$\mathbb{L}^2$- COHOMOLOGY FOR GROUPS OF REAL RANK ONE

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Let G be the group of real-valued points on a semi-simple algebraic group defined over $\mathbb{Q}$, which I will assume to be of rank one over both $\mathbb{Q}$ and $\mathbb{R}$. Further let

Γ = an arithmetic subgroup of G
K = a maximal compact subgroup
X = the symmetric space of G, which may be identified with G/K.
E = a finite-dimensional vector space over $\mathbb{C}$, on which G acts by a rational representation.

Let $\mathfrak{g}, \mathfrak{k}$ be the complexified Lie algebras of G, K and $\mathfrak{s}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then E possesses a Hermitian metric preserved by K and with respect to which $\mathfrak{s}$ acts by Hermitian matrices. Associated to E and this choice of metric is a locally constant system $\mathcal{E}$ over $\Gamma\backslash X$ and an inner product on it. The $\mathbb{L}^2$-cohomology $H_{(2)}(\Gamma\backslash X, \mathcal{E})$ is defined to be the cohomology of the complex of C^∞ forms ω with values in $\mathcal{E}$ such that ω and $d\omega$ are both square-integrable. It is known (from Borel-Casselman [1983]) that it is finite-dimensional precisely when G possesses a compact Cartan subgroup, <u>which I assume to be the case from now on</u>. With this assumption, the dimension of X is known to be even.

The space $\Gamma\backslash X$ possesses a rather simple compactification $\overline{\Gamma\backslash X}$ obtained by adjoining to it a finite number of points parameterized by the Γ-conjugacy classes of proper rational parabolic subgroups of G. What I am going to do in this paper is to discuss a result due to Zucker [1982] in some cases and Borel in the rest: <u>the $\mathbb{L}^2$-cohomology $H_{(2)}(\Gamma\backslash X, \mathcal{E})$ may be identified with the middle intersection homology $IH^\bullet(\overline{\Gamma\backslash X}, \mathcal{E})$</u> (as defined by Goresky-MacPherson [1980],[1981]).

The only thing new in my treatment will be that instead of proceeding case-by-case for the crux of the argument, as Borel and Zucker do, I will prove a simple geometric lemma which I hope to be more satisfactory.

When X is a Hermitian symmetric space, $\overline{\Gamma\backslash X}$ is the Baily-Borel compactification of $\overline{\Gamma\backslash X}$. When $\Gamma\backslash X$ is in addition a Shimura variety, the result above will presumably play a role in describing the Hasse-Weil ζ-function of $\overline{\Gamma\backslash X}$ (see Brylinski-Labesse [1982] for some results along these lines). I should also mention that even when G has real rank more than one, but X is Hermitian symmetric, Zucker has conjectured that the $\mathbb{L}^2$-cohomology and the middle intersection cohomology of the Baily-Borel compactification coincide. Borel has proven this for all groups of rational rank one.

1.

Let me begin with a few words about middle intersection (co) homology.

The result to be proven is perhaps best thought of as a generalization of the classical de Rham theorem. For this, beginning as naively as possible, one starts with a manifold M and triangulates it. Then one defines the homology of M (with coefficents in $\mathbb{R}$) in terms of the chain complex of this triangulation. Integration of forms over chains gives a map from the de Rham complex of C^∞ forms on M into the cochain complex, which is asserted to be an isomorphism. Thus the <u>construction</u> of the isomorphism is elementary. But the clearest and perhaps most satisfactory <u>proof</u> introduces sheaf theory. If $\Omega^\bullet$ is any complex of fine sheaves on M resolving $\mathbb{R}$, then the cohomology of M with coefficients in $\mathbb{R}$ is just the cohomology of the complex $\Gamma(M,\Omega^\bullet)$. Given this very general fact, what is left in order to prove de Rham's theorem is that the sheaf of germs of C^∞ forms is fine (partition of unity) and resolves $\mathbb{R}$ (Poincaré Lemma).

The original definition of intersection (co)homology was in terms of triangulations, but eventually Goresky and MacPherson were able to characterize it much more elegantly in terms of sheaves. For our purposes, it suffices to consider just the simplest stiuation, where only point singularities occur. Thus, let V be a $2n$-dimensional manifold with a set V_0 of isolated singularities called <u>cusps</u> (and

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assuming as a regularity condition, say, that V may be triangulated with cusps included as vertices). Let $i\colon V - V_0 \to V$ be the inclusion map, and let E be a locally constant sheaf of finite-dimensional vector spaces over $\mathbb{C}$, defined on $V - V_0$.

1.1 Theorem (Goresky-MacPherson). *Suppose $L^\bullet$ to be a non-negative complex of fine sheaves on V such that*

(1) *There exists an inclusion of $L^\bullet$ into $i_*(\Omega^\bullet(E))$ over all of V which induces an isomorphism $H^\bullet(L^\bullet) \cong E$ over $V - V_0$;*

(2) *At a cusp the cohomology of the stalk of $L^\bullet$ vanishes in dimension $\geqslant n$ and in dimensions $< n$ the inclusion of $L^\bullet$ into $i_*(\Omega^\bullet(E))$ induces an isomorphism of cohomology.*

Then the cohomology of the complex $\Gamma(V, L^\bullet)$ may be identified with $IH^\bullet(V,E)$.

In other words, one does not have to know exactly what intersection cohomology is in order to recognize when one has it to hand.

In our case, take V to be $\Gamma\backslash X$. The sheaf $L^\bullet = L^\bullet_{(2)}(E)$ is to be defined in a somewhat complicated manner. Recall (from Borel-Wallach [1980], Section 2 of Chapter VII) that lifting forms along the canonical projection $pr\colon \Gamma\backslash G \to \Gamma\backslash X$ allows one to identify C^∞ forms on any open set $U \subseteq \Gamma\backslash X$ with elements of $\mathrm{Hom}_K(\Lambda^\bullet(\mathfrak{g}\backslash\mathfrak{k}), C^\infty(pr^{-1}(U)) \otimes E)$. For any open set $S \subseteq \Gamma\backslash G$ define $\mathbb{L}^{2,\infty}(S)$ to be the space of all functions F on S such that F and all its right derivatives $R_X F$ $(X \in U(\mathfrak{g}))$ are square-integrable. Then the subsheaf $L \subseteq i_*(\Omega^\bullet(E))$ is defined by the condition that $\omega \in \Gamma(U, L^\bullet)$ if and only if the corresponding element of $\mathrm{Hom}_K(\Lambda^\bullet(\mathfrak{g}\backslash\mathfrak{k}), C^\infty(pr^{-1}(U)) \otimes E)$ actually lies locally in $\mathrm{Hom}_K(\Lambda^\bullet(\mathfrak{g}\backslash\mathfrak{k}), \mathbb{L}^{2,\infty}(pr^{-1}(U)) \otimes E)$. The complex of global sections of $L^\bullet$ may be identified with the Lie algebra cohomology $H^\bullet(\mathfrak{g}, K, \mathbb{L}^{2,\infty}(\Gamma\backslash G))$, and it is known (see Borel-Casselman [1983]) that this is the same as the $\mathbb{L}^2$-cohomology of E. For this sheaf, condition (1) of Theorem 1.1 is immediate. Hence, in order to prove the claim made in the introduction it suffices now to show that $L^\bullet$ is fine and that it satisfies condition (2) of Theorem 1.1.

In order to show $L^\bullet_{(2)}(E)$ to be fine, it suffices to find for each cusp functions f with support arbitrarily near the cusp, identically one near the cusp, which when lifted to $\Gamma\backslash G$ lie in

$\mathbb{L}^{2,\infty}(\Gamma\backslash G)$. This is done in Zucker [1982] (Proposition 4.4); it is somewhat technical, but not deep.

2.

It is condition (2) that is the crux of the matter. Suppose x to be a cusp of $\overline{\Gamma\backslash X}$, and say it corresponds to the rational parabolic P with Levi factorization $P = MN$. Let

A = The connected component of the maximal $\mathbb{Q}$-split torus in the center of M.

δ = The modulus character $|\det \mathrm{Ad}_{\mathfrak{n}}| : P \to \mathbb{R}^{pos}$ and for each $t > 0$ let

$$P^+(t) = \delta^{-1}(t,\infty)$$

$$A^+(t) = A \cap P^+(t), \text{ etc.}$$

Then as a basis of neighborhoods of x in $\overline{\Gamma\backslash X}$ one may take the images of the sets

$$X_P^+(t) = \Gamma \cap P\backslash P^+(t)/K \cap P$$

in $\Gamma\backslash X$, adjoined to $\{x\}$, for $t >> 0$.

Since $X = G/K$ is also $P/K\cap P$, a form ω on $\Gamma\backslash X$ can be lifted to one on $\Gamma\backslash P$, and one on $X_P^+(t)$ to one on $\Gamma \cap P\backslash P^+(t)$, and these to an element of $\mathrm{Hom}_{K_P}(\Lambda^\bullet(\mathfrak{p}\backslash \mathfrak{k}_P), C^\infty(\Gamma\cap P\backslash P^+(t))\otimes E)$. If it corresponds to a section of $L^\bullet$, then it induces an element of

$$\mathrm{Hom}_{K_P}(\Lambda^\bullet(\mathfrak{p}\backslash\mathfrak{k}_P), \mathbb{L}^{2,\infty}_{\text{left}}(\Gamma \cap P \backslash P^+(t')) \otimes E).$$

for $t' > t$. Here the "left" refers to a measure on $\Gamma \cap P\backslash P$ induced by a left-invariant Haar measure on P. Since a function f on $\Gamma \cap P\backslash P$ lies in $\mathbb{L}^2{}_{\text{left}}$ if and only if $f\,\delta^{-1/2}$ lies in $\mathbb{L}^2_{\text{right}}$ this corresponds in turn to an element of

$$\mathrm{Hom}_{K_P}(\Lambda^\bullet(\mathfrak{p}\backslash\mathfrak{k}_P), \mathbb{L}^{2,\infty}_{\text{right}}(\Gamma \cap P\backslash P^+(t')) \otimes E \otimes \delta^{1/2}).$$

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This is just the Koszul complex of the $(\mathfrak{p},K_P)$-module $\mathbb{L}^{2,\infty}_{\text{right}} \otimes E \otimes \delta^{1/2}$. Since in considering condition (2) of Theorem 1.1 we are looking at the stalk of $L^\bullet$, we are only interested in the direct limit of cohomology as $t \to \infty$, and it can be verified without too much trouble that what remains to be done is to prove:

2.1. Proposition. Let $2n = \dim X$, $t > 0$.

(1) The cohomology

$$H^\bullet(\mathfrak{p},K_P,\ \mathbb{L}^{2,\infty}_{\text{right}}(\Gamma \cap P\backslash P^+(t)) \otimes E \otimes \delta^{1/2}$$

vanishes in dimensions $\geq n$, and

(2) The map of cohomology from this to

$$H^\bullet(\mathfrak{p},K_P,\ C^\infty_*\ (\Gamma \cap P\backslash P^+(t)) \otimes E \otimes \delta^{1/2})$$

induced by the natural inclusion is an isomorphism in dimensions $< n$. Here C^∞_* denotes functions which are the restrictions of functions which are C^∞ on all $\Gamma \cap P\backslash P$.

Let for the moment V be either of the two $(\mathfrak{p},K_P)$-modules $\mathbb{L}^{2,\infty}_{\text{right}}(\Gamma \cap P\backslash P^+(t))$ or $C^\infty_*(\Gamma \cap P\backslash P^+(t))$. Then by Hochschild-Serre there exists a spectral sequence converging to $H^\bullet(\mathfrak{p},K_P,V \otimes E \otimes \delta^{1/2})$ with E_2-term

$$H^\bullet(\mathfrak{m},K_M,H^\bullet(\mathfrak{n},V \otimes E \otimes \delta^{1/2})).$$

2.2 Lemma. (1) The inclusion of V^N into V induces an isomorphism of $\mathfrak{n}$-cohomology $H^\bullet(\mathfrak{n},V^N \otimes E) \cong H^\bullet(\mathfrak{n},V \otimes E)$;

(2) The space $\mathbb{L}^{2,\infty}_{\text{right}}\ (\Gamma \cap P\backslash P^+(t))^N$ (resp. $C^\infty_*(\Gamma \cap P\backslash P^+(t))^N$) may be identified with $\mathbb{L}^{2,\infty}(\Gamma_M\backslash M^+(t))$ (resp. $C^\infty_*(\Gamma_M\backslash M^+(t))$.

Here Γ_M is the image in M of $\Gamma \cap P$. Since G has real rank one, this is at most a finite group, and may be taken to be trivial if we pass to at worst a subgroup of finite index in Γ. This is harmless, so I assume $\Gamma_M = \{1\}$.

Part (2) of this is straightforward. Part (1) is of the type of result first proven by Nomizu [1954] and van Est [1958]. A proof is given in Section 1 of Brylinski-Labesse [1982]. Something equivalent can be found in Zucker [1982] (Proposition 4.24) and Borel has also proven a similar result.

2.3 Corollary. There exists a spectral sequence converging to

$$H^{\bullet}(\mathfrak{p},K_p,\ \mathbb{L}^{2,\infty}_{\mathrm{right}}(\Gamma\cap P\backslash P^+(t))\otimes E\otimes\delta^{1/2})$$

with E_2-term

$$\mathrm{Ext}^{\bullet}_{(\mathfrak{m},K_M)}\ (H_{\bullet}(\mathfrak{n},\hat{E})\otimes\delta^{-1/2},\ \mathbb{L}^{2,\infty}(M^+(t)))$$

Similarly for $H^{\bullet}(\mathfrak{p},K_p,\ C^{\infty}_{*}(\Gamma\cap P\backslash P^+(t))\otimes E\otimes\delta^{1/2})$.

Here $\hat{E}$ is the contragredient of E.

Since G has real rank one, $M\cong K_M\times A$ and this E_2-term can be simplified even further to

$$\mathrm{Ext}^{\bullet}_{\mathfrak{a}}\ (H_{\bullet}(\mathfrak{n},\hat{E})\otimes\delta^{-1/2},\ \mathbb{L}^{2,\infty}(A^+(t))).$$

and similarly for C^{∞}_{*} instead of $\mathbb{L}^{2,\infty}$.

The action of A on $H(\mathfrak{n},\hat{E})$ is semi-simple, hence a sum of characters $\chi\colon A\to\mathbb{C}^*$. Therefore, Proposition 2.1 follows from the results:

2.4 Lemma. Let χ be any character: $A\to\mathbb{C}^*$. Then

(a) if $|\chi|>1$ on $A^+(1)$,

$$\mathrm{Ext}^{\bullet}_{\mathfrak{a}}\ (\chi,\mathbb{L}^{2,\infty}(A^+(t)))=0;$$

(b) if $|\chi|=1$ on $A^+(1)$ then $\mathrm{Ext}^0=0$ while Ext^1 is infinite-dimensional;

(c) if $|\chi|<1$ on $A^+(1)$ then $\mathrm{Ext}^1=0$ while $\mathrm{Ext}^0=\mathbb{C}$.

Furthermore,

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$$\mathrm{Ext}_{\mathfrak{a}}^{M}(\chi, C^\infty(A^+(t))) = \begin{cases} \mathbb{C}, & m=0 \\ 0, & m=1. \end{cases}$$

for all χ.

2.5 Proposition. *Let* χ *be an eigencharacter of* $H_m(\mathfrak{n},\hat{E}) \otimes \delta^{-1/2}$. *Then* χ *is never unitary, and* $|\chi| < 1$ *on* $A^+(1)$ *precisely when* $m < n$.

The first result will be the concern of the next section.

As for the second, note that $X \cong AN$, since $G = PK$ and $P = ANK_M$. Therefore, the dimension of $\mathfrak{n}$ is $2n - 1$, and the condition $m < n$ is equivalent to $m < 1/2(\dim \mathfrak{n})$.

It is illuminating to follow Zucker and look first at the most elementary case, $G = SO(2n,1)$ and $E = \mathbb{C}$. Then $H_\bullet(\mathfrak{n},\mathbb{C}) \cong \Lambda^\bullet\mathfrak{n} \cong \Lambda^\bullet(\mathbb{C}^{2n-1})$, $A \cong \mathbb{R}^{pos}$ acts as a^m on H_m, $\delta = a^{2n-1}$, and the Proposition is clear. Note that if n were odd then A would act on the middle dimensional homology exactly as $\delta^{1/2}$, and thus by 2.4 the local $L^\bullet$-cohomology would be infinite-dimensional.

Continuing case-by-case is not too difficult since there are only a few cases to be dealt with, but I prefer something else. The following is somewhat stronger than is needed here, but provides perhaps some insight into the mechanism of Zucker's conjecture.

2.6 Proposition. *Let* G *be an arbitrary semi-simple Lie group, and*

P = *maximal proper parabolic subgroup of* G *with real Levi decomposition* $P = MN$
δ_P = *modulus character of* P
$M(1) = M \cap \ker(\delta_P)$
E = *a finite-dimensional complex representation of* G *isomorphic to its own complex contragredient.*

Thus $M \cong M(1) \times A$ *with* $A \cong \mathbb{R}^{pos}$. *Let* σ *be an irreducible* M-*constituent of* $H_m(\mathfrak{n},E)$ *whose restriction to* $M(1)$ *is isomorphic to its own complex contragredient, and let* ε *be the character by which*

A act on σ. Then the restriction of $|\varepsilon|$ to $A^+(1)$ is

$$>, =, \text{ or } < \delta_P^{1/2}$$

according to whether m is

$$>, =, \text{ or } < 1/2(\dim \mathfrak{n}).$$

In the case at hand the condition on σ is automatic since $M(1) = K_M$ is compact, and that on E is automatic since we have assumed G to possess a compact Cartan. But the more general result stated here is tailored as closely as possible to prove many other cases of Zucker's conjecture when G has rational, but not necessarily real, rank one.

The proof is almost a matter of geometry, given Kostant's well-known result on the structure of $H_\bullet(\mathfrak{n},E)$.

Choose a Borel subgroup B of $G(\mathbb{C})$ contained in $P(\mathbb{C})$ and a maximal torus $T \subseteq M(\mathbb{C}) \cap B$. Let $T(1)$ be the Zariski closure of $M(1) \cap T$.

It is easy to see from the hypothesis on E that since σ is a constituent of $H_m(\mathfrak{n},E)$, $\overline{\sigma}^\wedge \otimes \delta_P$ is one of $H_{n-m}(\mathfrak{n},E)$, where now (for the moment) $n = \dim \mathfrak{n}$ and $\overline{\sigma}^\wedge$ is the complex contragredient of σ. Let χ be the lowest weight of σ (for (B,T)), τ that of $\overline{\sigma}^\wedge \otimes \delta_P$. Then again by hypothesis, χ and τ have the same restriction to $T(1)$. Or: $\chi = \tau\delta_P^s$ for some $s \in \mathbb{R}$. (Here we have $\mathbb{R}$ and not $\mathbb{C}$ because $T/T(1)$ is a $\mathbb{Q}$-split torus).

Let $X(T)$ be the lattice of holomorphic characters of T. In the vector space $X(T) \otimes \mathbb{R}$ let C be the interior of the cone of weights anti-dominant with respect to B. Thus $\delta_B^{-1/2}$ lies in B. For χ to be a lowest weight with respect to $M(1) \cap B$ means that $\chi\delta_P^{-t}$ lies in $\overline{C}$ for $t >> 0$.

There are three cases now to be distinguished: $s > 0$, $s = 0$, and $s < 0$. Since $\delta_P > 1$ on $A^+(1)$ these correspond respectively to $|\chi| > |\tau|$, $|\chi| = |\tau|$, or $|\chi| < |\tau|$ on $A^+(1)$. Since τ is a weight of $\overline{\sigma}^\wedge \otimes \delta_P$, the restriction of τ to A is $\varepsilon^{-1}\delta_P$, where ε is that of χ to A. So these conditions are in turn equivalent to $|\varepsilon|\, \delta_P^{-1/2} > 1$, $= 1$, or < 1 on $A^+(1)$.

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Say for example that $s < 0$. Then $\chi \neq \tau$ at least, and the line from τ through χ continues on to intersect $\bar{C}$. This line shifted by $\delta_B^{-1/2}$, that is to say the line from $\tau\delta_B^{-1/2}$ through $\chi\delta_B^{-1/2}$, will meet not only $\bar{C}$ but even C.

What we now need from Kostant's results on the structure of $H_\bullet(\mathfrak{n},E)$ are these: (1) the characters $\chi\delta_B^{-1/2}$ and $\tau\delta_B^{-1/2}$ are regular; (2) if $\chi \neq \tau$ they lie in different Weyl chambers; (3) the number m, the dimension of the homology in which χ occurs, is the same as the number of root hyperplanes separating χ from C. Similarly for τ. In other words, if χ lies in wC with w in the complex Weyl group, then $m = \ell(w)$.

Because χ lies on a line between τ and C, any hyperplane which separates χ and C must also separate τ and C, so by (3) we have $(n-m) \geqslant m$. But by (2) there must exist at least one hyperplane separating τ from C but not χ from C. So in fact $n-m > m$ or $m < \frac{1}{2}(\dim \mathfrak{n})$.

The cases $\delta = 0$, $\delta < 0$ give no new trouble.

Q.E.D.

Given Proposition 2.6, all that remains in order to prove 2.5 is the claim that unitary characters never occur. This is not at all trivial—it is one of the main results of Borel-Casselman [1983].

3.

This section will be spent on Lemma 2.4. I will give two proofs. The first will follow Zucker (Proposition 2.39 of Zucker [1982]) and is entirely elementary. The second is much more complicated, but illustrates what I believe to be an important technique.

First proof. Since Ext^0 is Hom and χ is square-integrable on $A^+(t)$ if and only if $|\chi| < 1$ on $A^+(1)$, all the assertions about Ext^0 are trivial.

As for Ext^1, I use the explicit isomorphism

$$\begin{aligned}\mathrm{Ext}^1_{\mathfrak{a}}(\chi,V) &\cong H^1(\mathfrak{a},\chi^{-1}\otimes V)\\ &\cong V/(\alpha - d\chi(\alpha))V\end{aligned}$$

where α is a generator of $\mathfrak{a}$. If χ is a character of A, then identifying A with $\mathbb{R}^{pos}$ we have

$$\chi(x) = x^s$$

for some $s \in \mathbb{C}$. The three cases of Lemma 2.5 correspond to the trichotomy $Re(s) > 0$, $Re(s) = 0$, $Re(s) < 0$. The differential $d\chi$ takes $\alpha = xd/dx$ to s. Thus describing Ext^1 is related to solving the differential equation

$$x\frac{d\phi}{dx} - s\phi = f$$

given $f \in \mathbb{L}^{2,\infty}$.

First of all say $Re(s) > 0$. Then for any $f \in \mathbb{L}^{2,\infty}$ we can choose

$$\phi(x) = -x^s \int_x^\infty u^{-s} f(u)(du/u) \quad (x > t)$$

(as we tell our students in the first week of D.E.'s). The integral makes sense since du/u is multiplicatively invariant and both x^{-s} and $f(x)$ are square-integrable. But why does ϕ lie in $\mathbb{L}^{2,\infty}$? For ϕ itself, note that the above integral is the convolution of the $\mathbb{L}^2$ function f with the $\mathbb{L}^1$ function β:

$$\beta(x) = \begin{cases} x^s & x \leqslant 1 \\ 0 & x > 1 \end{cases}$$

and hence is itself $\mathbb{L}^2$. I leave it as an exercise to check the property for all the derivatives $(xd/dx)^n\phi$.

For 6.1(c), given $f \in \mathbb{L}^{2,\infty}(A^+(t))$ define

$$\phi(x) = x^s \int_t^x u^{-s} f(u)(du/u).$$

This is the convolution of f with a new β:

$$\beta(x) = \begin{cases} x^s & x \geqslant 1 \\ 0 & x < 1 \end{cases}$$

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$x \geq 1$

$x < 1$

so the same reasoning applies.

As for 6.1(b), we do not really need it, and I leave it also as as an exercise. (Hint: one may as well assume $s = 0$.)

To conclude the proof of 2.4 all that remains are the assertions about $C_*^\infty(A^+(t))$. But these are clear, after applying the second integral above.

This concludes my first proof of Lemma 2.4.

The second proof will illustrate a principle I hope will play a role in many different situations involving cohomology and automorphic forms. The point is that from the considerations above it is more or less clear that calculating Ext-groups is closely related to solving differential equations, and as is well known one technique in doing this is to apply the Fourier transform.

What I am interested in is a characterization of the functions in $\mathbb{L}^{2,\infty}(A^+(t))$ in terms of their multiplicative Fourier transforms. It will do no harm to set $t = 1$, and it will simplify notation to work with the additive group of $\mathbb{R}$ rather than the multiplicative group $\mathbb{R}^{pos}$. This means only that I must choose a new coordiante by exponentiation. In other words, I will look at $\mathbb{L}^2(0,\infty)$ with respect to the measure dx.

The starting point is a classical result of Paley-Wiener (I refer to Chapter 3, Section 3.4 of Dym-McKean [1972] for a simplified proof). Given $f \in \mathbb{L}^2(0,\infty)$ its Fourier transform

$$\hat{f}(s) = \int_0^\infty e^{-sx}f(x)dx$$

is defined and holomorphic in the region $\mathrm{Re}(s) > 0$. The boundary value of $\hat{f}(s)$ along $i\mathbb{R}$ is the usual $\mathbb{L}^2$ Fourier transfrom of f. The content of the theorem of Paley-Wiener is that the Fourier transforms $F(s)$ of all of $\mathbb{L}^2(0,\infty)$ are characterized by these two properties:

(a) $F(s)$ is analytic for $\mathrm{Re}(s) > 0$;

(b) There exists a common bound for all the integrals

$$\int_{\sigma - i\infty}^{\sigma + i\infty} |F(s)|^2 d|s| \qquad (\sigma > 0).$$

This upper bound may be taken as the $\mathbb{L}^2$-norm of the ordinary transform of f. Such functions are called Hardy functions.

The subspace $\mathbb{L}^{2,\infty}(0,\infty)$ may be topologized by the $\mathbb{L}^2$-norms of its derivatives. Call $H^{2,\infty}$ its topological dual: this, rather than $\mathbb{L}^{2,\infty}$, is the space I really want to look at. Since truncation in $(-\infty,0)$ gives a continuous $\mathfrak{a}$-morphism from the Schwartz space $S(\mathbb{R})$ into $\mathbb{L}^{2,\infty}(0,\infty)$, the space $H^{2,\infty}$ may be considered as a subspace of that of tempered distribution with support on $[0,\infty)$. Hence any $\Phi \in H^{2,\infty}$ also possesses as Fourier transform a function $F(s)$ analytic for $\mathrm{Re}(s) > 0$ (and whose boundary value on $i\mathbb{R}$ is a tempered distribution, the usual transform of Φ).

There are two kinds of derivatives of a function in $\mathbb{L}^{2,\infty}$. On the one hand if $f \in \mathbb{L}^{2,\infty}$ then by definition so do all the $(d/dx)^n f$, again considered as functions on $(0,\infty)$. But f may also be considered as a distribution, and its derivative as such is not the same. It is easy to see that any function in $\mathbb{L}^{2,\infty}$ is in the neighborhood of 0 the restriction of a C^∞ function on all of $\mathbb{R}$, and as a distribution the derivative of f is $f' + f(0)\,\delta_0$, its second derivative $f'' + f(0)\,\delta_0' + f'(0)\,\delta_0$, etc. In other words the $\mathfrak{a}$-module $\mathbb{L}^{2,\infty}(0,\infty)$ ($\mathfrak{a}$ acting by the first differentiation) is not self-dual.

For this reason, the Fourier transforms of functions in $\mathbb{L}^{2,\infty}$ are difficult to characterize: the Laplace transform of f' is $s\hat{f}(s) - f(0)$. But those of $H^{2,\infty}$ are no problem.

3.1 Proposition. A function $F(s)$ in $\mathrm{Re}(s) > 0$ is the Fourier transform of some $\Phi \in H^{2,\infty}$ if an only if $(1 + s)^{-n}F(s)$ is a Hardy function for some $n \geqslant 0$.

Proof. Let $AC^1(0,\infty)$ be the space of all $f \in \mathbb{L}^2(0,\infty)$ which are absolutely continuous and such that f' also lies in $\mathbb{L}^2(0,\infty)$. Let D be the operator $f \longrightarrow f'$ with domain $AC^1(0,\infty)$. Define $AC^n (n > 1)$ inductively: $f \in AC^n$ if and only if $f \in AC^1$ and Df lies in AC^{n-1}. Thus $\mathbb{L}^{2,\infty} = \cap AC^n$ $(n > 0)$.

3.2 Lemma. For any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, $(D-\lambda)$ is a bijection of $AC^1(0,\infty)$ and $\mathbb{L}^2(0,\infty)$.

In other words, the spectrum of D has no points in the right half-plane. The proof is trivial: we clearly have no kernel, and

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$$(D-\lambda)^{-1}f(x) = -\int_x^\infty e^{-\lambda(t-x)}f(t)\,dt.$$

Composition with the truncation map gives a diagram

$$S(\mathbb{R}) \longrightarrow AC \xrightarrow{n(D-\lambda)^n} \mathbb{L}^2$$

and dual to this is

$$\mathbb{L}^2 \xrightarrow{(-d/dx-\lambda)^n} \text{dual of } AC^n \to \text{tempered distributions}$$

Applying the Lemma n times, one sees that the dual of AC^n consists of all tempered distributions which can be written as $(d/dx + \lambda)^n$ applied to some $f \in \mathbb{L}^2$. Since the dual of $\mathbb{L}^{2,\infty}$ is just the union of the duals of the AC^n, and since under Fourier transformation d/dx corresponds to multiplication by s, this proves the Proposition.

Precisely because d/dx corresponds to multiplication by s under Fourier transformation, this Proposition presents a fairly clear picture of $H^{2,\infty}$ as an $\mathfrak{a}$-module. In particular:

3.3 Proposition. *The operator* $d/dx - \lambda$, *for* $\operatorname{Re}(\lambda) \neq 0$, *is an injection of* $H^{2,\infty}$ *into itself with closed image.*

3.4 Corollary. *For* $\operatorname{Re}(\lambda) \neq 0$ *the operator* $d/dx - \lambda$ *is a surjection from* $\mathbb{L}^{2,\infty}$ *onto itself. In other words,* $\operatorname{Ext}^1_{\mathfrak{a}}(\chi, \mathbb{L}^{2,\infty}(0,\infty)) = 0$ *for* $|\chi| \neq 1$.

The reasoning used here is hardly a simple proof of Lemma 2.4. But Proposition 3.3 strikes me as interesting in its own right. Similar results in other contexts will always have cohomological significance. I summarize it by saying that $H^{2,\infty}$ is a Paley-Wiener module with respect to $\mathfrak{a}$ over the spectrum $\mathbb{C} - i\mathbb{R}$. (For another cohomological application of a type of Paley-Wiener theorem, see Casselman [1983]).

4. SOME FINAL REMARKS.

The method used by Borel to prove Zucker's conjecture for all rational rank one groups are along the same lines used above, except in one very interesting case. Suppose D is a central devision algebra, say of degree d, over a quadratic imaginary extension of $\mathbb{Q}$. Let τ be the involution of D which amounts to conjugation of matrix coefficients on $D_{\mathbb{R}} \cong M_d(\mathbb{C})$. Let $\mathbb{Q}$ be the Hermitian form $xx^{\tau} - yy^{\tau}$ on D^2, and let $G = SU(Q)$. The corresponding real group is $SU(n,n)$, but the rational rank of G is one. The Baily-Borel compactification is obtained by adding cusps, and the corresponding parabolic subgroups P have $N \cong$ {Hermitian dxd matrices}, $M \cong GL_d(\mathbb{C})$. In this case, Borel tells me, one needs the very precise vanishing theorems of Enright [1980] in order to prove Zucker's conjecture. This strongly suggests an important role for vanishing theorems when the rational rank is more than one.

Another open question is a more direct construction of the isomorphism of $\mathbb{L}^2$ - cohomology and $IH^{\bullet}$. For example, Cheeger [1980] was able to integrate $\mathbb{L}^2$-cocycles over almost all the chains satisfying the suitable perversity condition. Can something similar be done here? Along the same lines, one knows that certain $\mathbb{L}^2$-cohomology arises from residues of Eisenstein's series. Can one relate these residues to specific cycles on $\Gamma\backslash X$?

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REFERENCES

[1] A. Borel and W. Casselman, L^2-cohomology of locally symmetric manifolds of finite volume, preprint, 1983.

[2] A. Borel and N. Wallach, Continuous cohomology, discrete subgroups, and representations of reductive groups, Ann. of Math. Studies 94, Princeton University Press, Princeton, 1980.

[3] J.L. Brylinski and J.P. Labesse, Cohomologie d'intersection et fonctions L de certaines variétés de Shimura, preprint, 1982.

[4] W. Casselman, Automorphic forms and a Hodge theory for congruence subgroups of $SL_2(\mathbb{Z})$, preprint, 1983.

[5] J. Cheeger, Geometry of the Laplace operator, in Proc. Symp. Pure Math. XXXVI, A.M.S., Providence, 1980.

[6] J. Cheeger, M. Goresky, R. MacPherson, L^2-cohomology and intersection homology of singular algebraic varieties, in Seminar on Differential Geometry, Ann. of Math. Studies, Princeton University Press, Princeton, 1982.

[7] H. Dym and H.P. McKean, Fourier series and integrals, Academic Press, New York, 1972.

[8] T. Enright, Relative Lie algebra cohomology and unitary representations of complex Lie groups, Duke Math. J. 46(1979), 513-525.

[9] W.T. van Est, A generalization of the Cartan-Leray spectral sequence II, Proc. Koninkl. Ned. Ak. v. Wet. Amsterdam, series A, 61(1958), pp. 406-413.

[10] M. Goresky and R. MacPherson, Intersection homology theory, Topology 19 (1980), pp. 135-162.

[11] M. Goresky and R. MacPherson, Intersection Homology II, Invent. Math. 72(1983), 77-129.

[12] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. of Math. 59(1954), pp. 531-538.

[13] M. Reed and B. Simon, Methods of modern mathematical physics I., Academic Press, New York, 1972.

[14] S. Zucker, L^2-cohomology of warped products and arithmetic groups, Invent. Math. 70(1982), 169-218.