}(\bar{T}) \cap Z$ is a nonempty compact set. Let s be a smooth section in $\mathcal{S}(U,E)$. Then s vanishes with all of its derivatives on $\operatorname{supp}(\bar{T}) \cap Z$. Since V can be covered by a coordinate chart, we can assume that V is an open set in $\mathbb{R}^n$. Let d be the distance function on R^n. Also, for any x in V let $d(\operatorname{supp}(\bar{T}) \cap Z, x)$ be the distance of x from $\operatorname{supp}(\bar{T}) \cap Z$. As in the proof of [**24**, Ch. III, Th. XXXV], for small $\epsilon > 0$, one can construct a smooth compactly supported function φ_ϵ which satisfies: $\varphi_\epsilon(x) = 1$ if $d(\operatorname{supp}(\bar{T}) \cap Z, x) < \epsilon/2$ and $\varphi_\epsilon(x) = 0$ if $d(\operatorname{supp}(\bar{T}) \cap Z, x) > \epsilon$, and such that $\varphi_\epsilon s$ and all of its derivatives tend to 0 uniformly on V as $\epsilon \to 0$. By extending φ_ϵ by zero outside V we see that smooth sections $\varphi_\epsilon s$ converge to 0 in $\mathcal{S}(U,E)$ as $\epsilon \to 0$. Therefore, it follows that $\lim_{\epsilon\to 0} \bar{T}(\varphi_\epsilon s) = 0$. If we put $\phi_\epsilon = 1 - \varphi_\epsilon$, we get

$$\lim_{\epsilon\to 0} (\phi_\epsilon \bar{T})(s) = T(s) - \lim_{\epsilon\to 0} \bar{T}(\varphi_\epsilon s) = T(s),$$

On the other hand, $\phi_\epsilon \bar{T}$ is a distribution with support contained in Y. Therefore, its restriction T_ϵ to U is a tempered distribution in $\mathcal{T}(Y,E)$ with compact support. By the above argument T_ϵ tends weakly to T as $\epsilon \to 0$. Clearly, T_ϵ vanish on $\mathcal{M}_{0p}$. Hence, by the first part of the proof, they are perpendicular to $\mathcal{M}_p$. It follows that T is also perpendicular to $\mathcal{M}_p$. This completes the proof for $\bar{T}$ with "small" support. The general case is reduced to this situation by a partition of unity argument. ☐

By a partition of unity argument, from 2.9, we conclude that $T \in \mathcal{T}(Y,E)$ is in $\mathrm{F}_p\,\mathcal{T}(Y,E)$ if for any $y \in Y$ there exists an open neighborhood $V \subset U$ of y such that T vanishes on all sections of $\mathcal{M}_p$ with support in V. This immediately implies that the filtration $\mathrm{F}\,\mathcal{T}(Y,E)$ is independent of the choice of the subanalytic neighborhood U.

If T is in $\mathrm{F}_p\,\mathcal{T}(Y,E)$, we say that the *transversal degree* of T is $\leq p$.

Lemma 2.10. *The space $\mathcal{T}(Y,E)$ is a filtered right $\operatorname{Diff}(E)$-module.*

Proof. Let $D \in \operatorname{Diff}_s(E)$ for $s \in \mathbb{Z}_+$. Then $D(\mathcal{M}_{p+s}) \subset \mathcal{M}_p$ for any $p \in \mathbb{Z}$. Hence, $D^t(\mathrm{F}_p\,\mathcal{T}(Y,E)) = D^t(\mathcal{M}_p^{\perp}) \subset \mathcal{M}_{p+s}^{\perp} = \mathrm{F}_{p+s}\,\mathcal{T}(Y,E)$ for all $p \in \mathbb{Z}$. ☐

Example 2.11. Consider again the situation described in 2.3. Let S^{n-1} be a $(n-1)$-dimensional sphere in S^n passing through the pole p and such that under the stereographic projection $S^{n-1} - \{p\}$ corresponds to the copy Y of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ defined by the equation $x_n = 0$. Clearly, the restriction of smooth functions on S^n to S^{n-1} induces the restriction of Schwartz functions on $\mathbb{R}^n$ to Schwartz functions on $\mathbb{R}^{n-1}$. The adjoint of this map is a natural inclusion of the space of tempered distributions on $\mathbb{R}^{n-1}$ into the space $\mathcal{T}(Y)$ of tempered distributions on $\mathbb{R}^n$ with support in Y. The filtration $\mathrm{F}\,\mathcal{T}(Y)$ is described by

$$\mathrm{F}_p\,\mathcal{T}(Y) = \{T \in \mathcal{S}(\mathbb{R}^n)' \mid x_n^{p+1} T = 0\}$$

for $p \in \mathbb{Z}_+$. In particular, the space $\mathrm{F}_0\,\mathcal{T}(Y)$ is equal to the image of the inclusion of $\mathcal{S}(\mathbb{R}^{n-1})'$ into $\mathcal{S}(\mathbb{R}^n)'$. Then, as it is well-known [**24**, Ch. III, §10], we have

$$\mathcal{T}(Y) = \bigoplus_{i=0}^{\infty} \partial_n^i\,\mathrm{F}_0\,\mathcal{T}(Y) \text{ and } \mathrm{F}_p\,\mathcal{T}(Y) = \bigoplus_{i=0}^{p} \partial_n^i\,\mathrm{F}_0\,\mathcal{T}(Y)$$

for any $p \in \mathbb{Z}$.

3. Filtrations attached to a stratification

Let $Z_1 \subset Z_2 \subset \cdots \subset Z_n \subset X$ be an increasing family of closed subanalytic sets in X (to simplify the notation, we assume that $Z_p = \emptyset$ for $p \leq 0$ and $Z_p = X$ for $p > n$). Let $U_p = X - Z_p$ be their complements. They form a decreasing family of open sets in X. Assume that, for each p, $Z_p - Z_{p-1} = Z_p \cap U_{p-1}$ is a submanifold in X.

Then we can define a natural decreasing filtration $\mathrm{F}\,C^\infty(E)$ of $C^\infty(E)$ by the Schwartz subspaces

$$C^\infty(E) = \mathcal{S}(X,E) \supset \mathcal{S}(U_1,E) \supset \cdots \supset \mathcal{S}(U_n,E) \supset \{0\},$$

i.e., we put $\mathrm{F}_p\,C^\infty(E) = C^\infty(E)$ for $p \leq 0$, $\mathrm{F}_p\,C^\infty(E) = \mathcal{S}(U_p,E)$ for $1 \leq p \leq n$ and $\mathrm{F}_p\,C^\infty(E) = \{0\}$ for $p > n$. We want to describe the graded object $\mathrm{Gr}\,C^\infty(E)$ corresponding to this filtration.

The orthogonals $(\mathrm{F}_p\,C^\infty(E))^\perp$ to the subspaces $\mathrm{F}_p\,C^\infty(E)$ in the filtration are closed subspaces in the space $C^\infty(E)'$ of E-distributions on X. Moreover, they form an increasing filtration of $C^\infty(E)'$, i.e.,

$$\{0\} = (\mathrm{F}_0\,C^\infty(E))^\perp \subset (\mathrm{F}_1\,C^\infty(E))^\perp \subset \ldots \subset (\mathrm{F}_n\,C^\infty(E))^\perp \subset C^\infty(E)'.$$

Of course, the spaces $(\mathrm{F}_p\,C^\infty(E))^\perp$ are just the spaces of E-distributions supported in Z_p. By our previous discussion, we have the following short exact sequences

$$0 \longrightarrow (\mathrm{F}_p\,C^\infty(E))^\perp \longrightarrow C^\infty(E)' \longrightarrow \mathcal{S}(U_p,E)' \longrightarrow 0.$$

By our assumption, $Z_{p+1} - Z_p$ are closed submanifolds in U_p. Moreover, they are subanalytic sets in X. Therefore, for each p, U_p is a subanalytic neighborhood of $Z_{p+1} - Z_p$ in X. Hence, the above short exact sequences give the following exact sequences

$$0 \longrightarrow (\mathrm{F}_p\,C^\infty(E))^\perp \longrightarrow (\mathrm{F}_{p+1}\,C^\infty(E))^\perp \longrightarrow \mathcal{T}(Z_{p+1} - Z_p, E) \longrightarrow 0.$$

Consider now short exact sequences

$$0 \longrightarrow \mathrm{F}_{p+1}\,C^\infty(E) \longrightarrow \mathrm{F}_p\,C^\infty(E) \longrightarrow \mathrm{Gr}^p\,C^\infty(E) \longrightarrow 0.$$

Since we have $(\mathrm{F}_p\,C^\infty(E))' = C^\infty(E)'/(\mathrm{F}_p\,C^\infty(E))^\perp$, it follows that

$$(\mathrm{Gr}^p\,C^\infty(E))' = (\mathrm{F}_{p+1}\,C^\infty(X))^\perp/(\mathrm{F}_p\,C^\infty(X))^\perp = \mathcal{T}(Z_{p+1} - Z_p, E)$$

i.e., the dual of $\mathrm{Gr}^p\,C^\infty(E)$ is the space of all tempered E-distributions supported in $Z_{p+1} - Z_p$. This establishes the following result.

THEOREM 3.1. *The graded object corresponding to the filtration* $\mathrm{F}\, C^\infty(E)$ *of* $C^\infty(E)$ *is*

$$\operatorname{Gr} C^\infty(E) = \bigoplus_{p\in\mathbb{Z}} \mathcal{T}(Z_p - Z_{p-1}, E)'$$

as a left $\operatorname{Diff}(E)$*-module and an NF-space.*

A typical application of this result is in the following situation. Let G be a real Lie group. Let X be a compact analytic manifold with an analytic action of G. Let E be a homogeneous analytic vector bundle on X. Then G acts naturally on the space $C^\infty(E)$ of all smooth sections of E. Therefore, $C^\infty(E)$ is a representation of G. Let $\mathfrak{g}$ be the complexified Lie algebra of G and $\mathcal{U}(\mathfrak{g})$ its enveloping algebra. By differentiating the action of G on sections of E we get a homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\operatorname{Diff}(E)$, i.e, $C^\infty(E)$ has a natural structure of a $\mathcal{U}(\mathfrak{g})$-module. Moreover, the composition of the principal antiautomorphism of $\mathcal{U}(\mathfrak{g})$ with the natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\operatorname{Diff}(E)$ defines a homomorphism of $\mathcal{U}(\mathfrak{g})$ into the opposite algebra of $\operatorname{Diff}(E)$. Therefore, the right $\operatorname{Diff}(E)$-module of E-distributions $C^\infty(E)'$ can be viewed as a left $\mathcal{U}(\mathfrak{g})$-module. The action of $\mathcal{U}(\mathfrak{g})$ on this space is just the contragredient action.

Let N be a connected closed subgroup of G. We assume in the following that N acts on X with finitely many orbits. Therefore, the orbits of N in X are connected submanifolds [**30**, 5.2.4.1], and subanalytic sets in X [**5**].

LEMMA 3.2. *Let O be an N-orbit in X. The boundary $\partial O = \bar{O} - O$ of O is a union of finitely many N-orbits of dimension strictly smaller than* $\dim O$.

PROOF. The closure $\bar{O}$ is a N-invariant subanalytic set in X. Therefore, it is a union of finitely many N-orbits. Since $\partial O = \bar{O} - O$ is a subanalytic set of dimension $\le \dim O - 1$ [**27**, 1.16.(3)], we see that ∂O is a union of finitely many N-orbits of dimension strictly smaller than $\dim O$. □

For any $p \in \mathbb{Z}$ denote by Z_p the union of all N-orbits of dimension $< p$. By 3.2, it follows that Z_p are closed subanalytic sets in X. We define the filtration of $C^\infty(E)$ as in the above discussion. Clearly, the closed subspaces $\mathrm{F}_p\, C^\infty(E)$ are not G-invariant. On the other hand, they are invariant for the action of $\mathcal{U}(\mathfrak{g})$. Therefore, $\mathrm{F}\, C^\infty(E)$ is a decreasing filtration of $C^\infty(E)$ by left $\mathcal{U}(\mathfrak{g})$-submodules. In addition to 3.1, we have the following result.

LEMMA 3.3. *The space of all tempered distributions supported in $Z_{p+1} - Z_p$ is equal to*

$$\bigoplus_{\dim O = p} \mathcal{T}(O, E)$$

where O are N-orbits in X.

PROOF. Let $O_1, O_2, \dots, O_q$ be all p-dimensional N-orbits in X. Then, for any $1 \le i \le q$,

$$U_i = X - \left(Z_p \cup \bigcup_{j\neq i} O_j \right)$$

is an open subanalytic set in X containing O_i. Moreover, by 3.2, the orbit O_i is closed in U_i, i.e., U_i is a subanalytic neighborhood of O_i. On the other hand,

$U_i \cap O_j = \emptyset$ for $i \neq j$. Since the union U_i, $1 \leq i \leq q$, is equal to $X - Z_p$, by 2.4, we see that

$$\mathcal{T}(Z_{p+1} - Z_p, E) = \bigoplus_{i=1}^{q} \mathcal{T}(O_i, E).$$

□

Denote by $J(O,E)$ the strong dual of $\mathcal{T}(O,E)$. This finally implies the following result.

THEOREM 3.4. *The graded $\mathcal{U}(\mathfrak{g})$-module corresponding to $C^\infty(E)$ equipped with the filtration $\mathrm{F}\, C^\infty(E)$ is equal to*

$$\operatorname{Gr} C^\infty(E) = \bigoplus_{p\in\mathbb{Z}} \left(\bigoplus_{\dim O = p} J(O,E) \right).$$

In particular, $\mathcal{T}(O,E)$ has a natural structure of a filtered left $\mathcal{U}(\mathfrak{g})$-module.

Finally, we have the following simple observation. We recall the definition of the filtration by the transversal degree from §2.

LEMMA 3.5. *The subspaces $\mathrm{F}_q\, \mathcal{T}(O,E)$, $q \in \mathbb{Z}$, of $\mathcal{T}(O,E)$ are $\mathcal{U}(\mathfrak{n})$-invariant.*

PROOF. Consider the space $\mathcal{M}_p$ of all smooth sections of E vanishing with their derivatives of order $\leq p$ along the orbit O. Then, $\mathcal{M}_p$ is obviously N-invariant. Therefore, it is invariant for the action of $\mathfrak{n}$. Therefore, the assertion follows from the definition of the filtration. □

4. Bruhat filtration

Let G be a reductive Lie group of Harish-Chandra class [**15**]. Denote by $\mathfrak{g}_0$ its Lie algebra. Let K be a maximal compact subgroup of G and $\mathfrak{k}_0$ its Lie algebra. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{q}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{a}_0$ be a maximal abelian Lie subalgebra of $\mathfrak{q}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{a}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{a}_0$ respectively. Let R be the system of (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$ in $\mathfrak{a}^*$. Let $W = W(R)$ be the Weyl group of R. Fix a set of positive roots R^+ in R. Let $\mathfrak{n}$ be the subalgebra of $\mathfrak{g}$ spanned by the root subspaces of $\mathfrak{g}$ corresponding to the roots in R^+. Put $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$. Let A be the connected abelian Lie subgroup of G with Lie algebra $\mathfrak{a}_0$. Let N be the connected nilpotent Lie subgroup of G with Lie algebra $\mathfrak{n}_0$. Then we have the Iwasawa decomposition $G = KAN$ of G.

Let P be the minimal parabolic subgroup of G containing N and $P = MAN$ its Langlands decomposition. Denote by $X = P\backslash G$ the corresponding real flag variety. The actions of groups P and N by the right translations on G induce the actions on X. The orbits of these actions are identical, and by Bruhat lemma, parametrized by the Weyl group W. The group G acts by conjugation on the set of all minimal parabolic subgroups of G and this action is transitive. Since minimal parabolic subgroups are their own normalizers, the map $x \longmapsto S_x$ which attaches to a point in X its stabilizer S_x in G is a bijection of X onto the set of all minimal parabolic subgroups in G. For $w \in W$, let $\mathfrak{n}^w$ be the Lie algebra spanned by the root subspaces of $\mathfrak{g}$ corresponding to the roots in $w(R^+)$. Let $\mathfrak{n}_0^w = \mathfrak{n}^w \cap \mathfrak{g}_0$. We denote by N^w the connected nilpotent subgroup of G with Lie algebra $\mathfrak{n}_0^w$. Then $P^w = MAN^w$ is a minimal parabolic subgroup in G which determines a point x_w in X. Let $C(w)$ be the N-orbit of x_w. It is called the Bruhat cell attached to w.

The map $W \ni w \longmapsto C(w)$ is the bijection between W and the set of N-orbits in X. For any $p \in \mathbb{Z}_+$, we denote by $W(p)$ the set of all elements $w \in W$ such that $\dim C(w) = p$. Also, as in the example of the orbit stratification in the last section, we denote by Z_p the union of all Bruhat cells $C(w)$ for $w \in W(q)$, $q < p \in \mathbb{Z}_+$. Let $n = \dim X$.

Let E be the homogeneous vector bundle on X attached to a finite-dimensional representation σ of P trivial on N. Then, the representation $C^\infty(E)$ of G is nothing else than the *(smooth) induced representation* $\operatorname{Ind}^\infty(\sigma)$ of G. Denote by $C_p^\infty(\sigma)$ the space of all smooth sections of $\operatorname{Ind}^\infty(\sigma)$ which vanish with all of their derivatives on Z_p. As we explained in the preceding section, then

$$\operatorname{Ind}^\infty(\sigma) = C_0^\infty(\sigma) \supset C_1^\infty(\sigma) \supset \cdots \supset C_p^\infty(\sigma) \supset \cdots \supset C_{n+1}^\infty(\sigma) = \{0\}.$$

is a finite decreasing filtration of $\operatorname{Ind}^\infty(\sigma)$ by $\mathcal{U}(\mathfrak{g})$-submodules. This filtration is called the *Bruhat filtration* of $\operatorname{Ind}^\infty(\sigma)$.

As a specialization of 3.4, we get the following result (here we put $J(w,\sigma) = J(C(w), E)$).

THEOREM 4.1. *The graded module corresponding to the Bruhat filtration of* $\operatorname{Ind}^\infty(\sigma)$ *is the* $\mathcal{U}(\mathfrak{g})$*-module*

$$\bigoplus_{p\in\mathbb{Z}_+}\left(\bigoplus_{w\in W(p)} J(w,\sigma)\right).$$

More precisely, the following sequences

$$0 \to C_{p+1}^\infty(\sigma) \to C_p^\infty(\sigma) \to \bigoplus_{w\in W(p)} J(w,\sigma) \to 0$$

are exact for $p \in \mathbb{Z}_+$.

In the preceding section we described a natural increasing filtration of $J(w,\sigma)'$. Now we want to describe the graded modules in this filtration more precisely. This will follow from a more explicit description of $\mathcal{T}(C(w), E)$ which we are going to discuss now.

First, we fix a subanalytic neighborhood of $C(w)$. Clearly, the stabilizer of x_w in N is $N \cap N^w$. Let w_0 be the longest element of W. Then $w_0(R^+) = -R^+$. Let $U^w = N^{ww_0}$. Then the complexified Lie algebra of U^w is spanned by root subspaces corresponding to the roots in $-w(R^+)$. Therefore $C(w)$ is actually the orbit of x_w under the action of $N \cap U^w$. The map $u \longmapsto u \cdot x_w$ is a diffeomorphism of U^w onto the U^w-orbit V^w of x_w which is open and dense in X. Therefore, V^w is an open subanalytic set in X. By [**1**, V.21.14], the isomorphism $U^w \longrightarrow V^w$ maps the closed connected subgroup $N \cap U^w$ of U^w onto $C(w)$, i.e., $C(w)$ is a closed submanifold in V^w. Therefore, V^w is a subanalytic neighborhood of the Bruhat cell $C(w)$.

In discussion of the geometry of $X = P\backslash G$ we can assume without any loss of generality that G is a group of real points of a complex reductive group $\mathbf{G}$ defined over $\mathbb{R}$. Then P is the group of real points of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over $\mathbb{R}$. Therefore X is the space of real points of $\mathbf{P}\backslash\mathbf{G}$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$. Then its group of real points is equal to N. The group U^w is the group of real points of a unipotent subgroup $\mathbf{U}^w$ of $\mathbf{G}$ defined over $\mathbb{R}$. Let $\mathfrak{u}^w$ be the Lie algebra of $\mathbf{U}^w$. The exponential map $\exp : \mathfrak{u}^w \longmapsto \mathbf{U}^w$ is an algebraic isomorphism of the Lie algebra $\mathfrak{u}^w$ onto the group $\mathbf{U}^w$. The restriction of the exponential map to the Lie algebra $\mathfrak{u}_0^w$ of U^w is a diffeomorphism onto U^w.

In [**12**, §1], Du Cloux defined the notion of the Schwartz space of functions on the space of real points of a smooth algebraic variety defined over $\mathbb{R}$. By [**12**, 1.3.6.(iii)], his definition of the Schwartz space for U^w agrees with the "classic" one after the identification of U^w with $\mathfrak{u}_0^w$ under the exponential map.

Moreover, in [**12**, 1.2.4.(i)], he proves that the pullback by the orbit map defines a topological isomorphism of the Schwartz space $\mathcal{S}(V^w)$ in sense of Section 2 with the Schwartz space $\mathcal{S}(U^w)$ on U^w we just described. Therefore, the relative Schwartz space $\mathcal{S}(V^w)$ is isomorphic to the Schwartz space on $\mathfrak{u}_0^w$ viewed as an euclidean space.

Let E_w be the fiber of E at x_w. The stabilizer P^w of x_w in G acts on E_w. We denote by σ^w this representation of P^w. Let $g \in G$ be an element such that $x_w = g \cdot x_1$. Then $P = g^{-1} P^w g$. Since all Levi factors in P are N-conjugate, without loss of generality we can assume that g normalizes the common Levi factor MA of P and P^w. The representation σ^w is equivalent to $P_w \ni p \longmapsto \sigma(g^{-1}pg)$.

To any section s of $C^\infty(E)$ we can attach the function ϕ_s given by $V^w \ni u \cdot x_w \longmapsto u^{-1} s(u \cdot x_w) \in E_w$. Clearly, ϕ_s can be viewed as a function in $C^\infty(V^w) \otimes_{\mathbb{C}} E_w$. Actually, by [**12**, 2.2.4], the map $s \longmapsto \phi_s$ is an isomorphism of the NF-space $\mathcal{S}(V^w, E)$ onto $\mathcal{S}(V^w) \otimes_{\mathbb{C}} E_w$.

The adjoint of this map is an isomorphism of $\mathcal{S}(V^w)' \otimes_{\mathbb{C}} E_w^*$ with $\mathcal{S}(V^w, E)'$. Therefore, we get the following result.

LEMMA 4.2. *The space of all tempered distributions $\mathcal{T}(C(w), E)$ supported in the Bruhat cell $C(w)$ is isomorphic to the tensor product of the space of all tempered distributions $\mathcal{T}(N \cap U^w)$ on U^w with support in $N \cap U^w$ with E_w^*.*

The restriction of a Schwartz function on U^w is a Schwartz function on $N \cap U^w$. This gives a natural surjection of $\mathcal{S}(U^w)$ onto $\mathcal{S}(N \cap U^w)$. The dual map defines an injection of $\mathcal{S}(N \cap U^w)'$ into $\mathcal{S}(U^w)'$. Clearly, under the above isomorphism, $\mathcal{S}(N \cap U^w)' \otimes_{\mathbb{C}} E_w^*$ corresponds to $\mathrm{F}_0\, \mathcal{T}(C(w), E)$. The group MA acts on the space E_w^* by the contragredient of σ^w. We can extend this action to a representation of P trivial on N. Then the above isomorphism of $\mathrm{Gr}^0\, \mathcal{T}(C(w), E) = \mathrm{F}_0\, \mathcal{T}(C(w), E)$ with $\mathcal{S}(C(w))' \otimes_{\mathbb{C}} E_w^*$ is an isomorphism of $\mathcal{U}(\mathfrak{p})$-modules.

Our final result in this section is a generalization of this module isomorphism to $\mathrm{Gr}^p\, \mathcal{T}(C(w), E)$, $p \in \mathbb{Z}$. For a real affine space Y we denote by $R(Y)$ the ring of polynomials with complex coefficients on Y. Let J denote the ideal of all polynomials in $R(U^w)$ vanishing along $N \cap U^w$. Then, by the above isomorphism, the multiplication map defines a map $J \otimes_{R(U_w)} \mathrm{F}_p\, \mathcal{T}(C(w), E) \longrightarrow \mathrm{F}_{p-1}\, \mathcal{T}(C(w), E)$ of $\mathcal{U}(\mathfrak{p})$-modules. By iteration, we get a map $J^p \otimes_{R(U_w)} \mathrm{F}_p\, \mathcal{T}(C(w), E) \longrightarrow \mathrm{F}_0\, \mathcal{T}(C(w), E)$. Moreover, this map factors through

$$J^p/J^{p+1} \otimes_{R(U_w)} \mathrm{Gr}^p\, \mathcal{T}(C(w), E) = J^p/J^{p+1} \otimes_{R(N \cap U^w)} \mathrm{Gr}^p\, \mathcal{T}(C(w), E).$$

Therefore, we get a $\mathcal{U}(\mathfrak{p})$-module homomorphism

$$J^p/J^{p+1} \otimes_{R(N \cap U^w)} \mathrm{Gr}^p\, \mathcal{T}(C(w), E) \longrightarrow \mathrm{F}_0\, \mathcal{T}(C(w), E).$$

This in turn leads to a $\mathcal{U}(\mathfrak{p})$-module homomorphism

$$\alpha_p : \mathrm{Gr}^p\, \mathcal{T}(C(w), E) \longrightarrow \mathrm{Hom}_{R(N \cap U^w)}(J^p/J^{p+1}, \mathrm{F}_0\, \mathcal{T}(C(w), E)).$$

By a result giving the description of the space of all tempered distributions on $\mathbb{R}^n$ with support in $\mathbb{R}^k$ analogous to 2.11, we easily see that this map is an isomorphism, i.e., we have the following result.

We put $L_p(w) = \operatorname{Hom}_{R(C(w))}(J^p/J^{p+1}, R(C(w)))$. Clearly, this is an N-equivariant $R(C(w))$-module.

LEMMA 4.3. *For any $p \in \mathbb{Z}$, we have a natural isomorphism*

$$\alpha_p : \operatorname{Gr}^p \mathcal{T}(C(w), E) \longrightarrow L_p(w) \otimes_{R(C(w))} \mathcal{S}(C(w))' \otimes_{\mathbb{C}} E_w^*$$

of $\mathcal{U}(\mathfrak{p})$-modules.

Clearly, $L_p(w)$ is a free $R(C(w))$-module of finite rank. We shall need also the following observation.

LEMMA 4.4. *The $R(C(w))$-module $L_p(w)$ has a finite increasing filtration by N-equivariant $R(C(w))$-submodules $\mathrm{F}\, L_p(w)$ such that the graded module is isomorphic, as an N-equivariant $R(C(w))$-module, to a direct sum of copies of $R(C(w))$.*

PROOF. If we consider N as the group of real points of an unipotent complex algebraic group $\mathbf{N}$ defined over $\mathbb{R}$, $S^w = U^w \cap N$ is the subgroup of N of real points of an algebraic subgroup $\mathbf{S}^w$ of $\mathbf{N}$. The homogeneous space $\mathbf{S}^w\backslash\mathbf{N}$ is an affine variety [**1**, 21.14], and we can view $R(C(w))$ as the ring of regular functions on $\mathbf{S}^w\backslash\mathbf{N}$. The $R(C(w))$-module $L_p(w)$ is the module of global sections of a free $\mathcal{O}_{\mathbf{S}^w\backslash\mathbf{N}}$-module $\mathcal{V}_p$ of finite rank. This $\mathcal{O}_{\mathbf{S}^w\backslash\mathbf{N}}$-module is a sheaf $\mathcal{V}_p$ of local sections of a $\mathbf{N}$-homogeneous algebraic vector bundle V_p on $\mathbf{S}^w\backslash\mathbf{N}$. Since $\mathbf{S}^w$ is unipotent, its representation on the geometric fiber of $\mathcal{V}_p$ at the identity coset is unipotent. Therefore, V_p has an increasing finite filtration by $\mathbf{N}$-homogeneous algebraic vector subbundles $\mathrm{F}\, V_p$ such that $\mathrm{F}_q V_p / \mathrm{F}_{q-1} V_p$ are line bundles corresponding to the trivial representation of $\mathbf{S}^w$, i.e., trivial line bundles on $\mathbf{S}^w\backslash\mathbf{N}$. The sheaves $\mathrm{F}_q \mathcal{V}_p$ of local sections of $\mathrm{F}_q V_p$ define an increasing filtration of the $\mathcal{O}_{\mathbf{S}^w\backslash\mathbf{N}}$-module $\mathcal{V}_p$, and we have the exact sequences

$$0 \longrightarrow \mathrm{F}_{q-1} \mathcal{V}_p \longrightarrow \mathrm{F}_q \mathcal{V}_p \longrightarrow \mathcal{O}_{\mathbf{S}^w\backslash\mathbf{N}} \longrightarrow 0$$

of locally free coherent $O_{\mathbf{S}^w\backslash\mathbf{N}}$-modules. Since $\mathbf{S}^w\backslash\mathbf{N}$ is affine, by Serre's theorem, this leads to the exact sequences of the corresponding global sections

$$0 \longrightarrow \Gamma(\mathbf{S}^w\backslash\mathbf{N}, \mathrm{F}_{q-1} \mathcal{V}_p) \longrightarrow \Gamma(\mathbf{S}^w\backslash\mathbf{N}, \mathrm{F}_q \mathcal{V}_p) \longrightarrow R(C(w)) \longrightarrow 0.$$

As a result, the submodules $\Gamma(\mathbf{S}^w\backslash\mathbf{N}, \mathrm{F}_q \mathcal{V}_p)$ of the $R(C(w))$-module $L_p(w) = \Gamma(\mathbf{S}^w\backslash\mathbf{N}, \mathcal{V}_p)$ define an exhaustive increasing finite filtration $\mathrm{F}\, L_p(w)$ of $L_p(w)$ such that

$$0 \longrightarrow \mathrm{F}_{p-1} L_p(w) \longrightarrow \mathrm{F}_p L_p(w) \longrightarrow R(C(w)) \longrightarrow 0$$

is exact. From the construction it is evident that this is a filtration by $\mathbf{N}$-equivariant $R(C(w))$-submodules. □

5. Lie algebra homology of the twisted regular representation on the Schwartz space

For any $\mathcal{U}(\mathfrak{n})$-module V we denote by $H_p(\mathfrak{n}, V)$ (resp. $H^p(\mathfrak{n}, V)$), $p \in \mathbb{Z}_+$, the p-th Lie algebra (co)homology group with coefficients in V.

In this section we prove a vanishing theorem for the Lie algebra homology of some twisted regular representations of nilpotent Lie groups on Schwartz spaces on their homogeneous spaces.

Let N be a simply connected, connected nilpotent Lie group. Let $\mathfrak{n}_0$ be its Lie algebra. We denote by $\mathfrak{n}$ its complexification, and by $\mathcal{U}(\mathfrak{n})$ the universal enveloping algebra of $\mathfrak{n}$. Let U be a connected Lie subgroup of N. Let $U\backslash N$ be the right coset space of N with respect to U. Then N acts differentiably on it by right translations. This action defines the right regular action of N on the space $C^\infty(U\backslash N)$ of smooth complex valued functions on $U\backslash N$. This action is clearly differentiable and leads to a $\mathcal{U}(\mathfrak{n})$-module structure on $C^\infty(U\backslash N)$. We say that a smooth function f on $U\backslash N$ is a *polynomial* if its annihilator in $\mathcal{U}(\mathfrak{n})$ contains $\mathfrak{n}^k$ for sufficiently large $k \in \mathbb{Z}_+$. Clearly, polynomials on $U\backslash N$ form a subalgebra of $C^\infty(U\backslash N)$ stable under the action of N. The action of $\mathfrak{n}$ on $U\backslash N$ is given by a morphism τ of $\mathfrak{n}$ into the Lie algebra of tangent vector fields on $U\backslash N$. Let $\mathrm{Diff}(U\backslash N)$ be the algebra of all differential operators on $U\backslash N$ with smooth coefficients. Let $D_{U\backslash N}$ be the subalgebra of $\mathrm{Diff}(U\backslash N)$ generated by the vector fields in $\tau(\mathfrak{n})$ and polynomials on $U\backslash N$. We call $D_{U\backslash N}$ the *algebra of polynomial differential operators* on $U\backslash N$. Clearly, $D_{U\backslash N}$ is stable under the action of N.

A *Schwartz function* on $U\backslash N$ is a smooth function f on $U\backslash N$ such that Df is bounded on $U\backslash N$ for any polynomial differential operator $D \in D_{U\backslash N}$ [**12**]. Let $\mathcal{S}(U\backslash N)$ be the space of all Schwartz functions on $U\backslash N$. Then, we can equip it with a locally convex topology given by the seminorms

$$f \longmapsto \|f\|_D = \sup_{x\in U\backslash N} |Df(x)|.$$

The following result is established in [**12**].

LEMMA 5.1. (i) *$\mathcal{S}(U\backslash N)$ is an NF-space.*

(ii) *The action of N on $\mathcal{S}(U\backslash N)$ defines a differentiable representation ρ of N. The differential of this representation is given by the natural action of $\mathfrak{n}$.*

Continuous linear forms on $\mathcal{S}(U\backslash N)$ are called *tempered distributions* on $U\backslash N$.

Let $\mathfrak{u}_0$ be the Lie algebra of U. Denote by $\mathfrak{u}$ its complexification. Let $\mathfrak{c}_0$ be a one-dimensional central subalgebra of $\mathfrak{n}_0$. Assume that $\mathfrak{c}_0 \not\subset \mathfrak{u}_0$. Then $\mathfrak{v}_0 = \mathfrak{u}_0 \oplus \mathfrak{c}_0$ is a Lie subalgebra of $\mathfrak{n}_0$ and $\dim_{\mathbb{R}} \mathfrak{v}_0 = \dim_{\mathbb{R}} \mathfrak{u}_0 + 1$. Denote by C and V the connected Lie subgroups corresponding to $\mathfrak{c}_0$ and $\mathfrak{v}_0$ respectively. Clearly, C is isomorphic to $\mathbb{R}$. Then the fibers of natural projection $p : U\backslash N \to V\backslash N$ are exactly the C-orbits in $U\backslash N$, i.e., $V\backslash N$ is the quotient of $U\backslash N$ under the action of C. Therefore, the map $g \longmapsto g \circ p$ defines a homomorphism of $C^\infty(V\backslash N)$ into the subalgebra of $C^\infty(U\backslash N)$ of all smooth C-invariant functions on $U\backslash N$.

We fix a Haar measure on C. The following results can be proved easily using simple facts on the structure of $U\backslash N$ (see, for example, [**30**, 1.1.4.1]) or deduced from the induction in stages result in [**12**, 2.1.6].

LEMMA 5.2. *Let f be a Schwartz function on $U\backslash N$. Then*

(i) *The integral*

$$\bar{f}(x) = \int_C (\rho(c)f)(x)\, dc$$

is finite for any $x \in U\backslash N$.

(ii) *The function $\bar{f}$ is a C-invariant smooth function on $U\backslash N$.*

(iii) *Let f_C be the unique smooth function on $V\backslash N$ such that $\bar{f} = f_C \circ p$. Then f_C is a Schwartz function.*

Therefore, we have a natural linear map $\pi : f \longmapsto f_C$ of $\mathcal{S}(U\backslash N)$ into $\mathcal{S}(V\backslash N)$. On the other hand, let ξ be a nonzero element of $\mathfrak{c}_0$. Then it defines a polynomial differential operator $\tau(\xi)$ on $U\backslash N$. Moreover, $\tau(\xi)$ is a linear endomorphism of $\mathcal{S}(U\backslash N)$.

LEMMA 5.3.

$$0 \to \mathcal{S}(U\backslash N) \xrightarrow{\tau(\xi)} \mathcal{S}(U\backslash N) \xrightarrow{\pi} \mathcal{S}(V\backslash N) \to 0$$

is an exact sequence in the category of smooth representations of N.

This can be interpreted in the following way. Let $\mathfrak{c}$ be the complexification of the Lie algebra $\mathfrak{c}_0$.

PROPOSITION 5.4.

$$H_p(\mathfrak{c}, \mathcal{S}(U\backslash N)) = \begin{cases} 0 & \text{if } p \neq 0; \\ \mathcal{S}(V\backslash N) & \text{if } p = 0. \end{cases}$$

Finally, the group N is unimodular. We fix a Haar measure on N. Then, for any Schwartz function f on N, the integral $\int_N f(n)\,dn$ is finite. Moreover, we have the following result.

LEMMA 5.5. *The map $f \longmapsto \int_N f(n)\,dn$ is a tempered distribution on N.*

PROOF. We prove this statement by induction in $\dim N$. If $\dim N = 1$, $N \cong \mathbb{R}$ and the statement is obvious.

Let $\dim N > 1$. Then we can pick a one-dimensional central Lie subalgebra $\mathfrak{c}_0$ of $\mathfrak{n}_0$ and denote by C the corresponding connected Lie subgroup. Then, with an appropriate choice of a Haar measure on $C\backslash N$, we have

$$\int_N f(n)\,dn = \int_{C\backslash N} \left(\int_C f(nc)dc \right) d\dot{n}.$$

Hence, the statement follows from 5.2 and the induction assumption. □

Since this tempered distribution is N-invariant, it vanishes on all Schwartz functions of the type $\xi \cdot f$, $\xi \in \mathfrak{n}$, $f \in \mathcal{S}(N)$.

Let ψ be a character of N, i.e., a differentiable homomorphism of N into $\mathbb{C}^*$. Let η be the differential of ψ extended to a complex valued linear form η on $\mathfrak{n}$. Since ψ is a character, η vanishes on $[\mathfrak{n}, \mathfrak{n}]$. The character ψ is unitary if and only if $\operatorname{Re} \eta = 0$.

Denote by $\mathbb{C}_\eta$ the one-dimensional $\mathcal{U}(\mathfrak{n})$-module with action η. Consider the tensor product $\mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta$. This is the tensor product of the regular representation of N on $\mathcal{S}(U\backslash N)$ with the one-dimensional representation given by character ψ. We call this representation a *twisted regular representation* of N. Clearly, it is a differentiable representation of N. Therefore, it has a natural $\mathcal{U}(\mathfrak{n})$-module structure. In this section we establish the following result for the Lie algebra homology of twisted regular representations.

THEOREM 5.6. (i) *$H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ are finite-dimensional for all $p \in \mathbb{Z}_+$.*
(ii) *$H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for $p > \dim U$.*
(iii) *If either $\operatorname{Re} \eta \neq 0$ or $\eta|\mathfrak{u} \neq 0$, we have $H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$.*

First we consider the case of abelian N. In this situation, $N = U \times V$ for some complementary abelian Lie subgroup V. Hence, $\mathcal{S}(U\backslash N) = \mathcal{S}(V)$ and the action of U on this space is trivial. Let $\mathfrak{v}_0$ be the Lie algebra of V and $\mathfrak{v}$ its complexification.

Assume that $\eta|\mathfrak{u} \neq 0$. Then $\dim U > 0$ and we can pick a one-dimensional connected Lie subgroup D of U with Lie algebra $\mathfrak{d}_0$ such that $\eta|\mathfrak{d}_0 \neq 0$. Let $\mathfrak{d}$ be the complexification of $\mathfrak{d}_0$. Let ξ be a generator of $\mathfrak{d}$. Then, $H_p(\mathfrak{d}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ is the cohomology of the complex

$$\cdots \to 0 \to \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta \xrightarrow{\eta(\xi)} \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta \to 0 \to \dots,$$

i.e., all of them vanish. By the Hochschild-Serre spectral sequence

$$H_p(\mathfrak{n}/\mathfrak{d}, H_q(\mathfrak{d}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta)) \Rightarrow H_{p+q}(\mathfrak{n}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta),$$

we see that $H_p(\mathfrak{n}, \mathcal{S}(U \backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$.

Therefore, we can assume that $\eta | \mathfrak{u} = 0$. In this case, the action of U on $\mathcal{S}(U \backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta$ is trivial, and from the standard complex of Lie algebra homology we see that

$$H_p(\mathfrak{u}, \mathcal{S}(U \backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \bigwedge^p \mathfrak{u} \otimes_{\mathbb{C}} \mathcal{S}(U \backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta$$

for $p \in \mathbb{Z}_+$. To complete the argument, we have to calculate $H_p(\mathfrak{v}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ for $p \in \mathbb{Z}_+$.

Lemma 5.7. (i) *If* $\operatorname{Re} \eta \neq 0$*, we have*

$$H_p(\mathfrak{v}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$$

for $p \in \mathbb{Z}_+$.

(ii) *If* $\operatorname{Re} \eta = 0$*, we have*

$$H_p(\mathfrak{v}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \begin{cases} 0 & \text{for } p \neq 0; \\ \mathbb{C} & \text{for } p = 0. \end{cases}$$

We prove this lemma by induction in dimension of V. Clearly, $V = \mathbb{R}^n$. Let $\mathfrak{d}_0$ be a one-dimensional Lie subalgebra of $\mathfrak{v}_0$ and D the corresponding connected Lie subgroup. Let $\mathfrak{d}$ be the complexification of $\mathfrak{d}_0$.

Lemma 5.8. (i) $H_p(\mathfrak{d}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ *for* $p > 0$.

(ii) *If* $\operatorname{Re} \eta | \mathfrak{d} \neq 0$*, we have*

$$H_0(\mathfrak{d}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0.$$

Proof. Without any loss of generality we can assume that D is the first coordinate axis in $\mathbb{R}^n$. Then the unit vector $(1, 0, \dots, 0)$ in $\mathfrak{d}$ acts on $\mathcal{S}(\mathbb{R}^n)$ as ∂_1 and on $\mathbb{C}_\eta$ by the first coordinate $\eta_1 = \eta(1, 0, \dots, 0)$ of η. It follows that the Lie algebra homology is the cohomology of the complex

$$0 \to \mathcal{S}(\mathbb{R}^n) \xrightarrow{\partial_1 + \eta_1} \mathcal{S}(\mathbb{R}^n) \to 0.$$

If f is in the kernel of $D = \partial_1 + \eta_1$, it must satisfy $\partial_1 f = -\eta_1 f$. By integration we see that $f(x_1, x_2, \dots, x_n) = e^{-\eta_1 x_1} f(0, x_2, \dots, x_n)$. Hence, f can be rapidly decreasing if and only if $f = 0$. Therefore, the kernel of D is zero, and (i) follows.

It remains to study the image of D. Consider the equation $Df = g$ where $f, g \in \mathcal{S}(\mathbb{R}^n)$. Define the Fourier transform

$$\hat{h}(t_1, t_2, \dots, t_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} h(x_1, x_2, \dots, x_n) e^{-i(x_1 t_1 + x_2 t_2 + \dots + x_n t_n)} dx_1 dx_2 \dots dx_n.$$

Then the Fourier transform of our equation looks like

$$(it_1 + \eta_1) \hat{f}(t_1, t_2, \dots, t_n) = \hat{g}(t_1, t_2, \dots, t_n), \ (t_1, t_2, \dots, t_n) \in \mathbb{R}^n.$$

Assume now that $\operatorname{Re}\eta_1 \neq 0$. Then the function $(t_1, t_2, \ldots, t_n) \longmapsto \frac{1}{t_1 - i\eta_1}$ is a smooth function on $\mathbb{R}^n$ and all of its derivatives are bounded on $\mathbb{R}^n$. Therefore, in this case, since $\hat{g}$ is in $\mathcal{S}(\mathbb{R}^n)$, the function

$$\hat{f} : (t_1, t_2, \ldots, t_n) \longmapsto -i\frac{\hat{g}(t_1, t_2, \ldots, t_n)}{t_1 - i\eta_1}$$

is a Schwartz function $\hat{f}$ on $\mathbb{R}^n$. Its inverse Fourier transform is a Schwartz function f which satisfies $Df = g$. Hence, in this case, $\operatorname{coker} D = 0$. □

Assume that $\operatorname{Re}\eta \neq 0$. Then we can pick $\mathfrak{d}_0$ so that $\operatorname{Re}\eta|\mathfrak{d}_0 \neq 0$. By the Hochschild-Serre spectral sequence

$$H_p(\mathfrak{v}/\mathfrak{d}, H_q(\mathfrak{d}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta)) \Rightarrow H_{p+q}(\mathfrak{v}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta),$$

and we see that

$$H_p(\mathfrak{v}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0, \text{ for all } p \in \mathbb{Z}_+.$$

Therefore, we can assume that $\operatorname{Re}\eta = 0$. In this case, the function

$$(x_1, x_2, \ldots, x_n) \longmapsto e^{-(\eta_1 x_1 + \eta_2 x_2 + \cdots + \eta_n x_n)}$$

is smooth and all its partial derivatives are bounded. Hence multiplication by this function defines a linear automorphism of $\mathcal{S}(\mathbb{R}^n)$. This automorphism intertwines the representation on $\mathcal{S}(\mathbb{R}^n) \otimes_{\mathbb{C}} \mathbb{C}_\eta$ with the regular representation on $\mathcal{S}(\mathbb{R}^n)$. Therefore, to prove the statement we can assume that $\eta = 0$.

Now we can prove the rest of 5.7 by induction in the dimension of V. If $\dim V = 1$, the statement follows from 5.4.

Let $\mathfrak{d}_0$ be one-dimensional Lie subalgebra in $\mathfrak{v}_0$ and D the corresponding connected closed Lie subgroup of V. Denote by $\mathfrak{d}$ the complexification of $\mathfrak{d}_0$. Then, by 5.4, we have

$$H_p(\mathfrak{d}, \mathcal{S}(V)) = \begin{cases} 0 & \text{for } p \neq 0; \\ \mathcal{S}(V/D) & \text{for } p = 0. \end{cases}$$

By the Hochschild-Serre spectral sequence, we have

$$H_p(\mathfrak{v}/\mathfrak{d}, H_q(\mathfrak{d}, \mathcal{S}(V))) \Rightarrow H_{p+q}(\mathfrak{v}, \mathcal{S}(V)).$$

Hence by the above result, this spectral sequence degenerates and we have

$$H_p(\mathfrak{v}/\mathfrak{d}, \mathcal{S}(V/D)) = H_p(\mathfrak{v}, \mathcal{S}(V)).$$

Moreover, V/D is a connected and simply connected abelian Lie group. Therefore, the assertion of 5.7 follows from the induction assumption.

Now we go back to the proof of the theorem for abelian N. By the Hochschild-Serre spectral sequence, we have

$$\begin{aligned} H_p(\mathfrak{v}, \mathcal{S}(V) \otimes_{\mathbb{C}} \mathbb{C}_\eta) \otimes_{\mathbb{C}} \textstyle\bigwedge^q \mathfrak{u} &= H_p(\mathfrak{v}, H_q(\mathfrak{u}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta)) \\ &\Rightarrow H_{p+q}(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta). \end{aligned}$$

Hence, if $\operatorname{Re}\eta \neq 0$, by 5.7, we get $H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$. If $\operatorname{Re}\eta = 0$, we see that

$$H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \textstyle\bigwedge^p \mathfrak{u}$$

for all $p \in \mathbb{Z}_+$. This establishes 5.6 for abelian N.

Now, we prove the theorem by a downward induction in $\dim N$. In addition, we can assume that N is not abelian.

Let $\mathfrak{d}_0$ be a one-dimensional Lie subalgebra of the center of $\mathfrak{n}_0$. Since $\mathfrak{n}_0$ is not abelian, we can also assume that $\mathfrak{d}_0 \subset [\mathfrak{n}_0, \mathfrak{n}_0]$. Therefore, $\eta|\mathfrak{d}_0 = 0$. Let D be the corresponding connected Lie subgroup (which is in the center of N). Denote by $\mathfrak{d}$ the complexification of $\mathfrak{d}_0$.

Assume first that $D \subset U$. Then the action of D on $U\backslash N$ is trivial. Hence, the action on $\mathcal{S}(U\backslash N)$ is also trivial. As before, this implies the following result.

Lemma 5.9. *Assume that* $\mathfrak{d} \subset \mathfrak{u}$. *Then*

$$H_p(\mathfrak{d}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \begin{cases} 0 & \text{for } p \neq 0, 1; \\ \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta & \text{for } p = 0, 1; \end{cases}$$

as a $\mathfrak{n}/\mathfrak{d}$*-module.*

Let $N' = N/D$ and $U' = U/D$. Denote by $\mathfrak{n}'_0$ and $\mathfrak{u}'_0$ their Lie algebras, and by $\mathfrak{n}'$ and $\mathfrak{u}'$ their complexifications. Since $\eta|\mathfrak{d} = 0$, it defines a linear form η' on $\mathfrak{n}'$. Clearly, $U\backslash N = U'\backslash N'$. By the Hochschild-Serre spectral sequence we have

$$H_p(\mathfrak{n}', H_q(\mathfrak{d}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta)) \Rightarrow H_{p+q}(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta).$$

Hence, by 5.9, we see that

$$H_p(\mathfrak{n}', H_q(\mathfrak{d}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta)) = H_p(\mathfrak{n}', \mathcal{S}(U'\backslash N') \otimes_{\mathbb{C}} \mathbb{C}_{\eta'})$$

for $q = 0, 1$ and 0 otherwise.

If η doesn't vanish on $\mathfrak{u}$, η' doesn't vanish on $\mathfrak{u}'$. Therefore, by the induction assumption we get that the E^2-term in the above spectral sequence vanishes. Hence, the vanishing theorem follows in this case.

If $\operatorname{Re}\eta \neq 0$, we see that $\operatorname{Re}\eta' \neq 0$, and by the induction assumption the E^2-term in the above spectral sequence vanishes again.

If neither of these conditions is satisfied, by the induction assumption, we see that the E^2-terms are finite-dimensional and can be nonzero only for $0 \leq p \leq \dim U'$ and $q = 0, 1$. Therefore, $H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ are finite-dimensional and must vanish for $p > \dim U' + 1 = \dim U$.

This completes the induction step in this case.

Assume now that $D \not\subset U$. In this case, $\mathfrak{v}_0 = \mathfrak{u}_0 + \mathfrak{d}_0$ is a Lie subalgebra of $\mathfrak{n}_0$ and $\mathfrak{u}_0$ is a Lie subalgebra of $\mathfrak{v}_0$ of codimension 1. Put $\mathfrak{v}'_0 = \mathfrak{v}_0/\mathfrak{d}_0$. Then $\mathfrak{v}'_0$ is a Lie subalgebra of $\mathfrak{n}'_0$. Denote by V' the corresponding connected Lie subgroup of N'. Also, denote by $\mathfrak{v}$ and $\mathfrak{v}'$ the complexifications of $\mathfrak{v}_0$ and $\mathfrak{v}'_0$ respectively.

Then we have the following result.

Lemma 5.10. *We have*

$$H_p(\mathfrak{d}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$$

for $p > 0$ *and*

$$H_0(\mathfrak{d}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \mathcal{S}(V'\backslash N') \otimes_{\mathbb{C}} \mathbb{C}_{\eta'}.$$

Proof. This is essentially 5.4, since η vanishes on $\mathfrak{d}$. □

Therefore, by the Hochschild-Serre spectral sequence we have

$$H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes \mathbb{C}_\eta) = H_p(\mathfrak{n}', \mathcal{S}(V'\backslash N') \otimes \mathbb{C}_{\eta'}).$$

Clearly, if $\eta|\mathfrak{u} \neq 0$, we have $\eta|\mathfrak{v} \neq 0$ and $\eta'|\mathfrak{v}' \neq 0$. Hence, the vanishing follows by the induction assumption. The same applies in the case $\operatorname{Re}\eta \neq 0$.

If neither condition is satisfied, by the induction assumption $H_p(\mathfrak{n}', \mathcal{S}(V'\backslash N') \otimes \mathbb{C}_{\eta'})$ are finite-dimensional and vanish for $p > \dim V'$. But $\dim V' = \dim V - 1 =$

$\dim U$. Hence, we see that $H_p(\mathfrak{n}, \mathcal{S}(U\backslash N) \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ are finite-dimensional and vanish for $p > \dim U$.

This completes the proof of 5.6.

Let $\mathcal{S}(U\backslash N)'$ be the strong dual of $\mathcal{S}(U\backslash N)$. The contragredient action to the regular action on $\mathcal{S}(U\backslash N)$ defines a natural structure of a $\mathcal{U}(\mathfrak{n})$-module on $\mathcal{S}(U\backslash N)'$. By dualizing, 5.6 implies the following result on the Lie algebra cohomology of $\mathcal{S}(U\backslash N)' \otimes_{\mathbb{C}} \mathbb{C}_\eta$. Its proof is based on the following observation.

LEMMA 5.11. *Let E be a $\mathcal{U}(\mathfrak{n})$-module. Assume that E is an NF-space (or a DNF-space) and that $\mathfrak{n}$ acts by continuous linear transformations on E. Denote by E' the strong dual of E equipped with the contragredient action.*

(i) *Assume that $H_p(\mathfrak{n}, E)$, $p \in \mathbb{Z}_+$, are finite-dimensional. Then, the Lie algebra cohomology spaces $H^p(\mathfrak{n}, E')$ are linear duals of $H_p(\mathfrak{n}, E)$ for all $p \in \mathbb{Z}_+$.*
(ii) *Assume that $H^p(\mathfrak{n}, E)$, $p \in \mathbb{Z}_+$, are finite-dimensional. Then, the Lie algebra homology spaces $H_p(\mathfrak{n}, E')$ are linear duals of $H^p(\mathfrak{n}, E)$ for all $p \in \mathbb{Z}_+$.*

PROOF. We prove (i), the proof of (ii) is analogous. The Lie algebra homology of E is calculated from the standard complex $C_\cdot(\mathfrak{n}, E)$ with $C_p(\mathfrak{n}, E) = \bigwedge^{-p} \mathfrak{n} \otimes_{\mathbb{C}} E$, $p \in \mathbb{Z}$. By our assumption, this is a complex of NF-spaces (or DNF-spaces), its differentials are continuous and its cohomology spaces are finite-dimensional. By A.1, the differentials in this complex are all strict. Therefore, by A.2, the cohomology of the strong dual $C_\cdot(\mathfrak{n}, E)'$ is equal to $H_\cdot(\mathfrak{n}, E)^*$. On the other hand, by [**7**, Ch. XIII, §8], the complex $C_\cdot(\mathfrak{n}, E)'$ is equal to the complex $C^\cdot(\mathfrak{n}, E')$ with $C^p(\mathfrak{n}, E) = \mathrm{Hom}_{\mathbb{C}}(\bigwedge^p \mathfrak{n}, E')$, $p \in \mathbb{Z}$, which calculates Lie algebra cohomology of E'. ◻

Therefore, 5.6 and 5.11 imply the following result.

THEOREM 5.12. (i) *$H^p(\mathfrak{n}, \mathcal{S}(U\backslash N)' \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ are finite-dimensional for all $p \in \mathbb{Z}_+$.*
(ii) *$H^p(\mathfrak{n}, \mathcal{S}(U\backslash N)' \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for $p > \dim U$.*
(iii) *If either $\mathrm{Re}\,\eta \neq 0$ or $\eta|\mathfrak{u} \neq 0$, we have $H^p(\mathfrak{n}, \mathcal{S}(U\backslash N)' \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$.*

Finally, we want to prove a nonvanishing result for Lie algebra homology of twisted regular representations. Assume that U is trivial. Then, by 5.6, we have $H_p(\mathfrak{n}, \mathcal{S}(N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ for $p > 0$. Moreover, $H_0(\mathfrak{n}, \mathcal{S}(N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ if $\mathrm{Re}\,\eta \neq 0$. The only remaining case is treated in the following result.

PROPOSITION 5.13. *Assume that $\mathrm{Re}\,\eta = 0$. Then*

$$H_0(\mathfrak{n}, \mathcal{S}(N) \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \mathbb{C}.$$

This isomorphism is induced by the map

$$f \longmapsto \int_N f(n)\psi(n)^{-1} dn$$

which is a tempered distribution on N.

PROOF. Since $\mathrm{Re}\,\eta = 0$, ψ is a unitary character. Clearly, ψ is smooth and all its partial derivatives are bounded. Hence multiplication by this function defines a linear automorphism of $\mathcal{S}(N)$ with itself. This automorphism intertwines the representation on $\mathcal{S}(N)$ with the representation on $\mathcal{S}(N) \otimes_{\mathbb{C}} \mathbb{C}_\eta$. Therefore, to prove the statement we can assume that $\eta = 0$.

If $\dim N = 1$, N is abelian and, by 5.4, we have $H_0(\mathfrak{n}, \mathcal{S}(N)) = \mathbb{C}$.

Let $\mathfrak{d}_0$ be a one-dimensional central subalgebra of $\mathfrak{n}_0$. Denote by D the corresponding connected Lie subgroup. Let $\mathfrak{d}$ be the complexification of $\mathfrak{d}_0$. Then, by 5.4, we have $H_0(\mathfrak{d}, \mathcal{S}(N)) = \mathcal{S}(N/D)$. In addition, we have $H_0(\mathfrak{n}/\mathfrak{d}, H_0(\mathfrak{d}, \mathcal{S}(N))) = H_0(\mathfrak{n}, \mathcal{S}(N))$. Therefore, it follows that $H_0(\mathfrak{n}, \mathcal{S}(N)) = H_0(\mathfrak{n}/\mathfrak{d}, \mathcal{S}(N/D))$, and by induction in dimension of N we conclude that $H_0(\mathfrak{n}, \mathcal{S}(N)) = \mathbb{C}$.

On the other hand, by 5.5, $f \longmapsto \int_N f(n)\, dn$ is a tempered distribution on N, i.e., a continuous linear form on $\mathcal{S}(N)$. As we already remarked, it vanishes on $\mathfrak{n} \cdot \mathcal{S}(N)$, i.e., it factors through $H_0(\mathfrak{n}, \mathcal{S}(N))$. $\square$

As before, using 5.11, we deduce from this the following result.

Corollary 5.14. *Assume that* $\operatorname{Re} \eta = 0$. *Then* $H^0(\mathfrak{n}, \mathcal{S}(N)' \otimes_{\mathbb{C}} \mathbb{C}_\eta)$ *is one-dimensional and spanned by the tempered distribution*

$$f \longmapsto \int_N f(n)\psi(n)dn.$$

6. Whittaker vectors for smooth principal series

In this section we prove a generalization of a result of Kostant on the space of Whittaker vectors for smooth principal series [**21**, Theorem 6.6.2].[3]

We return to the setting of §4. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be an one-dimensional representation of $\mathfrak{n}$ (i.e., a complex linear form which vanishes on $[\mathfrak{n}, \mathfrak{n}]$). Let B be the basis of the root system R corresponding to the set of positive roots R^+. Since the root subspaces $\mathfrak{g}_\alpha$, $\alpha \in B$, span a complement of $[\mathfrak{n}, \mathfrak{n}]$, η is completely determined by its restrictions to these subspaces. We say that η is *nondegenerate* if $\eta|\mathfrak{g}_\alpha \neq 0$ for all $\alpha \in B$.

Any η can be uniquely written as $\eta = \operatorname{Re} \eta + i \operatorname{Im} \eta$ for some complex linear forms $\operatorname{Re} \eta$ and $\operatorname{Im} \eta$ on $\mathfrak{n}$ which take real values on $\mathfrak{n}_0$.

Let σ be an irreducible finite-dimensional representation of P. Denote by $\operatorname{Ind}^\infty(\sigma)$ the corresponding smooth principal series representation of G. Let $\operatorname{Ind}^\infty(\sigma)'$ be the strong dual of $\operatorname{Ind}^\infty(\sigma)$ equipped by the adjoint action of $\mathcal{U}(\mathfrak{g})$. We say that a distribution $\phi \in \operatorname{Ind}^\infty(\sigma)'$ is *a Whittaker vector* corresponding to a one-dimensional representation η of $\mathfrak{n}$ if $\xi \cdot \phi = \eta(\xi)\phi$ for $\xi \in \mathfrak{n}$.

The main result of this section is the following theorem.

Theorem 6.1. *Let* η *be a nondegenerate one-dimensional representation of* $\mathfrak{n}$. *Then, for arbitrary irreducible finite-dimensional representation* σ *of* P*:*

(i) *The space of all Whittaker vectors in* $\operatorname{Ind}^\infty(\sigma)'$ *is zero if* η *is not purely imaginary.*
(ii) *If* η *is purely imaginary, the dimension of the space of all Whittaker vectors in* $\operatorname{Ind}^\infty(\sigma)'$ *is equal to* $\dim_{\mathbb{C}}(\sigma)$.

If we consider the tensor product $\operatorname{Ind}^\infty(\sigma)' \otimes_{\mathbb{C}} \mathbb{C}_{-\eta}$ as a $\mathcal{U}(\mathfrak{n})$-module, the linear map $\phi \longmapsto \phi \otimes 1$ is a linear isomorphism of the space of all Whittaker vectors in $\operatorname{Ind}^\infty(\sigma)'$ with the space of all $\mathfrak{n}$-invariants in $\operatorname{Ind}^\infty(\sigma)' \otimes_{\mathbb{C}} \mathbb{C}_{-\eta}$.

Therefore, 6.1 follows immediately from the following result (after replacing η by its negative).

Theorem 6.2. *Let* η *be a nondegenerate one-dimensional representation of* $\mathfrak{n}$. *Then, for arbitrary irreducible finite-dimensional representation* σ *of* P*, we have:*

[3] Compare also [**22**].

(i) $H^p(\mathfrak{n}, \mathrm{Ind}^\infty(\sigma)' \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$ *for all* $p > 0$.
(ii) *If* η *is not purely imaginary,* $H^0(\mathfrak{n}, \mathrm{Ind}^\infty(\sigma)' \otimes_{\mathbb{C}} \mathbb{C}_\eta) = 0$.
(iii) *If* η *is purely imaginary,* $\dim_{\mathbb{C}} H^0(\mathfrak{n}, \mathrm{Ind}^\infty(\sigma)' \otimes_{\mathbb{C}} \mathbb{C}_\eta) = \dim_{\mathbb{C}}(\sigma)$.

Let $\mathrm{Ind}^\infty(\sigma) = C_0^\infty(\sigma) \supset \ldots \supset C_n^\infty(\sigma) \supset \{0\}$ be the Bruhat filtration of $\mathrm{Ind}^\infty(\sigma)$. The last nontrivial subspace $C_n^\infty(\sigma)$ is also $J(w_0, \sigma)$ where w_0 is the longest element in the Weyl group W. The adjoint of the inclusion $J(w_0, \sigma) \to \mathrm{Ind}^\infty(\sigma)$ is the restriction map $\mathrm{res} : \mathrm{Ind}^\infty(\sigma)' \to J(w_0, \sigma)'$ of distributions to the open Bruhat cell $C(w_0)$.

The main step in the proof of 6.2 is the following lemma.

Lemma 6.3. *The map* $\mathrm{res} \otimes 1 : \mathrm{Ind}^\infty(\sigma)' \otimes \mathbb{C}_\eta \to J(w_0, \sigma)' \otimes \mathbb{C}_\eta$ *induces isomorphisms on Lie algebra cohomology, i.e.,*

$$H^p(\mathrm{res} \otimes 1) : H^p(\mathfrak{n}, \mathrm{Ind}^\infty(\sigma)' \otimes \mathbb{C}_\eta) \to H^p(\mathfrak{n}, J(w_0, \sigma)' \otimes \mathbb{C}_\eta)$$

for $p \in \mathbb{Z}_+$, *are isomorphisms.*

In particular, the restriction map on Whittaker vectors in $\mathrm{Ind}^\infty(\sigma)'$ is injective, i.e., the support of any nontrivial Whittaker vector is equal to X.

Assuming 6.3 for a moment, we can complete the proof of 6.2. Since $C(w_0)$ is open in X, the filtration by the transversal degree on $J(w_0, \sigma)'$ is trivial. Therefore, by 4.3, $\mathcal{U}(\mathfrak{n})$-module $J(w_0, \sigma)'$ is isomorphic to a direct sum of $\dim_{\mathbb{C}}(\sigma)$ copies of $\mathcal{S}(C(w_0))'$. Moreover, the stabilizer in N of any point in $C(w_0)$ is trivial. Hence, from 5.12, we first conclude that $H^p(\mathfrak{n}, J(w_0, \sigma)' \otimes \mathbb{C}_\eta) = 0$ for $p > 0$. This implies (i) in 6.2. Moreover, if $\mathrm{Re}\,\eta \neq 0$, we have $H^0(\mathfrak{n}, J(w_0, \sigma)' \otimes \mathbb{C}_\eta) = 0$ and (ii) in 6.2 holds. Finally, if $\mathrm{Re}\,\eta = 0$, by 5.14, we have $\dim_{\mathbb{C}} H^0(\mathfrak{n}, J(w_0, \sigma)' \otimes \mathbb{C}_\eta) = \dim_{\mathbb{C}}(\sigma)$. This implies (iii) in 6.2 and completes its proof.

It remains to establish 6.3. Denote by $C_p^\infty(\sigma)^\perp$ the space off all distributions in $\mathrm{Ind}^\infty(\sigma)$ vanishing on $C_p^\infty(\sigma)$, i.e., the distributions supported in the union of Bruhat cells of dimension $< p$. Then, we have an increasing filtration of $\mathrm{Ind}^\infty(\sigma)'$ given by $\{0\} = C_0^\infty(\sigma)^\perp \subset C_1^\infty(\sigma)^\perp \subset \ldots \subset C_n^\infty(\sigma)^\perp \subset \mathrm{Ind}^\infty(\sigma)'$. We have the obvious exact sequence

$$0 \to C_n^\infty(\sigma)^\perp \to \mathrm{Ind}^\infty(\sigma)' \xrightarrow{\mathrm{res}} C_n^\infty(\sigma)' \to 0.$$

By tensoring with $\mathbb{C}_\eta$ we get the exact sequence

$$0 \to C_n^\infty(\sigma)^\perp \otimes \mathbb{C}_\eta \to \mathrm{Ind}^\infty(\sigma)' \otimes \mathbb{C}_\eta \xrightarrow{\mathrm{res} \otimes 1} C_n^\infty(\sigma)' \otimes \mathbb{C}_\eta \to 0.$$

From the long exact sequence of the Lie algebra cohomology corresponding to this short exact sequence of $\mathcal{U}(\mathfrak{n})$-modules we see that 6.3 follows immediately from the following result.

Lemma 6.4. $H^p(\mathfrak{n}, C_n^\infty(\sigma)^\perp \otimes \mathbb{C}_\eta) = 0$ *for all* $p \in \mathbb{Z}_+$.

This is just a special case of the following assertion, which we are going to prove by induction in q:

$$H^p(\mathfrak{n}, C_q^\infty(\sigma)^\perp \otimes \mathbb{C}_\eta) = 0$$

for all $p \in \mathbb{Z}_+$ and $0 \le q \le n$.

This statement is evident for $q = 0$. Assume that it holds for q, $n > q \ge 0$. By 4.1 and dualizing, we have the short exact sequence

$$0 \to \bigoplus_{w \in W(q)} J(w, \sigma)' \to C_q^\infty(\sigma)' \to C_{q+1}^\infty(\sigma)' \to 0.$$

This leads immediately to the short exact sequence

$$0 \to C_q^\infty(\sigma)^\perp \to C_{q+1}^\infty(\sigma)^\perp \to \bigoplus_{w\in W(q)} J(w,\sigma)' \to 0.$$

By tensoring with $\mathbb{C}_\eta$ we get the short exact sequence

$$0 \to C_q^\infty(\sigma)^\perp \otimes \mathbb{C}_\eta \to C_{q+1}^\infty(\sigma)^\perp \otimes \mathbb{C}_\eta \to \bigoplus_{w\in W(q)} J(w,\sigma)' \otimes \mathbb{C}_\eta \to 0.$$

In addition, we have the following vanishing result, the proof of which we postpone for a moment.

Lemma 6.5. *Assume that $w \in W$ is not the longest element in W. Then $H^p(\mathfrak{n}, J(w,\sigma)' \otimes \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$.*

From the long exact sequence of Lie algebra cohomology corresponding to the above short exact sequence of $\mathcal{U}(\mathfrak{n})$-modules, using the induction assumption and 6.5, we see that $H^p(\mathfrak{n}, C_{q+1}^\infty(\sigma)^\perp \otimes \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$. This proves the above assertion and completes the proof of 6.4.

It remains to establish 6.5. The proof is based on the following elementary fact.

Lemma 6.6. *Let V be a $\mathcal{U}(\mathfrak{n})$-module. Assume that $\mathrm{F}\,V$ is an increasing exhaustive filtration of V by $\mathcal{U}(\mathfrak{n})$-modules such that $\mathrm{F}_p\,V = \{0\}$ for sufficiently negative $p \in \mathbb{Z}$.*

Let $H^q(\mathfrak{n}, \mathrm{Gr}\,V) = 0$ for all $q \in \mathbb{Z}$. Then, we have $H^q(\mathfrak{n}, V) = 0$ for all $q \in \mathbb{Z}$.

Proof. First, consider the short exact sequence

$$0 \to \mathrm{F}_{p-1}\,V \to \mathrm{F}_p\,V \to \mathrm{Gr}^p\,V \to 0$$

of $\mathcal{U}(\mathfrak{n})$-modules. By the assumption, $H^q(\mathfrak{n}, \mathrm{Gr}^p\,V) = 0$ for all $p, q \in \mathbb{Z}$. Hence, from the long exact sequence of Lie algebra cohomology attached to the above short exact sequence, we conclude that the natural map $H^q(\mathfrak{n}, \mathrm{F}_{p-1}\,V) \to H^q(\mathfrak{n}, \mathrm{F}_p\,V)$ is an isomorphism for all $p, q \in \mathbb{Z}$. Since $F_pV = \{0\}$ for sufficiently negative $p \in \mathbb{Z}$, obviously we have $H^q(\mathfrak{n}, \mathrm{F}_p\,V) = 0$ for all $q \in \mathbb{Z}$ and sufficiently negative $p \in \mathbb{Z}$. By induction in p, we see that $H^q(\mathfrak{n}, \mathrm{F}_p\,V) = 0$ for all $p, q \in \mathbb{Z}$.

We claim that the standard complex $C^\cdot(\mathfrak{n}, V)$ is acyclic. The increasing family of submodules $\mathrm{F}_p\,V$ defines an increasing exhaustive filtration of the standard complex $C^\cdot(\mathfrak{n}, V)$ by subcomplexes $C^\cdot(\mathfrak{n}, \mathrm{F}_p\,V)$. We just established that all these subcomplexes are acyclic.

Let $u \in C^q(\mathfrak{n}, V)$ be such that $du = 0$. Then, since the filtration of the standard complex is exhaustive, $u \in C^q(\mathfrak{n}, \mathrm{F}_p\,V)$ for some large $p \in \mathbb{Z}$. By the above remarks, there exists $v \in C^{q-1}(\mathfrak{n}, \mathrm{F}_p\,V) \subset C^{q-1}(\mathfrak{n}, V)$ such that $u = dv$. Therefore, $H^q(\mathfrak{n}, V) = 0$. □

As explained in §2, $J(w,\sigma)'$ has a natural increasing filtration $\mathrm{F}\,J(w,\sigma)'$ by the transversal degree. This filtration is exhaustive by 2.7, and $\mathrm{F}_p\,J(w,\sigma)'$ are $\mathcal{U}(\mathfrak{n})$-modules for all $p \in \mathbb{Z}$ by 3.5. Therefore it induces an analogous filtration $\mathrm{F}(J(w,\sigma)' \otimes \mathbb{C}_\eta)$ given by $\mathrm{F}_p(J(w,\sigma)' \otimes \mathbb{C}_\eta) = \mathrm{F}_p\,J(w,\sigma)' \otimes \mathbb{C}_\eta$ for all $p \in \mathbb{Z}$. The corresponding graded object is $\mathrm{Gr}\,J(w,\sigma)' \otimes \mathbb{C}_\eta$.

Therefore, by 6.6, to establish 6.5, it is enough to prove the following result.

Lemma 6.7. *Assume that $w \in W$ is not the longest element in W. Then $H^p(\mathfrak{n}, \mathrm{Gr}^q\,J(w,\sigma)' \otimes \mathbb{C}_\eta) = 0$ for all $p, q \in \mathbb{Z}_+$.*

PROOF. Let w be an element of the Weyl group W. If $w \neq w_0$, the intersection $\mathfrak{n} \cap \mathfrak{n}^w$ is spanned by root subspaces corresponding to the roots in $R^+ \cap w(R^+)$. Clearly, $R^+ \cap w(R^+) = R^+ \cap (-ww_0(R^+))$, where ww_0 is different from the identity in W. Let $s_1 s_2 \dots s_q$, $q \geq 1$, be a reduced expression of $w_0 w^{-1} = (ww_0)^{-1}$ by simple reflections in W. Let $\alpha_1, \alpha_2, \dots, \alpha_q$ be the simple roots corresponding to the reflections $s_1, s_2, \dots, s_q$. By [**3**, Ch. VI, §1, no. 6, Cor. 2 of Prop. 17], $-w_0 w^{-1}(\alpha_q)$ is a positive root. Therefore, α_q is in $R^+ \cap w(R^+)$, i.e., the root subspace corresponding to the simple root α_q is in $\mathfrak{n} \cap \mathfrak{n}^w$. It follows that the restriction of η to $\mathfrak{n} \cap \mathfrak{n}^w$ is nonzero. Hence, by 5.12.(iii), $H^p(\mathfrak{n}, \mathcal{S}(C(w))' \otimes \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}_+$. Now, by 4.3 and 4.4, $\operatorname{Gr}^q J(w,\sigma)'$ has a finite increasing filtration $\operatorname{F} \operatorname{Gr}^q J(w,\sigma)'$ by closed $\mathcal{U}(\mathfrak{n})$-modules such that the corresponding graded module is a direct sum of copies of $\mathcal{S}(C(w))'$, i.e., we have exact sequences

$$0 \longrightarrow \operatorname{F}_{s-1} \operatorname{Gr}^q J(w,\sigma)' \longrightarrow \operatorname{F}_s \operatorname{Gr}^q J(w,\sigma)' \longrightarrow \mathcal{S}(C(w))' \otimes E_w^* \longrightarrow 0$$

for all $p \in \mathbb{Z}$. By the above vanishing result and the long exact sequence of Lie algebra cohomology attached to this short exact sequence, we conclude that the natural morphism $H^p(\mathfrak{n}, \operatorname{F}_{s-1} \operatorname{Gr}^q J(w,\sigma)' \otimes \mathbb{C}_\eta) \longrightarrow H^p(\mathfrak{n}, \operatorname{F}_s \operatorname{Gr}^q J(w,\sigma)' \otimes \mathbb{C}_\eta)$ is an isomorphism for all $p \in \mathbb{Z}$. Since $\operatorname{F}_s \operatorname{Gr}^q J(w,\sigma)' = 0$ for very negative s, we see that $H^p(\mathfrak{n}, \operatorname{F}_s \operatorname{Gr}^q J(w,\sigma)' \otimes \mathbb{C}_\eta) = 0$ for all $p, s \in \mathbb{Z}$. Therefore, since the filtration is finite, we have $H^p(\mathfrak{n}, \operatorname{Gr}^q J(w,\sigma)' \otimes \mathbb{C}_\eta) = 0$ for all $p \in \mathbb{Z}$. ◻

7. Holomorphic continuation of Jacquet integrals

In this section we prove a variant of 6.1. It gives a description of Whittaker vectors for smooth principal series as holomorphic continuation of certain integrals considered by Jacquet [**18**].

Let $P = MAN$ be the Langlands decomposition of P. Then M is a compact group. For a linear form λ on $\mathfrak{a}$ we denote, by abuse of notation, by λ the one-dimensional representation $a \longmapsto e^{\lambda(\log a)}$. We can view it as a representation of P by $man \longmapsto \lambda(a)$, for $m \in M$, $a \in A$ and $n \in N$. Therefore, σ determines a holomorphic family of irreducible representations $\sigma \otimes \lambda$ of P on $\mathfrak{a}^*$. Since $X = P\backslash G = M\backslash K$, all representations $\operatorname{Ind}^\infty(\sigma \otimes \lambda)$ can be realized on the same space of smooth sections $C^\infty(\sigma)$ of the homogeneous vector bundle on $X = M\backslash K$ determined by σ restricted to M. In this way, we can view $\operatorname{Ind}^\infty(\sigma \otimes \lambda)$ as a holomorphic family of representations. The Lie algebra $\mathfrak{g}$ acts on $C^\infty(\sigma)$ by first order differential operators depending holomorphically on $\lambda \in \mathfrak{a}^*$. Moreover, the continuous duals of the spaces of $\operatorname{Ind}^\infty(\sigma \otimes \lambda)$ can be identified with the continuous dual $C^\infty(\sigma)'$ of $C^\infty(\sigma)$. The contragredient action of $\mathfrak{g}$ on $C^\infty(\sigma)'$ is also given by first order differential operators depending holomorphically on $\lambda \in \mathfrak{a}^*$.

Let U be an open set in $\mathfrak{a}^*$. Denote by $\mathcal{H}(U)$ the ring of all holomorphic functions on U equipped with the topology of uniform convergence on compact sets in U. Let $\mathcal{H}(U, C^\infty(\sigma)')$ be the space of all holomorphic functions on U with values in $C^\infty(\sigma)'$ endowed with the topology of uniform convergence on compact sets in U. By the results discussed in the appendix, we know that $\mathcal{H}(U, C^\infty(\sigma)') = \mathcal{H}(U) \hat{\otimes} C^\infty(\sigma)'$. The action of $\mathfrak{g}$ extends by linearity to an action by differential operators on $\mathcal{H}(U, C^\infty(\sigma)')$.

Fix a nondegenerate $\eta \in \mathfrak{n}^*$. For each $\lambda \in \mathfrak{a}^*$, we denote by $W_\eta(\sigma \otimes \lambda)$ the linear subspace of Whittaker vectors in $C^\infty(\sigma)'$ for the representation $\operatorname{Ind}^\infty(\sigma \otimes \lambda)$. By 6.1, $W_\eta(\sigma \otimes \lambda) = 0$ if $\operatorname{Re} \eta \neq 0$. Therefore, we can assume that $\operatorname{Re} \eta = 0$ in the following. Denote by ψ the unitary character of N with the differential η.

Finally, let $\mathcal{W}_\eta(U,\sigma)$ be the subspace of $\mathcal{H}(U, C^\infty(\sigma)')$ consisting of all holomorphic maps $F : U \to C^\infty(\sigma)'$ such that $F(\lambda) \in W_\eta(\sigma\otimes\lambda)$ for all $\lambda \in U$. Evidently, $\mathcal{W}_\eta(U,\sigma)$ is a module over the ring $\mathcal{H}(U)$ of holomorphic functions on U. Moreover, we have

$$\mathcal{W}_\eta(U,\sigma) = H^0(\mathfrak{n}, \mathcal{H}(U, C^\infty(\sigma)') \otimes_{\mathbb{C}} \mathbb{C}_{-\eta}).$$

The assignment $U \longmapsto \mathcal{W}_\eta(U,\sigma)$ defines a sheaf of modules $\mathcal{W}_\eta(\sigma)$ over the sheaf of holomorphic functions $\mathcal{H}$ on $\mathfrak{a}^*$.

By a variant of an argument in the last section we are going to describe the structure of this sheaf. The main difference is that we have to keep track of the dependence on the parameter λ. First we have to make the above isomorphisms more explicit. Let $E_0 = E_{x_1}$ be the fiber of E at the point $x_1 \in X$ corresponding to the parabolic subgroup P. The spaces $\mathrm{Ind}^\infty(\sigma\otimes\lambda)$, $\lambda\in\mathfrak{a}^*$, can be identified with the spaces of smooth functions $F : G \longrightarrow E_0$ satisfying $F(pg) = \sigma(p)\lambda(p)F(g)$ for all $g \in G$ and $p \in P$. The restriction map $F \longmapsto F|_K$ defines the isomorphism with functions from K to E_0 which correspond to the sections in $C^\infty(\sigma)$. The inverse map sends a function f on K to a function $nak \longmapsto \sigma(a)\lambda(a)f(k)$, $n \in N$, $a \in A$ and $k \in K$, on G. In the following, we are going to view $\mathrm{Ind}^\infty(\sigma\otimes\lambda)$ and $C^\infty(\sigma)$ as these spaces of functions.

We fix representatives $k_w \in K$ of $w \in W$. As we described in §4, the space of Schwartz sections of $\mathrm{Ind}^\infty(\sigma\otimes\lambda)$ on V^w can be identified with $\mathcal{S}(U^w)\otimes E_0$, i,e., the space of Schwartz functions on U^w with values in E_0. This isomorphism is given by the restriction map, i.e., it attaches to $F \in \mathrm{Ind}^\infty(\sigma\otimes\lambda)$ the function $u \longmapsto F(k_w u)$ on U^w. The inverse of the above map sends the function $\varphi \in \mathcal{S}(U^w)\otimes E_0$ into the function on G which is obtained by extension by zero from the function $namk_w u \longmapsto \sigma(am)\lambda(a)\varphi(u)$, $n \in N$, $a \in A$, $m \in M$ and $u \in U^w$, on the open set PwU^w.

For a given λ we have the natural injective linear map

$$\mathcal{S}(U^w)\otimes E_0 \longrightarrow \mathrm{Ind}^\infty(\sigma\otimes\lambda) \longrightarrow C^\infty(\sigma).$$

These maps depend on λ but have the common image equal to the space of Schwartz sections on V^w. To see the dependence on λ explicitly, denote by $h : G \longrightarrow A$ and $\kappa : G \longrightarrow K$ the unique analytic maps such that $g \in Nh(g)\kappa(g)$ for $g \in G$. Then the above map attaches to $\varphi \in \mathcal{S}(U^w)\otimes E_0$ the function $K \ni \kappa(k_w u) \longmapsto \lambda(h(k_w u k_{w^{-1}}))^{-1}\varphi(u)$. The multiplier $u \longmapsto \lambda(h(k_w u k_{w^{-1}}))$ is a function of moderate growth on U^w which depends holomorphically on λ. Therefore, the above linear maps depend holomorphically on λ.

In particular, the bottom part $C_n^\infty(\sigma) = J(w_0,\sigma)$ in the Bruhat filtration is naturally isomorphic to $\mathcal{S}(N)\otimes E_0$ as a representation of N and this isomorphism depends holomorphically on λ. Therefore, the composition with the adjoint of the inclusion $J(w_0,\sigma) \longrightarrow C^\infty(\sigma)$ induces a linear map of $W_\eta(\sigma\otimes\lambda)$ into the space of ψ-covartiant continuous linear forms on $\mathcal{S}(N)\otimes E_0$. By 5.13, the unique ψ-covariant continuous linear form on $\mathcal{S}(N)$ is equal to $f \longmapsto \int_N \psi^{-1}(n)f(n)\,dn$. Hence, we see that this space is isomorphic to E_0^*. Combining this with the above discussion, it follows that in this way we get a $\mathcal{H}$-module morphism Φ of $\mathcal{W}_\eta(\sigma)$ into $\mathcal{H}\otimes E_0^*$.

THEOREM 7.1. *Assume that $\eta \in \mathfrak{n}^*$ is nondegenerate and $\mathrm{Re}\,\eta = 0$. Then, for any irreducible representation σ of P, Φ is an isomorphism of $\mathcal{W}_\eta(\sigma)$ with $\mathcal{H}\otimes E_0^*$.*

In particular, $\mathcal{W}_\eta(\sigma)$ is a free $\mathcal{H}$-module of rank $\dim_{\mathbb{C}}(\sigma)$.

It follows that the space $\mathcal{W}_\eta(\mathfrak{a}^*, \sigma)$ of global sections of $\mathcal{W}_\eta(\sigma)$, i.e., of holomorphic maps from $\mathfrak{a}^*$ into $C^\infty(\sigma)'$ with values in Whittaker vectors is a free module over the entire functions on $\mathfrak{a}^*$ of rank $\dim_{\mathbb{C}}(\sigma)$. By 6.1, for an arbitrary fixed $\lambda \in \mathfrak{a}^*$, a basis of $\mathcal{W}_\eta(\mathfrak{a}^*, \sigma)$ (as a free $\mathcal{H}(\mathfrak{a}^*)$-module) evaluated at λ, forms a basis of the space of Whittaker vectors for $\operatorname{Ind}^\infty(\sigma \otimes \lambda)$.

Without any loss of generality, we can assume that σ is a unitary representation on a finite-dimensional inner product space E_0. Consider the linear map $\Omega_{\psi,\lambda} : \operatorname{Ind}^\infty(\sigma \otimes \lambda) \longrightarrow E_0^*$ given by

$$\Omega_{\psi,\lambda}(f) = \int_N \psi^{-1}(n) f(k_{w_0} n)\, dn.$$

If this integral converges and defines a continuous linear map, $\omega \longmapsto \omega \circ \Omega_{\psi,\lambda}$ is a linear map from E_0^* into $W_\eta(\sigma \otimes \lambda)$. Clearly, we have

$$\|\Omega_{\psi,\lambda}(f)\| \leq C_1 \int_N \|f(k_{w_0} n)\|\, dn$$

for some $C_1 > 0$. Since $n \longmapsto k_{w_0} n k_{w_0}^{-1}$ is an isomorphism of N onto $\bar{N}$, we have

$$\begin{aligned}\|\Omega_{\psi,\lambda}(f)\| &\leq C_1 \int_{\bar{N}} \|f(\bar{n} k_{w_0})\|\, d\bar{n} \\ &= C_1 \int_{\bar{N}} |\lambda(h(\bar{n}))|\ \|f(\kappa(\bar{n}) k_{w_0})\|\, d\bar{n} \leq C_1 \max_{k \in K} \|f(k)\|\ \int_{\bar{N}} |\lambda(h(\bar{n}))|\, d\bar{n}\end{aligned}$$

with an appropriate choice of the Haar measure on $\bar{N}$. Let ρ be the half-sum of positive roots (counted with their multiplicities). Denote by C_ρ the set of all $\lambda \in \mathfrak{a}^*$ such that $\operatorname{Re} \alpha\check{}(\lambda - \rho) > 0$ for all dual roots $\alpha\check{}$ of $\alpha \in R^+$. By the standard arguments from the theory of intertwining operators [**16**, I.5], it follows that the last integral converges for $\lambda \in C_\rho$. Therefore, $\lambda \longmapsto \Omega_{\psi,\lambda}$ is defined and holomorphic in C_ρ. Moreover, for $\lambda \in C_\rho$, $\omega \longmapsto \omega \circ \Omega_{\psi,\lambda}$ is an isomorphism of E_0^* onto the space $W_\eta(\sigma \otimes \lambda)$, i.e., all Whittaker vectors are given by the Jacquet integral $\Omega_{\psi,\lambda}$. Therefore, for any open set $U \subset C_\rho$, Jacquet integral defines the inverse of $\Phi(U) : \mathcal{W}_\eta(U, \sigma) \longrightarrow \mathcal{H}(U) \otimes E_0^*$. To sum up, 7.1 has the following immediate consequence.

THEOREM 7.2. *The Jacquet integral $\lambda \longmapsto \Omega_{\psi,\lambda}$ extends to a holomorphic map from $\mathfrak{a}^*$ into $\mathcal{W}_\eta(\mathfrak{a}^*, \sigma) \otimes E_0$.*

This result was established in [**29**] by a different method.

To prove 7.1 it is enough to show the following local result. Let λ be a point in $\mathfrak{a}^*$. Denote by $\mathcal{H}_\lambda(C^\infty(\sigma)')$ the space of all germs at λ of holomorphic maps with values in $C^\infty(\sigma)'$. It is a module over the ring $\mathcal{H}_\lambda$ of germs of holomorphic functions at λ. Clearly, $\mathfrak{n}$ acts on this space by $\mathcal{H}_\lambda$-module endomorphisms.

LEMMA 7.3. *For any $\lambda \in \mathfrak{a}^*$, the natural morphism*

$$\Phi_\lambda : H^0(\mathfrak{n}, \mathcal{H}_\lambda(C^\infty(\sigma)') \otimes \mathbb{C}_\eta) \longrightarrow \mathcal{H}_\lambda \otimes E_0^*$$

is an isomorphism.

It remains to establish 7.3.

First, let E be a DNF-space and a $\mathcal{U}(\mathfrak{n})$-module, such that $\mathfrak{n}$ acts on E by continuous linear transformations. Then this action defines, by linearity, a natural structure of $\mathcal{U}(\mathfrak{n})$-module on $\mathcal{H}_\lambda(E)$.

Lemma 7.4. *If the Lie algebra cohomology spaces $H^p(\mathfrak{n}, E)$ are finite-dimensional for all $p \in \mathbb{Z}_+$, we have*

$$H^p(\mathfrak{n}, \mathcal{H}_\lambda(E)) = \mathcal{H}_\lambda(H^p(\mathfrak{n}, E))$$

for all $p \in \mathbb{Z}_+$.

Proof. The standard complex of Lie algebra cohomology $C^{\cdot}(\mathfrak{n}, E)$ is a complex of DNF-spaces with continuous differentials. By A.1, all differentials are strict morphisms. Therefore, their kernels and images are DNF-spaces. The assertion follows by tensoring this complex by $\mathcal{H}_\lambda$ and using B.2. □

To establish 7.3 we again use the results about the Bruhat filtration from 4.1. By dualizing, we get the exact sequence

$$0 \to \bigoplus_{w \in W(p)} J(w, \sigma)' \to C_p^\infty(\sigma)' \to C_{p+1}^\infty(\sigma)' \to 0$$

for $p \in \mathbb{Z}_+$. Since these are DNF-spaces, by B.2, we get the exact sequence

$$0 \to \bigoplus_{w \in W(p)} \mathcal{H}_\lambda(J(w, \sigma)') \to \mathcal{H}_\lambda(C_p^\infty(\sigma)') \to \mathcal{H}_\lambda(C_{p+1}^\infty(\sigma)') \to 0.$$

The proof of 7.3 is based on the following vanishing result.

Lemma 7.5. *If $\eta \in \mathfrak{n}^*$ is nondegenerate and $w \in W$ different from w_0, we have*

$$H^p(\mathfrak{n}, \mathcal{H}_\lambda(J(w, \sigma)') \otimes \mathbb{C}_\eta) = 0$$

for all $p \in \mathbb{Z}_+$.

By 7.5 and the long exact sequence of cohomology, we conclude that

$$H^q(\mathfrak{n}, \mathcal{H}_\lambda(C_p^\infty(\sigma)') \otimes \mathbb{C}_\eta) = H^q(\mathfrak{n}, \mathcal{H}_\lambda(C_{p+1}^\infty(\sigma)') \otimes \mathbb{C}_\eta)$$

for all $0 \le p < n$ and $q \in \mathbb{Z}_+$. This implies that

$$H^q(\mathfrak{n}, \mathcal{H}_\lambda(C^\infty(\sigma)') \otimes \mathbb{C}_\eta) = H^q(\mathfrak{n}, \mathcal{H}_\lambda(C_n^\infty(\sigma)') \otimes \mathbb{C}_\eta)$$

for all $q \in \mathbb{Z}_+$. As we already remarked before, $C_n^\infty(\sigma) = J(w_0, \sigma)$. Hence, we get the following extension result.

Lemma 7.6. *Let $\eta \in \mathfrak{n}^*$ be nondegenerate. Then the restriction map $C^\infty(\sigma)' \to J(w_0, \sigma)'$ induces an isomorphism*

$$H^p(\mathfrak{n}, \mathcal{H}_\lambda(C^\infty(\sigma)') \otimes \mathbb{C}_\eta) \to H^p(\mathfrak{n}, \mathcal{H}_\lambda(J(w_0, \sigma)') \otimes \mathbb{C}_\eta)$$

for all $p \in \mathbb{Z}_+$.

First, we want to show that this result implies 7.3. As we already remarked, $\mathcal{U}(\mathfrak{n})$-module $J(w_0, \sigma)'$ is isomorphic to $\mathcal{S}(N)' \otimes E_0^*$ and this isomorphism depends holomorphically on λ. Hence, by 5.14 and 7.4, we immediately see that Φ_λ is an isomorphism.

It remains to establish 7.5. As we explained in §4, $J(w, \sigma)'$ has a natural filtration by closed linear subspaces $\mathrm{F}_p\, J(w, \sigma)'$, $p \in \mathbb{Z}$. Clearly, $\mathrm{F}_p\, J(w, \sigma)'$ are DNF-spaces for all $p \in \mathbb{Z}$, and $J(w, \sigma)' = \varinjlim_{p \in \mathbb{Z}_+} \mathrm{F}_p\, J(w, \sigma)'$ by 2.8. By B.3, we have

$$\mathcal{H}_\lambda(J(w, \sigma)') = \varinjlim_{k \in \mathbb{Z}_+} \mathcal{H}_\lambda(\mathrm{F}_k\, J(w, \sigma)').$$

Moreover, by exactness of tensoring, $\mathcal{H}_\lambda(\mathrm{F}_k\, J(w, \sigma)')$ are closed subspaces in the space $\mathcal{H}_\lambda(J(w, \sigma)')$, i.e., $\mathcal{H}_\lambda(\mathrm{F}\, J(w, \sigma)')$ is an exhaustive increasing filtration of

$\mathcal{H}_\lambda(J(w,\sigma)')$ by closed $\mathcal{U}(\mathfrak{n})$-modules. On the other hand, by B.2, from the short exact sequences

$$0 \longrightarrow \mathrm{F}_{p-1}\, J(w,\sigma)' \longrightarrow \mathrm{F}_p\, J(w,\sigma)' \longrightarrow \mathrm{Gr}^p\, J(w,\sigma)' \longrightarrow 0$$

we get the short exact sequences

$$0 \longrightarrow \mathcal{H}_\lambda(\mathrm{F}_{p-1}\, J(w,\sigma)') \longrightarrow \mathcal{H}_\lambda(\mathrm{F}_p\, J(w,\sigma)') \longrightarrow \mathcal{H}_\lambda(\mathrm{Gr}^p\, J(w,\sigma)') \longrightarrow 0.$$

Therefore, we have $\mathrm{Gr}^p\,\mathcal{H}_\lambda(J(w,\sigma)') = \mathcal{H}_\lambda(\mathrm{Gr}^p\, J(w,\sigma)')$ for all $p \in \mathbb{Z}_+$.

Now, by the discussion from the beginning of this section and 4.3, the action of $\mathfrak{n}$ on $\mathrm{Gr}^p\, J(w,\sigma)'$ is isomorphic to the natural action on $L_p(w)\otimes_{R(C(w))}\mathcal{S}(C(w))'\otimes_{\mathbb{C}} E_w^*$ and this isomorphism depends holomorphically on λ. Hence, by 7.4 and 6.7, we have

$$H^q(\mathfrak{n}, \mathrm{Gr}^p\,\mathcal{H}_\lambda(J(w,\sigma)')\otimes\mathbb{C}_\eta) = H^q(\mathfrak{n}, \mathcal{H}_\lambda(\mathrm{Gr}^p\, J(w,\sigma)')\otimes\mathbb{C}_\eta) = 0$$

for all $p, q \in \mathbb{Z}_+$. Finally, by 6.6, this completes the proof of 7.5.

8. The Whittaker functor

Let V be a $\mathfrak{n}$-module and η a nondegenerate linear form on $\mathfrak{n}$. Denote by $\mathbb{C}_\eta$ the one-dimensional $\mathfrak{n}$-module with action η. Then we define the functor $V \longmapsto \Psi_\eta(V) = H_0(\mathfrak{n}, V\otimes\mathbb{C}_\eta)$ from the category of $\mathfrak{n}$-modules into the category of complex linear spaces. We call Ψ_η the *Whittaker functor* corresponding to $\mathfrak{n}$ and η. This functor is right exact.

The following fact is evident, since the tensoring by $\mathbb{C}_\eta$ is an equivalence of categories.

LEMMA 8.1. (i) *The left cohomological dimension of the functor Ψ_η is $\leq \dim_{\mathbb{C}}\mathfrak{n}$.*

(ii) *The left derived functors $L^{-p}\Psi_\eta$, $p \in \mathbb{Z}_+$, of Ψ_η are given by $L^{-p}\Psi_\eta(V) = H_p(\mathfrak{n}, V\otimes\mathbb{C}_\eta)$.*

Let $\mathcal{M}(\mathfrak{g}, K)$ be the abelian category of Harish-Chandra modules for G and $\mathcal{M}_\infty(G)$ the corresponding abelian category of smooth representations of moderate growth and finite length in the sense of [**9**]. Denote by $V \longmapsto V^\infty$ the corresponding completion functor from $\mathcal{M}(\mathfrak{g}, K)$ into $\mathcal{M}_\infty(G)$. By [**9**] this functor is an equivalence of categories and its quasiinverse is the functor of K-finite vectors.

We want to establish the following result.

THEOREM 8.2. (i) *The functor Ψ_η is nonzero on the category $\mathcal{M}_\infty(G)$ if and only if $\mathrm{Re}\,\eta = 0$.*

(ii) *For any V in $\mathcal{M}_\infty(G)$, $\Psi_\eta(V)$ is finite-dimensional.*

(iii) *The functor Ψ_η is exact on $\mathcal{M}_\infty(G)$.*

Since V is a smooth representation, $\Psi_\eta(V)$ is the cokernel of the continuous map of the NF-space $V\otimes\mathfrak{n}$ into the NF-space V. By A.1, we see that the kernel of the map $V \to \Psi_\eta(V)$ is closed, and we have the following obvious consequence.

Let V be an object in $\mathcal{M}_\infty(G)$. A continuous linear form δ on V is called a *Whittaker vector* (with respect to $\mathfrak{n}$ and η) if δ spans a one-dimensional invariant subspace for the contragredient representation of $\mathfrak{n}$ with the action given by η.

COROLLARY 8.3. *For any V in $\mathcal{M}_\infty(G)$, the dimension of the space of Whittaker vectors with respect to $\mathfrak{n}$ and η is equal to $\dim_{\mathbb{C}}\Psi_\eta(V)$.*

The proof of 8.2 is by reduction to the case of smooth principal series.

We start with the following observation.

LEMMA 8.4. *Assume that any object V in $\mathcal{M}_\infty(G)$ is a subobject of an object W satisfying:*

(i) *W is acyclic for Ψ_η;*

(ii) *$\Psi_\eta(W)$ is finite-dimensional.*

Then the functor Ψ_η is exact on $\mathcal{M}_\infty(G)$ and $\Psi_\eta(V)$ is finite-dimensional.

PROOF. By our assumption, we have a short exact sequence

$$0 \to V \to W \to Q \to 0$$

where Q is the quotient of W by V. From the long exact sequence of derived functors of Ψ_η it follows that $L^{p-1}\Psi_\eta(Q) \cong L^p\Psi_\eta(V)$ for $p \le -1$. Since, by 8.1, the left cohomological dimension of Ψ_η is $\le \dim_{\mathbb{C}} \mathfrak{n}$, $L^q\Psi_\eta(Q) = 0$ for $q < -\dim_{\mathbb{C}} \mathfrak{n}$. By upward induction in p we see that $L^p\Psi_\eta(V) = 0$ for all $p < 0$ and V in $\mathcal{M}_\infty(G)$. It follows that the sequence

$$0 \to \Psi_\eta(V) \to \Psi_\eta(W) \to \Psi_\eta(Q) \to 0$$

is exact. Therefore $\dim_{\mathbb{C}} \Psi_\eta(V) \le \dim_{\mathbb{C}} \Psi_\eta(W) < \infty$. □

Let V be an object in $\mathcal{M}_\infty(G)$ and U the corresponding Harish-Chandra module of K-finite vectors. Then $U^\infty \cong V$. Let P be the minimal parabolic subgroup of G containing N. Let σ be a finite-dimensional representation of P. We denote by $\operatorname{Ind}^\infty(\sigma)$ the (smooth) induced representation of G defined by σ. Let $\operatorname{Ind}(\sigma)$ be the corresponding Harish-Chandra module. Then, by [**10**, 8.21], there exist σ and a monomorphism $U \to \operatorname{Ind}(\sigma)$. By the exactness of the completion functor, we have a monomorphism $V \cong U^\infty \to \operatorname{Ind}(\sigma)^\infty = \operatorname{Ind}^\infty(\sigma)$. If σ is irreducible, $\operatorname{Ind}^\infty(\sigma)$ is a smooth principal series representation of G.

LEMMA 8.5. *Let σ be an irreducible finite-dimensional representation of P and $\operatorname{Ind}^\infty(\sigma)$ the corresponding smooth principal series representation. Then*

(i) *$L^p\Psi_\eta(\operatorname{Ind}^\infty(\sigma)) = 0$ for $p < 0$;*

(ii) *$\Psi_\eta(\operatorname{Ind}^\infty(\sigma)) = 0$ if $\operatorname{Re}\eta \neq 0$;*

(iii) *$\dim_{\mathbb{C}} \Psi_\eta(\operatorname{Ind}^\infty(\sigma)) = \dim_{\mathbb{C}}(\sigma)$ for $\operatorname{Re}\eta = 0$.*

PROOF. As we remarked in 8.1, $L^{-p}\Psi_\eta(\operatorname{Ind}^\infty(\sigma)) = H_p(\mathfrak{n}, \operatorname{Ind}^\infty(\sigma) \otimes \mathbb{C}_\eta)$ for $p \in \mathbb{Z}_+$. Clearly, the topological dual of $\operatorname{Ind}^\infty(\sigma) \otimes \mathbb{C}_\eta$ is equal to $\operatorname{Ind}^\infty(\sigma)' \otimes \mathbb{C}_{-\eta}$ as a $\mathcal{U}(\mathfrak{n})$-module. By 6.2, $H^p(\mathfrak{n}, \operatorname{Ind}^\infty(\sigma)' \otimes \mathbb{C}_{-\eta})$, $p \in \mathbb{Z}_+$, are finite-dimensional. Therefore, by 5.11, we have

$$H_p(\mathfrak{n}, \operatorname{Ind}^\infty(\sigma) \otimes \mathbb{C}_\eta)^* = H_p(\mathfrak{n}, (\operatorname{Ind}^\infty(\sigma)' \otimes \mathbb{C}_{-\eta})')^* = H^p(\mathfrak{n}, \operatorname{Ind}^\infty(\sigma)' \otimes \mathbb{C}_{-\eta})$$

for all $p \in \mathbb{Z}_+$. Now, the statement follows immediately from 6.2. □

Now, we claim that then the statements of this lemma hold for arbitrary finite-dimensional representations σ of P. We prove this by induction in the length of σ. Let

$$0 \to \sigma' \to \sigma \to \sigma'' \to 0$$

be a short exact sequence of finite-dimensional representations of P. Assume that σ'' is irreducible. Then $\operatorname{length}(\sigma') < \operatorname{length}(\sigma)$. Moreover, by exactness of smooth induction we have an exact sequence

$$0 \to \operatorname{Ind}^\infty(\sigma') \to \operatorname{Ind}^\infty(\sigma) \to \operatorname{Ind}^\infty(\sigma'') \to 0.$$

By 8.5 and the induction assumption, we have $L^p\Psi_\eta(\mathrm{Ind}^\infty(\sigma')) = L^p\Psi_\eta(\mathrm{Ind}^\infty(\sigma''))$ $= 0$ for $p < 0$. Therefore, from the long exact sequence of derived functors of Ψ_η we conclude that $L^p\Psi_\eta(\mathrm{Ind}^\infty(\sigma)) = 0$ for all $p < 0$. Moreover, we have an exact sequence

$$0 \to \Psi_\eta(\mathrm{Ind}^\infty(\sigma')) \to \Psi_\eta(\mathrm{Ind}^\infty(\sigma)) \to \Psi_\eta(\mathrm{Ind}^\infty(\sigma'')) \to 0.$$

This clearly implies the vanishing of $\Psi_\eta(\mathrm{Ind}^\infty(\sigma))$ if $\mathrm{Re}\,\eta \neq 0$. If $\mathrm{Re}\,\eta = 0$, we have

$$\begin{aligned}\dim_{\mathbb{C}} \Psi_\eta(\mathrm{Ind}^\infty(\sigma)) &= \dim_{\mathbb{C}} \Psi_\eta(\mathrm{Ind}^\infty(\sigma')) + \dim_{\mathbb{C}} \Psi_\eta(\mathrm{Ind}^\infty(\sigma'')) \\ &= \dim_{\mathbb{C}}(\sigma') + \dim_{\mathbb{C}}(\sigma'') = \dim_{\mathbb{C}}(\sigma).\end{aligned}$$

Therefore, it follows that 8.5 implies that $\mathrm{Ind}^\infty(\sigma)$ satisfy the conditions (i) and (ii) in 8.4. Hence, the functor Ψ_η is exact on $\mathcal{M}_\infty(G)$ and $\Psi_\eta(V)$ is finite-dimensional for any object V in $\mathcal{M}_\infty(G)$. Moreover, if $\mathrm{Re}\,\eta \neq 0$, $\Psi_\eta(\mathrm{Ind}^\infty(\sigma)) = 0$ for any finite-dimensional representation σ of P. Since any object in $\mathcal{M}_\infty(G)$ is a subobject of some $\mathrm{Ind}^\infty(\sigma)$, this in turn implies that Ψ_η vanishes on $\mathcal{M}_\infty(G)$. This completes the proof of 8.2.

9. Linear quasisplit groups

As we remarked in the introduction, from the point of view of applications to automorphic forms the most important case is of the group G of real points of a connected complex reductive algebraic group $\mathbf{G}$ defined over $\mathbb{R}$ which is quasisplit, i.e., such that $\mathbf{G}$ contains a Borel subgroup defined over $\mathbb{R}$. This was the situation originally considered by Kostant.

For such group G, the Lie algebra $\mathfrak{n}$ is the nilpotent radical of a Borel subalgebra of $\mathfrak{g}$. The group of real points of the corresponding Borel subgroup of $\mathbf{G}$ is the minimal parabolic subgroup P containing N. If $P = MAN$ is the Langlands decomposition of P, we conclude that M is commutative. Hence, all irreducible finite-dimensional representations of P are one-dimensional. For purely imaginary η, 6.1 implies that the space of Whittaker vectors is one-dimensional. More precisely, by combining this observation with 7.1 and 7.2, we have the following result.

THEOREM 9.1. *Let ψ be a nondegenerate unitary character of N. Then, for arbitrary one-dimensional representation σ of P, the space of all Whittaker vectors in $\mathrm{Ind}^\infty(\sigma)'$ is one-dimensional. It is spanned by the holomorphic continuation of the Jacquet integral.*

In addition to this, as we remarked in the last section, any irreducible smooth representation V of G is a subrepresentation of some smooth principal series $\mathrm{Ind}^\infty(\sigma)$. Hence, we have $\dim_{\mathbb{C}} \Psi_\eta(V) \leq \dim_{\mathbb{C}} \Psi_\eta(\mathrm{Ind}^\infty(\sigma)) = 1$. Therefore, we have the following immediate consequence.

THEOREM 9.2. *Let η be a nondegenerate one-dimensional representation of $\mathfrak{n}$. Then, $\Psi_\eta(V)$ is either zero or one-dimensional for any irreducible smooth representation V in $\mathcal{M}_\infty(G)$.*

Appendix A. Some results on nuclear spaces

In this section we include some results on nuclear spaces we use in the main text. Most of them are well known to the experts, but for some we were not able to find a suitable reference. We thank Joe Taylor for his expert tutorial on this subject.

Let E be a Fréchet space. Then it is a bornological space [**4**, Ch. III, §2, Prop. 2]. Therefore, its strong dual E' is complete [**4**, Ch. III, §3, no. 8, Cor. 1 of Prop. 12]. Moreover, E is barreled [**4**, Ch. III, §4, no. 1, Cor. of Prop. 2].

A nuclear Fréchet space E is called an *NF-space* for short. Since, by [**14**, Ch. II, §2, no. 1, Cor. 1 of Lemma 3], any bounded set in E is relatively compact, we see that E is a Montel space. Therefore, E is reflexive [**4**, Ch. IV, §2, no. 3, Th. 2]. Its strong dual E' is a *DNF-space*. By the above remarks, E' is complete and reflexive. It is bornological and barreled [**4**, Ch. IV, §3, no. 4, Cor. of Prop. 4]. Also, it is a Montel space [**4**, Ch. IV, §2, no. 5, Prop. 9],. Moreover, it is nuclear by [**14**, Ch. II, §2, no. 1, Th. 7].

By [**23**, Ch. IV, §8, Examples 1 and 2], NF-spaces and DNF-spaces are Ptak. Therefore, if E and F are either NF-spaces or DNF-spaces, a surjective continuous linear map $T : E \to F$ is open [**23**, Ch. IV, §8, Cor. 1 of Th. 3].

Let $T : E \longrightarrow F$ be a continuous linear map between two hausdorff locally convex spaces. Then $\ker T$ is a closed linear subspace of E. Let $p : E \longrightarrow E/\ker T$ be the canonical projection. The image $\operatorname{im} T$ is a locally convex space with the topology induced by the topology of F. Denote by $i : \operatorname{im} T \to F$ the canonical inclusion. The map T factors as

$$E \xrightarrow{p} E/\ker T \xrightarrow{\tau} \operatorname{im} T \xrightarrow{i} F,$$

where τ is continuous linear isomorphism. We say that T is a *strict morphism* if τ is a homeomorphism.

Let E and F be NF-spaces, resp. DNF-spaces, and $T : E \longrightarrow F$ a continuous linear map. Let $T^t : F' \longrightarrow E'$ be the adjoint linear map between the strong duals. Then T^t is continuous [**4**, Ch. IV, §1, no. 3, Cor. of Prop. 5].

Let E be a nuclear space. Then its subspaces and hausdorff quotient spaces of E are nuclear [**14**, Ch. II, §2, no. 2, Th. 9]. Therefore, closed subspaces and hausdorff quotient spaces of NF-spaces are NF-spaces.

Now we prove that closed subspaces and hausdorff quotient spaces of DNF-spaces are DNF-spaces. Let E be a DNF-space. Let F be a closed subspace of E. Then F and E/F are nuclear spaces. Clearly, $E = G'$ for some NF-space G. Therefore, $E' = G'' = G$ is an NF-space. Let $F^\perp$ be the closed subspace of all linear forms in E' vanishing on F. Then we have the exact sequence

$$0 \to F^\perp \xrightarrow{\alpha} E' \xrightarrow{\beta} E'/F^\perp \to 0.$$

of NF-spaces. By dualizing we get the exact sequence of strong duals

$$0 \to (E'/F^\perp)' \xrightarrow{\beta^t} E'' \xrightarrow{\alpha^t} (F^\perp)' \to 0.$$

The injectivity of β^t is evident, the surjectivity of α^t follows from the Hahn-Banach theorem. If $f \in E''$ is in the kernel of α^t, this means that its restriction to $F^\perp$ vanishes. But then it defines a continuous linear form on $E'/F^\perp$, i.e., it is in the image of β^t. Therefore, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E/F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (E'/F^\perp)' \arrow[r, "\beta^t"'] & E'' \arrow[r, "\alpha^t"'] & (F^\perp)' \arrow[r] & 0
\end{tikzcd}.$$

The vertical arrows are the obvious natural maps. Since E is reflexive, the middle vertical arrow is an isomorphism, and $\operatorname{im}\beta^t$ corresponds to elements $u \in E$ which are in the kernels of all linear forms in $F^\perp$, i.e, $u \in F$. Hence the first vertical arrow is an isomorphism. Therefore, all vertical arrows are isomorphisms of linear spaces. On the other hand, as we remarked above, NF-spaces are Montel spaces. Hence, by [**4**, Ch. IV, §4, no. 2, Cor. 3 of Th.1], α^t and β^t are strict morphisms. This immediately implies that all vertical arrows are isomorphisms of topological linear spaces. In particular, F and E/F are DNF-spaces.

Let E and F be either NF-spaces or DNF-spaces and $T : E \to F$ a continuous linear map. Then, by the above discussion, T a strict morphism if and only if it has closed image. If T is strict, $\operatorname{im} T = E/\ker T$ is complete, and therefore closed in F. Conversely, if $\operatorname{im} T$ is closed, τ is a homeomorphism by the open mapping theorem.

The following simple sufficient criterion for strictness is critical in our applications.

Lemma A.1. *Assume that E and F are either NF-spaces or DNF-spaces. Let $T : E \to F$ be a continuous linear map. Assume that the image* $\operatorname{im} T$ *is of finite codimension in F. Then T is a strict morphism.*

Proof. By replacing E with $E/\ker T$ we can assume that T is injective.

Let H be a finite-dimensional direct complement of $\operatorname{im} T$ in F. Then, the natural map $i : H \to F$ is continuous. Therefore, $T \oplus i : E \oplus H \to F$ is a continuous bijective linear map. By the open mapping theorem, this map is also open. Therefore, it is an isomorphism of topological vector spaces. This implies that $\operatorname{im} T$ is closed in W. □

Moreover, the next result follows easily from the above observations by reduction to short exact sequences.

Lemma A.2. *Assume that $(C^\cdot, d)$ is a complex such that*

(i) *all C^p, $p \in \mathbb{Z}$, are NF-spaces (or DNF-spaces);*
(ii) *all differentials are strict morphisms.*

Then the dual complex $(C'^\cdot, d^t)$ is a complex with differentials which are strict morphisms and

$$H^p(C'^\cdot) = H^{-p}(C^\cdot)'.$$

For two locally convex spaces E and F, we equip the tensor product $E \otimes F$ with the finest locally convex topology such that the bilinear map $E \times F \to E \otimes F$ is continuous. If E and F are hausdorff, this topology on $E \otimes F$ is also hausdorff. We denote by $E \hat{\otimes} F$ completion of $E \otimes F$ with respect to this topology. The locally convex space $E \hat{\otimes} F$ is *the (projective) topological tensor product* of E and F.

If E and F are nuclear spaces, $E \hat{\otimes} F$ is nuclear [**14**, Ch. II, §, no. 1, Th. 9]. If E and F are both NF-spaces (resp. DNF-spaces), $E \hat{\otimes} F$ is an NF-space (resp. DNF-space) [**14**, Ch. I, §1, no. 3, Prop. 5]. Let E and F be NF-spaces (or DNF-spaces). Then we have a natural linear map $E' \otimes F' \to (E \hat{\otimes} F)'$. This map induces an isomorphism $E' \hat{\otimes} F' \to (E \hat{\otimes} F)'$ of locally convex spaces [**14**, Ch. II, §3, no. 2, Th. 12].

Let F be an NF-space (resp. DNF-space). Then, $E \longmapsto E \hat{\otimes} F$ is an exact functor on the category of NF-spaces (resp. DNF-spaces). By the above remarks it is enough to establish the following result.

LEMMA A.3. *Let*

$$0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{p} E_3 \to 0$$

be a short exact sequence of NF-spaces. Let F be an NF-space. Then

$$0 \to E_1 \hat{\otimes} F \xrightarrow{i \otimes 1_F} E_2 \hat{\otimes} F \xrightarrow{p \otimes 1_F} E_3 \hat{\otimes} F \to 0$$

is also a short exact sequence.

PROOF. By [**14**, Ch. I, §1, no. 2, Prop. 3] the map $i \otimes 1_F$ is injective and the map $p \otimes 1_F$ is surjective. Moreover, the kernel of $p \otimes 1_F$ is the closure of the subspace $\ker p \otimes F = \operatorname{im} i \otimes F$. Since i is strict, by [**14**, Ch. II, §3, no. 1, Cor. of Prop. 10], the map $i \otimes 1_F$ is a strict morphism. Therefore, its image is closed and it is an isomorphism of $E_1 \hat{\otimes} F$ onto the closure of $\operatorname{im} i \otimes F$. $\square$

A projective limit of nuclear spaces is nuclear [**14**, Ch. II, §2, no. 2, Th. 9]. Therefore, the projective limit of a countable projective system of NF-spaces is an NF-space.

Let $(E_i; i \in I)$ be an inductive system of nuclear spaces and $E = \varinjlim_{i \in I} E_i$ the inductive limit in the category of locally convex spaces. If I is countable and E is hausdorff, it is nuclear by [**14**, Ch. II, §2, no. 2, Cor. 1 of Th. 9]. Moreover, if E_i are DNF-spaces, E is a DNF-space [**13**, Ch. I, no. 4, Cor. of Th. 9].

LEMMA A.4. *Let $(E_i; i \in I)$ be an inductive system of DNF-spaces. Assume that I is countable and $E = \varinjlim_{i \in I} E_i$ is hausdorff. Let F be a DNF-space. Then $E \hat{\otimes} F = \varinjlim_{i \in I} (E_i \hat{\otimes} F)$.*

PROOF. Clearly, $(E_i \hat{\otimes} F; i \in I)$ is an inductive system of DNF-spaces. We denote by $\alpha_{ij} : E_i \to E_j$ for $i \leq j$ in I, and $\alpha_i : E_i \to E$ for $i \in I$, the corresponding canonical maps. Now consider the inductive system $(E_i \hat{\otimes} F; i \in I)$ with the maps $\beta_{ij} : E_i \hat{\otimes} F \to E_j \hat{\otimes} F$ which are the tensor products of α_{ij} with the identity map on F. Denote by β_i the canonical continuous linear maps $E_i \hat{\otimes} F \to \varinjlim_{i \in I} (E_i \hat{\otimes} F)$.

The continuous linear maps γ_i, $i \in I$, which are the tensor products of α_i with the identity on F define a family of maps from the inductive system $(E_i \hat{\otimes} F; i \in I)$ into $E \hat{\otimes} F$. Therefore, they define a continuous linear map $\gamma : \varinjlim_{i \in I} (E_i \hat{\otimes} F) \to E \hat{\otimes} F$.

We claim that γ is injective. Assume that $v \in \varinjlim_{i \in I} (E_i \hat{\otimes} F)$ is such that $\gamma(v) = 0$. Then, $v = \beta_i(w)$ for some $w \in E_i \hat{\otimes} F$. Moreover, we have $\gamma_i(w) = \gamma(\beta_i(w)) = 0$. By [**14**, Ch. I, §1, no. 2, Prop. 3], this implies that w is in the closure of the subspace $\ker \alpha_i \otimes F$ in $E_i \hat{\otimes} F$. Since $\ker \alpha_i$ is the union of $\ker \alpha_{ij}$ for $j > i$, we see that w is in the union of closures of $\ker \alpha_{ij} \otimes F$. Therefore, $w \in \ker \beta_{ij}$ for some $j > i$. This implies that $v = 0$ and γ is injective. Therefore, $\varinjlim_{i \in I} (E_i \hat{\otimes} F)$ is hausdorff. By the above remark, it follows that $\varinjlim_{i \in I} (E_i \hat{\otimes} F)$ is a DNF-space. Hence, it is complete. By [**14**, Ch. I, §4, no. 3, Cor. of Prop. 6], γ is a homeomorphism of $\varinjlim_{i \in I} (E_i \hat{\otimes} F)$ onto its image in $E \hat{\otimes} F$. Therefore, the image of γ must be closed in $E \hat{\otimes} F$. On the other hand, it is evident that the image of γ is dense in $E \hat{\otimes} F$. $\square$

If M is a compact manifold and E a vector bundle over M, we denote by $C^\infty(E)$ the space of all smooth sections of E equipped with the topology of uniform convergence of sections and their derivatives. In this way, $C^\infty(E)$ becomes an NF-space [**14**, Ch. II, §2, no. 3, Th. 10]. Therefore, its strong dual $C^\infty(E)'$ is a DNF-space.

Appendix B. Holomorphic vector valued functions

In this section we prove some technical results on holomorphic functions with values in DNF-spaces.

Let M be a complex manifold. Denote by $\mathcal{H}$ the sheaf of holomorphic functions on M. For any open set U in M, we denote by $\mathcal{H}(U)$ the ring of holomorphic functions on U. We equip $\mathcal{H}(U)$ with the locally convex topology of uniform convergence on compact subsets of U. In this way $\mathcal{H}(U)$ becomes an NF-space [**14**, Ch. II, §2, no. 3, Cor. of Th. 10]. If V is an open subset of U, the restriction map from $\mathcal{H}(U)$ into $\mathcal{H}(V)$ is continuous. If K is a compact subset of M, we define $\mathcal{H}(K)$ as the inductive limit of $\mathcal{H}(U)$ over the family of all open sets U containing K. In this way, $\mathcal{H}(K)$ becomes a locally convex space. By [**13**, §1, no. 4, Exemple b], $\mathcal{H}(K)$ is a DNF-space.

If m is a point in M, we can put $K = \{m\}$. In this way, the space $\mathcal{H}_m$ of germs of holomorphic functions at m becomes a DNF-space.

Let E be a complete locally convex space. Analogously, we denote by $\mathcal{H}(E)$ the sheaf of all holomorphic functions on M with values in E. In particular, for an open set U in M, we denote by $\mathcal{H}(U, E)$ the space of all holomorphic functions on U with values in E. By [**14**, Ch. II, §3, no. 3, Exemples], we know that $\mathcal{H}(U, E) = \mathcal{H}(U)\hat{\otimes}E$.

For a compact set K in M, we define $\mathcal{H}(K, E) = \varinjlim_{U \supset K} \mathcal{H}(U, E)$ where the inductive limit is taken over the directed system of all open sets U containing K. In particular, for a point $m \in M$, we denote by $\mathcal{H}_m(E)$ the space of all germs at m of holomorphic maps with values in E.

Let U be an open set in M and K a compact set in U. Then we have a natural continuous linear map $\mathcal{H}(U) \to \mathcal{H}(K)$. This induces a natural continuous linear map $\mathcal{H}(U)\hat{\otimes}E \to \mathcal{H}(K)\hat{\otimes}E$.

On the other hand, if V is an open set in M contained in K, by the definition of $\mathcal{H}(K)$ we have a continuous linear map $\mathcal{H}(K) \to \mathcal{H}(V)$ given as the inductive limit of restrictions $\mathcal{H}(U) \to \mathcal{H}(V)$ over the directed system of open sets $U \supset K$. It induces a continuous linear map $\mathcal{H}(K)\hat{\otimes}E \to \mathcal{H}(V)\hat{\otimes}E$.

Let $(U_n; n \in \mathbb{N})$ be a decreasing family of open neighborhoods of a compact set K in M which is cofinal in the family of all open sets containing K, satisfying the following conditions:

(i) U_n are relatively compact;
(ii) the closure $\bar{U}_n$ is contained in U_{n-1} for $n > 1$;
(iii) the intersection of all U_n, $n \in \mathbb{N}$, is equal to K.

Also, let $(K_n; n \in \mathbb{N})$ be a family of compact sets such that $U_n \supset K_n \supset U_{n+1}$. Consider two inductive systems $(\mathcal{H}(U_n)\hat{\otimes}E; n \in \mathbb{N})$ and $(\mathcal{H}(K_n)\hat{\otimes}E; n \in \mathbb{N})$. As we remarked above, we have natural continuous linear maps $\alpha_n : \mathcal{H}(U_n)\hat{\otimes}E \to \mathcal{H}(K_n)\hat{\otimes}E$ and $\beta_n : \mathcal{H}(K_n)\hat{\otimes}E \to \mathcal{H}(U_{n+1})\hat{\otimes}E$ for $n \in \mathbb{N}$. Clearly, these maps define the maps of corresponding inductive systems, Therefore, they induce continuous linear maps $\alpha : \varinjlim_{n\to\infty}(\mathcal{H}(U_n)\hat{\otimes}E) \to \varinjlim_{n\to\infty}(\mathcal{H}(K_n)\hat{\otimes}E)$ and $\beta : \varinjlim_{n\to\infty}(\mathcal{H}(K_n)\hat{\otimes}E) \to \varinjlim_{n\to\infty}(\mathcal{H}(U_n)\hat{\otimes}E)$ such that the compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ are identities. Therefore, we have

$$\begin{aligned}\mathcal{H}(K, E) = \varinjlim_{U \supset K} \mathcal{H}(U, E) = \varinjlim_{U \supset K} (\mathcal{H}(U)\hat{\otimes}E) \\ = \varinjlim_{n\to\infty}(\mathcal{H}(U_n)\hat{\otimes}E) = \varinjlim_{n\to\infty}(\mathcal{H}(K_n)\hat{\otimes}E).\end{aligned}$$

In particular, for $E = \mathbb{C}$, we see that $\mathcal{H}(K) = \varinjlim_{n\to\infty} \mathcal{H}(K_n)$. By A.4, this implies that

$$\mathcal{H}(K,E) = \varinjlim_{n\to\infty}(\mathcal{H}(K_n)\hat{\otimes}E) = (\varinjlim_{n\to\infty}\mathcal{H}(K_n))\hat{\otimes}E = \mathcal{H}(K)\hat{\otimes}E.$$

Therefore, we proved the following result.

Lemma B.1. *If E is a DNF-space, we have $\mathcal{H}(K,E) = \mathcal{H}(K)\hat{\otimes}E$ for any compact subset K in M.*

In particular, for any $m \in M$, we have $\mathcal{H}_m(E) = \mathcal{H}_m\hat{\otimes}E$.

This, in combination with A.4, has the following consequences.

Corollary B.2. *Let*

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

be a short exact sequence of DNF-spaces. Then, for any compact set K in M,

$$0 \longrightarrow \mathcal{H}(K,E_1) \longrightarrow \mathcal{H}(K,E_2) \longrightarrow \mathcal{H}(K,E_3) \longrightarrow 0$$

is exact.

In particular, for any $m \in M$,

$$0 \longrightarrow \mathcal{H}_m(E_1) \longrightarrow \mathcal{H}_m(E_2) \longrightarrow \mathcal{H}_m(E_3) \longrightarrow 0$$

is exact.

Corollary B.3. *Let $(E_i; i \in I)$ be a countable inductive system of DNF-spaces. Assume that $E = \varinjlim_{i\in I} E_i$ is hausdorff. Then, for any compact set K in M we have $\mathcal{H}(K,E) = \varinjlim_{i\in I}\mathcal{H}(K,E_i)$.*

In particular, for any $m \in M$, we have $\mathcal{H}_m(E) = \varinjlim_{i\in I}\mathcal{H}_m(E_i)$.

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