Jacquet modules and the asymptotic behaviour of matrix coefficients

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There is an intimate relationship between the asymptotic behaviour at infinity of matrix coefficients of admissible representations of both real and $\mathfrak{p}$-adic reductive groups and the way in which these representations embed into representations induced from parabolic subgroups. Weak versions of this were known for a long time for real groups but, until work of Jacquet on $\mathfrak{p}$-adic groups around 1970, one didn't really understand very well what was going on. Starting with Jacquet's observations, something now called the **Jacquet module** was constructed, first and most easily for $\mathfrak{p}$-adic groups, and then, with somewhat more difficulty, for real groups (see [Casselman:1974] and [Casselman:1979]). More or less by definition, the Jacquet module of a representation controls its embeddings into induced representations, and following another hint by Jacquet it was established without a lot of difficulty that algebraic properties of the Jacquet module also controlled the asymptotic behaviour of matrix coefficients. What characterizes this best is something called the **canonical pairing** between Jacquet modules associated to a representation and its contragredient. It is not difficult to define the canonical pairing abstractly and to relate it to matrix coefficients, but it is not so easy to determine it in cases where one knows the Jacquet modules explicitly. The formula of [Macdonald:1971] for spherical functions is a particular example that has been known for a long time, but I'm not aware that this has been generalized in the literature in the way that I'll do it. In this paper I'll sketch very roughly how things ought to go.

For $\mathfrak{p}$-adic groups, there exists also a relationship between the asymptotic behaviour of Whittaker functions and the Whittaker analogue of the Jacquet module. The best known example here is the formula found in [Casselman-Shalika:1980] for unramified principal series. I think it likely that a similar relationship exists for real groups, and that it will explain to some extent the recent work of [Hirano and Oda:2007] on Whittaker functions for $\mathrm{SL}_3(\mathbb{C})$. I'll make a few comments on this at the end of the paper.

The results discussed in this paper were originally commissioned, in a sense, by Jim Arthur many years ago. He subsequently used them, at least the ones concerned with real groups, in [Arthur:1983], to prove the Paley-Wiener theorem. His argument depended on Harish-Chandra's Plancherel formula for real reductive groups but in fact, with a little thought and a few observations about Plancherel measures, one can deduce that formula at the same time as following Jim's proof.

There is one intriguing question raised by the results I sketch here. One trend in representation theory over the past few years has been to replace analysis by algebraic geometry. This is particularly striking in the theory of unramified representations of $\mathfrak{p}$-adic groups, where sheaves replace functions, which are related to them by Grothendieck's dictionary. I have in mind the version of Macdonald's formula as a consequence of the 'geometric' Satake isomorphism of [Mirkovic & Vilonen:2000], for example. (There is an efficient survey of results about unramified representations in [Haines-Kottwitz-Prasad:2005].) What do these ideas have to say in the presence of ramification? Or about representations of real groups (which, according to [Manin:1991], ought to be considered infinitely ramified)?

As I have said, I shall give few details here. My principal purpose, rather, is to exhibit plainly the astonishing parallels between the real and $\mathfrak{p}$-adic cases.

Throughout this paper, G will be a reductive group defined over a local field. In addition:

$$\begin{aligned}
P = {} & \text{a parabolic subgroup} \\
N = N_P = {} & \text{its unipotent radical} \\
M = M_P = {} & \text{a subgroup of } P \text{ isomorphic to } P/N \\
A = A_P = {} & \text{maximal split torus in } M \\
\Sigma_P = {} & \text{eigencharacters of } \mathrm{Ad}_{\mathfrak{n}} \,\big|\, A \\
A^{--} = {} & \big\{ a \in A \,\big|\, |\alpha(a)| \le 1 \text{ for all } \alpha \in \Sigma_P \big\} \\
\overline{P} = {} & \text{opposite of } P \quad (\text{i. e. } P \cap \overline{P} = M) \\
\delta_P(p) = {} & |\det \mathrm{Ad}_{\mathfrak{n}}(p)| \\
W = {} & \text{the Weyl group with respect to } A\,.
\end{aligned}$$

Thus δ_P is the **modulus character** of P.

Part I. What happens for $\mathfrak{p}$-adic groups

1. Notation. Suppose $\mathfrak{k}$ to be a $\mathfrak{p}$-adic field, G the $\mathfrak{k}$-rational points on an unramified reductive group defined over $\mathfrak{k}$. In addition to basic notation:

$$A^{--}(\varepsilon) = \{a \in A \,\big|\, |\alpha(a)| < \varepsilon \text{ for all } \alpha \in \Sigma_P\}$$
$$K = K_{\mathfrak{o}} = \text{ what [Bruhat \& Tits:1966] call a `good' maximal compact } .$$

Thus $G = PK_{\mathfrak{o}}$ and if P is minimal we have the Cartan decomposition $G = K_{\mathfrak{o}}A^{--}K_{\mathfrak{o}}$.

I write $a \to_P 0$ for a in A_P if $|\alpha(a)| \to 0$ for all α in Σ_P. Because of the Cartan decomposition, this is one way points on G travel off to infinity. I'll say that a is near 0 if all of those same $|\alpha(a)|$ are small.

2. Admissible representations. In these notes a **smooth** representation (π, V) of G will be a representation of G on a complex vector space V with the property that the subgroup of G fixing any v in V is open. It is **admissible** if in addition the dimension of the subspace fixed by any open subgroup is finite.

The simplest examples are the principal series. If (σ, U) is an admissible representation of M, hence of P, the normalized induced representation is the right regular representation of G on

$$\mathrm{Ind}(\sigma \,|\, P, G) = \{f \in C^\infty(G, U) \,|\, f(pg) = \delta_P^{1/2}(p)\sigma(p)f(g)\} .$$

If (π, V) is an admissible representation of G, its **Jacquet module** V_N is the quotient of V by the linear span $V(N)$ of the vectors $v - \pi(n)v$, the universal N-trivial quotient of V. It is in a natural way a smooth representation of M, which turns out in fact to be admissible (see [Casselman:1974]). The **normalized** Jacquet module π_N is this twisted by $\delta_P^{-1/2}$.

The point of the normalization of the Jacquet module is that Frobenius reciprocity becomes

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma \,|\, P, G)\big) = \mathrm{Hom}_M(\pi_N, \sigma) .$$

3. Matrix coefficients. The contragredient $(\widetilde{\pi}, \widetilde{V})$ of an admissible representation (π, V) is the subspace of smooth vectors in its linear dual. The **matrix coefficient** associated to $\widetilde{v}$ in $\widetilde{V}$, v in V is the function

$$\Phi_{\widetilde{v},v}(g) = \langle \widetilde{v}, \pi(g)v \rangle .$$

The asymptotic behaviour of matrix coefficients at infinity on the $\mathfrak{p}$-adic group G is fairly simple, at least qualitatively. Jacquet first observed that if v lies in $V(N)$ then $\langle \pi(a)v, \widetilde{v}\rangle = 0$ for a near 0—i.e. if $|\alpha(a)|$ is small for all α in Σ_P. This implied, for example, that the matrix coefficients of a cuspidal representation had compact support modulo the centre of G. A refinement of Jacquet's observation is this:

> *There exists a unique pairing $\langle \widetilde{u}, u \rangle_{\mathrm{can}}$ of $\widetilde{V}_{\overline{N}}$ and V_N with this property: for each $\widetilde{v}$, v with images $\widetilde{u}$ in $\widetilde{V}_{\overline{N}}$, u in V_N there exists $\varepsilon > 0$ such that*
>
> $$\langle \widetilde{v}, \pi(a)v \rangle = \delta_P^{1/2}(a)\langle \widetilde{u}, \pi_N(a)u \rangle_{\mathrm{can}}$$
>
> *for a in $A^{--}(\varepsilon)$.*

This canonical pairing induces an isomorphism of $\widetilde{V}_{\overline{N}}$ with the admissible dual of V_N. It has a geometric interpretation. For example, if $G = \mathrm{SL}_2(\mathbb{Q}_p)$ and v is fixed on the right by $\mathrm{SL}_2(\mathbb{Z}_p)$, then the matrix coefficient becomes a function on certain vertices of the Bruhat-Tits tree of G, and their asymptotic behaviour is related to the embedding of π via boundary behaviour into a representation induced from P, a kind of complex line bundle over the copy of $\mathbb{P}^1(\mathbb{Q}_p)$ that compactifies the tree.

4. Principal series. One can describe rather explicitly the Jacquet modules of representations induced from parabolic subgroups. Can one then describe the canonical pairing explicitly?

I'll explain this problem by an example. Suppose P to be minimal and π to be the principal series $\mathrm{Ind}(\chi \,|\, P, G)$ with χ a generic character of M. Its contragredient may be identified with

$$\mathrm{Ind}(\chi^{-1} \,|\, P, G)$$

where the pairing is

$$\langle \varphi, f \rangle = \int_{P\backslash G} \varphi(x) f(x)\, dx = \int_K \varphi(k) f(k)\, dk\,.$$

Can we find an explicit formula for $\langle \varphi, R_a f\rangle$ as $a \to 0$?

Let me recall what we know about the Jacquet module in this situation. The Bruhat decomposition tells us that $G = \bigsqcup PwN$, with $PxN \subseteq \overline{PyN}$ if $x \le y$ in W. As explained in [Casselman:1974], the space $\mathrm{Ind}(\chi)$ is filtered by subspaces I_w of f with support on the closure of PwN. Similarly $G = \bigsqcup Pw\overline{N}$, and $\mathrm{Ind}(\chi^{-1})$ is filtered by the opposite order on W. These two double coset decompositions are transversal to one another. For each w in the Weyl group W we have a map

$$\Omega_w(f) = \int_{N\cap wNw^{-1}\backslash N} f(w^{-1}n)\, dn$$

well defined on $I_{w^{-1}}$. It extends generically (that is to say for generic χ) to all of $Ind(\chi)$, and determines an M-covariant map from π_N to $\mathbb{C}_{w\chi}$. All together, as long as χ is generic, these induce an isomorphism of the (normalized) Jacquet module π_N with $\oplus\, w\chi$. Similar functionals

$$\widetilde{\Omega}_w(f) = \int_{\overline{N}\cap wNw^{-1}\backslash \overline{N}} f(w^{-1}\overline{n})\, d\overline{n}$$

determine an isomorphism of $\widetilde{\pi}_{\overline{N}}$ with $\oplus\, w\chi^{-1}$.

The agreement of these formulas with the canonical pairing is clear—the two Jacquet modules are dual, piece by piece, but the duality is only determined up to scalar multiplication. We have therefore an asymptotic equality of the form

$$\langle \varphi, R_a f\rangle = \sum_w c_{w,\chi}\, \delta_P^{1/2}(a)\, w\chi(a) \cdot \widetilde{\Omega}_w(\varphi)\, \Omega_w(f)$$

for a near 0, with suitable constants $c_{w,\chi}$.

The problem we now pose is this: *What are those constants?*

There is a classical formula found in [Langlands:1988] that gives the leading term of the asymptotic behaviour for χ in a positive chamber. It involves an analytic estimate of an integral. (I'll present a simple case of Langlands' calculation later on, that of $\mathrm{SL}_2(\mathbb{R})$.) The result stated here is a stronger and more precise version of that result. What makes the new version possible is the apparently abstract result relating asymptotic behaviour to Jacquet modules. One point is that we don't have to find the asymptotic behaviour of all matrix coefficients, just enough to cover all the different components of the Jacquet modules. Another is that we just have to look at one component at a time.

5. Integration. The question about the constants $c_{w,\chi}$ is not quite precise, because we have to be more careful about what the integrals mean. The first point is that it is not functions on $P\backslash G$ that one integrates, but densities. These may be identified with functions in $\mathrm{Ind}(\delta^{1/2})$, but *the identification depend on a choice of measures, and is definitely not canonical*. There are two integral formulas commonly used to make the identification of densities with functions in $\mathrm{Ind}(\delta^{1/2})$, and I shall introduce a third.

The first formula depends on the factorization $G = PK$. The integral on densities must be K-invariant, so we must have

$$\int_{P\backslash G} f(x)\,dx = \textit{constant} \cdot \int_K f(k)\,dk\,,$$

where we take the total measure of K to be 1. Indeed, we can just define the integral by this formula, with this choice.

Since $P\overline{N}$ is open in G and the integral must be $\overline{N}$-invariant, we may also set

$$\int_{P\backslash G} f(x)\,dx = \textit{constant} \cdot \int_{\overline{N}} f(n)\,dn\,.$$

where now we can choose measure on $\overline{N}$ so that $K\cap\overline{N}$ has measure equal to 1. Understanding this second formula requires some work to show that the integral always converges.

The two formulas can only differ by a scalar. So we have

$$\int_K f(k)\,dk = \mu \int_{\overline{N}} f(\overline{n})\,d\overline{n}$$

for some constant μ, easy enough to determine explicitly in all cases:

> *Let B be an Iwahori subgroup of K and w_ℓ in K represent the the longest element of the Weyl group. Then $\mu = \mathrm{meas}(Bw_\ell B)/\mathrm{meas}(K)$.*

This is because $Bw_\ell B$ is completely contained in the single Bruhat double coset $Pw_\ell P$.

Integration over $\overline{N}$ (or, in a mild variation, over $w_\ell N$) is in many ways the more natural choice. It is, for example, the one that arises in dealing with Tamagawa measures (implicit in [Langlands:1966]). But it has a serious problem, and that is the question of convergence. Convergence shouldn't really arise here. The theory of admissible representations of $\mathfrak{p}$-adic groups is essentially algebraic, and one should be able to work with an arbitrary coefficient field, for which analysis is not in the toolbox. We would therefore like to modify the formula

$$\int_{P\backslash G} f(x)\,dx = \int_{\overline{N}} f(\overline{n})\,d\overline{n}\,.$$

so as to make all integrals into sums, and avoid all convergence considerations.

This is easy, and a very similar idea will reduce the analytical difficulties for real groups to elementary calculus. Choosing representatives of W in K, we get also measures and similar formulas on the translates $P\overline{N}w^{-1}$. The variety $P\backslash G$ is covered by these open translates, and we can express

$$f = \sum_w f_w \quad (\text{support of } f_w \text{ on } P\overline{N}w^{-1})$$

as a sum of functions f_w, each with compact support on one of them. Then

$$\int_{P\backslash G} f(x)\,dx = \sum_w \int_{\overline{N}} f_w(xw^{-1})\,dx = \sum_w \int_{w\overline{N}w^{-1}} f_w(w^{-1}x)\,dx\,.$$

All these integrals are now finite sums. This in turn gives explicit measures to choose for evaluating Ω_w and $\widetilde{\Omega}_w$ because if $N_w = w\overline{N}w^{-1}$ then

$$N_w = (N_w\cap N)(N_w\cap\overline{N})\,.$$

One can also choose measures on each one-dimensional unipotent root group compatibly with the action of w in K.

6. A sample calculation. To evaluate the canonical pairing, we may deal with each summand of the Jacquet module by itself. We can therefore choose both f and φ with support on $P\overline{N}w^{-1}$, in which case it is easy to see what the asymptotic behaviour of $\langle\varphi, R_a f\rangle$ is.

For f, φ with support on $P\overline{N}$ and a near 0 we have

$$\begin{aligned}\int_{\overline{N}} f(\overline{n}a)\varphi(\overline{n})\,d\overline{n} &= \int_{\overline{N}} f(a\cdot a^{-1}\overline{n}a)\varphi(\overline{n})\,d\overline{n}\\ &= \delta_P^{1/2}(a)\chi(a)\int_{\overline{N}} f(a^{-1}\overline{n}a)\varphi(\overline{n})\,d\overline{n}\\ &= \delta_P^{1/2}(a)\chi(a)f(1)\int_{\overline{N}}\varphi(\overline{n})\,d\overline{n}(\text{if } a \text{ is near } 0)\\ &= \delta_P^{1/2}(a)\chi(a)\cdot\Omega_1(f)\cdot\widetilde{\Omega}_1(\varphi)\end{aligned}$$

Similar calculations work for all principal series, and as this suggests the canonical pairing turns out to be that with $c_{\chi,w} = \mu$ for all w.

The most general result of this sort is that if

$$\pi = \mathrm{Ind}(\sigma \,|\, Q, G)$$

then the canonical pairing for the Jacquet module π_{N_P} can be expressed explicitly in terms of the canonical pairing for the Jacquet modules of σ and certain N_P-invariant functionals on π determined by integration over pieces of the Bruhat filtration, together with analogues for $\overline{P}$ for $\widetilde{\pi}$. This formula, an explicit formula for the canonical pairing, is too elaborate to present here.

7. Range of equality. *For what range of a does the 'asymptotic' equation hold?* The answer depends on the ramification of χ as well as on the particular f and φ. The most important result is that if χ is unramified and both f and φ are fixed by an Iwahori subgroup, then the equation is good on all of A^{--}. Macdonald's formula for the unramified spherical function is neither more nor less than the main formula together with this observation about the Iwahori-fixed case.

8. Whittaker functions. Let $P = MN$ be a Borel subgroup of a quasi-split group G, and let ψ be a non-degenerate character of the maximal unipotent subgroup N. One may define an analogue $V_{\psi,N}$ of the Jacquet module to be the quotient of V by the span of all $\big(\pi(n) - \psi(n)\big)v$.

The Whittaker functional on $V = \mathrm{Ind}(\sigma \,|\, P, G)$ is effectively in the dual of this, is defined formally by

$$\langle W_\psi, f\rangle = \int_N f(w_\ell n)\psi^{-1}(n)\,dn\,.$$

and the Whittaker function as $W_\psi(g) = \langle W_\psi, R_g f\rangle$. Finding the asymptotic behaviour of Whittaker functions at infinity, or equivalently finding $W_\psi(a)$ for $a \to 0$, is similar to that of finding asymptotic behaviour of matrix coefficients, with the Whittaker functional W_ψ replacing integration against φ. This is explained, maybe a bit hurriedly, first of all in [Casselman-Shalika:1980] and then in more detail in [Casselman-Shahidi:1998]. The approach given in that last paper was in fact motivated by the approach to matrix coefficients that I have used here. There exists in this situation a canonical map $V_N \to V_{\psi,N} \cong \mathbb{C}$ describing the asymptotic behaviour, roughly because as $a \to 0$ the value of $\psi(ana^{-1})$ becomes 1, and each component of this map is determined by the effect of the standard intertwining operators $T_w\colon \mathrm{Ind}(\chi) \to \mathrm{Ind}(w\chi)$ on the Whittaker functional W_ψ.

Part II. Real groups

9. Introduction. Let now G be the group of real points on a Zariski-connected reductive group defined over $\mathbb{R}$. In addition, let

$$\begin{aligned} K &= \text{a maximal compact subgroup} \\ \mathfrak{g}, \text{ etc.} &= \text{complex Lie algebra of } G \text{ etc.} \end{aligned}$$

In the first part, the simple nature of Jacquet modules as well as the phenomenon that 'asymptotic' expansions are asymptotic equalities made our task easy. For real groups, there are both algebraic and analytical complications:

- the behaviour of matrix coefficients at infinity on G is truly asymptotic, expressed in terms of Taylor series;
- the Jacquet module is, as it consequently has to be, more complicated;
- there is no Bruhat filtration for the usual representation of $(\mathfrak{g}, K)$ on the K-finite principal series. Instead, one has to consider certain smooth representations of G itself, for example the C^∞ principal series;
- there are now two Jacquet modules to be considered, one for K-finite, one for smooth spaces.

You can get a rough idea of what happens in general by looking at the case of harmonic functions on the unit disk D, a space on which $\mathrm{SL}_2(\mathbb{R})$ acts (since the Cayley transform $z \mapsto (z-i)/(z+i)$ takes the upper half plane $\mathcal{H}$ to D). There are two spaces of interest: (1) the finite sums of polynomials in z and their conjugates, a representation of $(\mathfrak{g}, K)$; (2) the space of harmonic functions which extend smoothly to $\overline{D}$, on which $\mathrm{SL}_2(\mathbb{R})$ itself acts. In either case, the constant functions are a stable subspace, and the quotient is the sum of two discrete series, holomorphic and anti-holomorphic.

The group P fixes the point 1 (corresponding to ∞ on the upper half-plane), and $\mathfrak{n}$ acts trivially on the tangent space there. This means that if I is the ideal of functions in the local ring $\mathcal{O}$ vanishing at 1, then $\mathfrak{n}$ takes $\mathcal{O}$ to I, and in general I^n to I^{n+1}. The asymptotic behaviour of a harmonic function at 1 is controlled by its Taylor series. The space of all Taylor series at 1 is a representation of P as well as the Lie algebra $\mathfrak{g}$. This space of formal power series is the correct analogue of the Jacquet module here. One thing that is deceptive here is that the K-finite harmonic functions are polynomials. This is unique to that case.

One feature seen here, a feature characteristic of real groups, is that the analogue of the Jacquet module has a simpler relationship to the representation on C^∞ functions than on K-finite ones—the map from the first onto Taylor series is actually surjective, while the second is not.

What I, and presumably everyone who works with both real and $\mathfrak{p}$-adic groups, finds remarkable is that in spite of great differences in technique required to deal with the two cases, the results themselves are uncannily parallel. It might incline some to believe that there is some supernatural being at work in this business.

10. The real Jacquet module. If V is a finitely generated Harish-Chandra module over $(\mathfrak{g}, K)$ it is finitely generated as a module over $U(\mathfrak{n})$, and its contragredient $\widetilde{V}$ is finitely generated over $U(\overline{\mathfrak{n}})$. Its **Jacquet module** is the completion $V_{[\mathfrak{n}]}$—the projective limit of the quotients $V/\mathfrak{n}^k V$—with respect to powers of $\mathfrak{n}$ (introduced in [Casselman:1979]). It is obviously a representation of $(\mathfrak{p}, K \cap P)$ and in fact one of P, even though V itself is not. Slightly more surprising is that it is a representation of all of $\mathfrak{g}$, although it is easy to verify. Not so surprising if you think about the example of harmonic functions, where this completion ts the space of harmonic Taylor series at 1 and the enveloping algebra $U(\mathfrak{g})$ acts by differential operators.

This Jacquet module is easily related to homomorphisms *via* Frobenius reciprocity from V to representations induced from finite-dimensional representations of P, since it is universal with respect to $\mathfrak{n}$-nilpotent modules.

The projective limit is a kind of non-abelian formal power series construction. As in the p-adic case:

10.1. Proposition. *The functor $V \to V_{[\mathfrak{n}]}$ is exact.*

Since a similar question will arise later in different circumstances, I recall how this goes. As proved in [McConnell:1967], the Artin-Rees Lemma holds for the augmentation ideal $(\mathfrak{n})$ of $U(\mathfrak{n})$. This is one example of the fact that much of the theory of commutative Noetherian rings remains valid for $U(\mathfrak{n})$. If

$$0 \to A \to B \to C \to 0$$

is exact then we have a right exact sequence

$$\ldots \to A/\mathfrak{n}^n A \to B/\mathfrak{n}^n B \to C/\mathfrak{n}^n C \to 0\,.$$

The left inclusion is not necessarily injective. But by Artin-Rees, there exists $k \geq 0$ such that then $A \cap \mathfrak{n}^n B \subseteq \mathfrak{n}^{n-k} A$ for $n \gg 0$. Suppose (a_n) lies in the projective limit of the quotients $A/\mathfrak{n}^n A$ with image 0 in $B/\mathfrak{n}^n B$. Then for n large, a_{n+k} lies in $\mathfrak{n}^n A$, so $a_n = 0$. qed

This argument will fail in later circumstances, but something close to it will succeed. For the moment, let R be the ring $U(\mathfrak{n})$, I the ideal generated by $\mathfrak{n}$. The long exact sequence above fits into

$$\ldots \to \mathrm{Tor}_1^R(R/I^n, B) \to \mathrm{Tor}_1^R(R/I^n, C) \to A/I^n A \to B/I^n B \to C/I^n C \to 0\,.$$

The following is equivalent to Artin-Rees.

10.2. Proposition. *If C is a finitely generated module over $U(\mathfrak{n})$, then for some k and $n \gg 0$, the canonical map from $\mathrm{Tor}_1^R(R/I^n, C)$ to $\mathrm{Tor}_1^R(R/I^{n-k}, C)$ is identically 0.*

Proof. Suppose

$$0 \to E \to F \to C \to 0$$

to be an exact sequence of finitely generated modules over $U(\mathfrak{n})$, where F is free. Choose k so that $E \cap I^n F \subseteq I^{n-k} E$ for $n \gg 0$. Since Tor of a free module vanishes, the proof follows from diagram chasing in:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Tor}_1^R(R/I^n, C) & \to & E/I^n E & \to & F/I^n F & & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Tor}_1^R(R/I^{n-k}, C) & \to & E/I^{n-k} E & \to & F/I^{n-k} F & & \ldots \end{array}$$

qed

10.3. Corollary. *If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of $U(\mathfrak{n})$ modules and C is finitely generated, then

$$0 \to A_{[\mathfrak{n}]} \to B_{[\mathfrak{n}]} \to C_{[\mathfrak{n}]} \to 0$$

is also exact.

Exactness for real groups is thus much more sophisticated than it is for p-adic ones. Still, that one can define a Jacquet module and that it again defines an exact functor seems almost miraculous. The one common feature in both cases is the connection with geometry—here with compactifications of symmetric spaces, in the other with compactifications of the building. But then the analogy between symmetric spaces and buildings is another miracle.

11. Verma modules. The Jacquet module for real groups is closely related to the more familiar Verma modules.

Traditionally, a Verma module is a representation of $\mathfrak{g}$ on a space

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$$

where V is an irreducible finite dimensional representation of $\mathfrak{p}$. Since any finite-dimensional representation of $\mathfrak{p}$ is necessarily annihilated by some power of $\mathfrak{n}$, every vector in such a space is also annihilated by some power of $\mathfrak{n}$. I shall therefore introduce a slightly more general notion with the same name—what I shall call a Verma module is a compatible pair of representations of $U(\mathfrak{g})$ and P on a space V which is finitely generated and has the property that every vector in V is annihilated by some power of $\mathfrak{n}$. In other words, V is the union $V^{[\mathfrak{n}]}$ of its $\mathfrak{n}$-torsion subspaces $V(\mathfrak{n}^n)$. The compatibility means that the representation of P agrees with that of its Lie algebra $\mathfrak{p}$ as a subalgebra of $\mathfrak{g}$.

Every Verma module is the quotient of one of the form $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$, where V is a finite dimensional representation of P. A Verma module will always have finite length as a module over $U(\mathfrak{g})$, and will be annihilated by some ideal of $Z(\mathfrak{g})$ of finite codimension.

How do Verma modules relate to the Jacquet module? If V is a Verma module, its linear dual $\widehat{V}$ is the projective limit of the duals of its finite-dimensional $\mathfrak{n}$-stable subspaces. In other words we know that V is the direct limit of finite-dimensional subspaces:

$$V = \varinjlim V(\mathfrak{n}^k)$$

which means that

$$\widehat{V} = \text{ the projective limit of the duals of the } V(\mathfrak{n}^k)\,.$$

Furthermore, we have an exact sequence

$$0 \to V(\mathfrak{n}^k) \to V \to (\text{ dual of } \mathfrak{n}^k) \otimes V$$

and we deduce the exact sequence

$$\mathfrak{n}^k \otimes \widehat{V} \to \widehat{V} \to \text{ dual of } V(\mathfrak{n}^k) \to 0\,.$$

so that $\widehat{V}$ is the projective limit of the finite-dimensional quotients $\widehat{V}/\mathfrak{n}^k\widehat{V}$. It is a finitely generated module over the completion $U_{[\mathfrak{n}]}$ of $U(\mathfrak{n})$ with respect to powers of $\mathfrak{n}^n$.

Conversely, the topological dual of this completion —i.e. the space of linear functionals vanishing on some $\mathfrak{n}^k\widehat{V}$—is the original Verma module. Thus, a natural and straightforward duality exhibits a close relationship between Verma modules and $\mathfrak{g}$-modules finitely generated over $U_{[\mathfrak{n}]}$. In particular, the Jacquet module of V is the linear dual of the space of $\mathfrak{n}$-torsion in the linear dual $\widehat{V}$ of V, which is a Verma module in the sense defined above.

There is another duality relationship between Verma modules for P and those for its opposite $\overline{P}$. Something like this is to be expected in view of the duality between Jacquet modules in the $\mathfrak{p}$-adic case, where N and $\overline{N}$ both occur in the description of asymptotic behaviour of matrix coefficients. It is easy to see that a Verma module V is always finitely generated over $U(\overline{\mathfrak{n}})$. It is in fact the submodule of $\mathfrak{n}$-finite vectors in its completion $\overline{V} = V_{[\overline{\mathfrak{n}}]}$. The continuous dual U of $\overline{V}$ is then in turn a Verma module for $\overline{\mathfrak{p}}$. If we now perform the same construction for U we get V back again. So the categories of Verma modules for $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are naturally dual to each other. This is crucial, as we shall see, in understanding the relationship between Jacquet modules and matrix coefficients.

12. Jacquet modules and matrix coefficients. Matrix coefficients satisfy certain differential equations which have regular singularities at infinity on G. This implies that we have a convergent expansion

$$\langle \widetilde{v}, \pi(a)v \rangle = \sum_{\varphi \in \Phi, n \geq 0} c_{\varphi,n}\, \varphi(a) \alpha^n(a)$$

where Φ is a finite collection of A-finite functions on A. If $A \cong \mathbb{R}^\times$, for example, functions in Φ will be of the form $|x|^s \log^m |x|$.

An analogue of Jacquet's observation holds:

For v in $\mathfrak{n}^k V$ or $\widetilde{v}$ in $\overline{\mathfrak{n}}^k \widetilde{V}$ the coefficient $c_{\varphi,n}$ vanishes for $n < k$.

This is easy to see for harmonic functions, since $\mathfrak{n}^k$ takes $\mathcal{O}$ to I^k.

In the limit we therefore get a pairing of $V_{[\mathfrak{n}]}$ with $\widetilde{V}_{[\overline{\mathfrak{n}}]}$ taking values in a space of formal series. The pairing of $V_{[\mathfrak{n}]}$ with $\widetilde{V}_{[\overline{\mathfrak{n}}]}$ is best expressed in terms of the duality explained in the previous section. If $\widetilde{v}$ is annihilated by $\overline{\mathfrak{n}}$ then the series associated to v and $\widetilde{v}$ will be finite, hence defining an A-finite function. So we are now in a situation much like that for p-adic groups. This A-finite function may be evaluated at 1, and in this we we get a 'canonical pairing' between $V_{[\mathfrak{n}]}$ and the $\mathfrak{n}$-torsion in $\widetilde{V}_{[\overline{\mathfrak{n}}]}$ ([Casselman:1979]).

That the pairing is in some strong sense non-degenerate is highly non-trivial, first proven in [Miličič:1977] His argument was rather indirect. It will also be a corollary of the computation of the canonical pairing for induced representations, which is what this paper is all about.

13. Langlands' calculation for $SL_2(\mathbb{R})$. The results for arbitrary reductive groups are quite complicated, even to state (and remain so far unpublished). To give you at least some idea of what goes on I'll look just at the principal series of $\mathrm{SL}_2(\mathbb{R})$. But in order to offer some contrast to what is to come later, I'll begin with a 'classical' argument to be found in [Langlands:1988] which is itself presumably based on earlier results of Harish-Chandra.

Suppose that f lies in $\mathrm{Ind}^\infty(\chi)$, φ in $\mathrm{Ind}^\infty(\chi^{-1})$. The associated matrix coefficient is

$$\langle \varphi, R_g f \rangle = \int_{P\backslash G} \varphi(x) f(xg)\, dx\,.$$

In the rest of this paper, let

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In the following result, let

$$a_t = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}.$$

13.1. Theorem. (Langlands) *If $\chi(x) = |x|^s$ with $\Re(s) > 0$ then*

$$\langle \varphi, R_{a_t} f \rangle \sim \delta^{1/2}(a_t) \chi^{-1}(a_t) \varphi(w) \int_N f(wn)\, dn$$

as $t \to 0$.

This is an asymptotic equation, with the interpretation that the limit of

$$\lim_{t \to 0} \frac{\langle \varphi, R_{a_t} f \rangle}{\delta^{1/2}(a_t) \chi^{-1}(a_t)} = \varphi(w) \int_N f(wn)\, dn\,.$$

It is well known (and we'll see in a moment) that for $\Re(s) > 0$ the integral

$$\Omega_w(f) = \int_N f(wn)\,dn$$

is absolutely convergent and defines an N-invariant functional on $\mathrm{Ind}(\chi)$.

Proof. We have

$$\begin{aligned}
\langle \varphi, R_{a_t} f\rangle &= \int_N f(wna)\varphi(wn)\,dn \\
&= \int_N f(wa_t w^{-1}\cdot w\cdot a_t^{-1} n a_t)\varphi(wn)\,dn \\
&= \delta^{-1/2}(a_t)\chi^{-1}(a_t)\int_N f(w\cdot a_t^{-1} n a_t)\varphi(wn)\,dn \\
&= \delta^{-1/2}(a_t)\chi^{-1}(a_t)\delta(a_t)\int_N f(wn)\varphi(w\cdot a_t n a_t^{-1})\,dn \\
&= \delta^{1/2}(a_t)\chi^{-1}(a_t)\int_N f(wn)\varphi(w\cdot a_t n a_t^{-1})\,dn\,.
\end{aligned}$$

We'll be through if I show that

$$\lim_{t\to 0}\int_N f(wn)\varphi(w\cdot a_t n a_t^{-1})\,dn = \varphi(w)\cdot\int_N f(wn)\,dn\,.$$

First I recall the explicit Iwasawa factorization for $\mathrm{SL}_2(\mathbb{R})$:

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} 1 & (ac+bd)/r^2 \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1/r & 0 \\ 0 & r\end{bmatrix}\begin{bmatrix} d/r & -c/r \\ c/r & d/r\end{bmatrix}$$

where $r = \sqrt{c^2+d^2}$. Thus we can write

$$wn = \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & x\end{bmatrix} = n_x\begin{bmatrix} (1+x^2)^{-1/2} & 0 \\ 0 & (1+x^2)^{1/2}\end{bmatrix} k_x$$

where n_x, k_x are continuous functions of x.

As a consequence, the integral is

$$\int_{\mathbb{R}} (x^2+1)^{-(s+1)/2} f(k_x)\big((t^2x)^2+1\big)^{(s-1)/2}\varphi(k_{t^2x})\,dx$$

The integrand converges to

$$(x^2+1)^{-(s+1)/2} f(k_x)\varphi(w)$$

as $t\to 0$. According to the dominated convergence theorem (elementary in this case, and justified in a moment) the integral converges to $\varphi(w)\cdot\Omega_w(f)$.

In later sections we'll see a more precise description of the behaviour of matrix coefficients at infinity on G.

I include here a Lemma needed to apply the dominated convergence theorem, a pleasant exercise in calculus.

13.2. Lemma. *For all $0 \le |t| \le 1$ and $0 < s$ the product*

$$(x^2+1)^{-(s+1)/2}\big((t^2x)^2+1\big)^{(s-1)/2}$$

lies between $(x^2+1)^{-(s+1)/2}$ and $(x^2+1)^{-1}$.

Proof. These are what you get for $t=0$, $t=1$, and the derivative with respect to t is always of constant sign in between.

We shall see later that this result gives only the leading term in an infinite asymptotic series.

14. The Bruhat filtration. Now let's begin a new analysis, following the p-adic case as closely as possible. We need first to say something about the Jacquet module for principal series. Here, as in the p-adic case, this depends on the Bruhat decomposition $G = P \bigsqcup PwN$.

Let V be $\mathrm{Ind}^\infty(\chi)$. For f in this space let Ω_f be the map from $U(\mathfrak{g})$ taking X to $Xf(1)$. This lies in a kind of **infinitesimal principal series**

$$V_1 = \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \mathbb{C}_{\chi\delta_P^{1/2}}) .$$

In this way, we get a $\mathfrak{p}$-covariant map

$$\Omega\colon\ f \longmapsto \Omega_f .$$

Let V_w be the subspace of functions in V vanishing of infinite order along P, which are the closure in $\mathrm{Ind}^\infty(\chi)$ of those functions with compact support on PwN. By a theorem of E. Borel this fits into a short exact sequence

$$0 \to V_w \to V \xrightarrow{\Omega} V_1 \to 0 .$$

I call this the **Bruhat filtration** of V.

It is important to realize—and originally, it took me quite a while to realize—that:

There does not exist such a sequence for the K-finite principal series.

After all, the K-finite functions are analytic, and an analytic function cannot vanish of infinite order anywhere. In other words, we do not have a good Bruhat filtration for representation of $(\mathfrak{g}, K)$ on the K-finite principal series. In spite of this, we *do* have however a filtration of the Jacquet module of the K-finite principal series, because according to the main result of [Casselman:1989], *the Jacquet modules of a K-finite principal series and its C^∞ version are the same*, in a very strong sense. The way in which this is phrased in [Casselman:1989] is that any $(\mathfrak{g}, K)$-covariant homomorphism from one K-finite principal series to another extends continuously to a map between the associated smooth principal series (the phenomenon called in [Wallach:1983] 'automatic continuity'). These two assertions are essentially equivalent because of Frobenius reciprocity. Another miracle to put in the pot.

As before, I define the Jacquet module of any representation U of $\mathfrak{g}$ to be the projective limit of the quotients $U/\mathfrak{n}^k U$. As in the p-adic case, the Bruhat filtration of V gives rise to a filtration of its Jacquet module.

14.1. Proposition. *The exact sequence defining the Bruhat filtration gives rise to an exact sequence of Jacquet modules*

$$0 \to (V_w)_{[\mathfrak{n}]} \to V_{[\mathfrak{n}]} \to (V_1)_{[\mathfrak{n}]} \to 0 .$$

AS we have seen, if the terms in the original exact sequence were finitely generated over $U(\mathfrak{n})$, this would be the consequence of the Artin-Rees Lemma. This would still be true if they were finitely generated over $U(\mathfrak{n})_{[\mathfrak{n}]}$. However, I have proved a variation of Artin-Rees which applies here, because of:

14.2. Lemma. *The space V_1 is the linear dual of the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{\chi^{-1}\delta_P^{1/2}}$.*

This is straightforward to verify, and well known. In particular, the space V_1 is finitely generated over $U(\mathfrak{n})_{[\mathfrak{n}]}$, and with the help of a few standard results comparing Tor for $U(\mathfrak{n})$ and its completion, we can deduce the exactness we want. qed

Incidentally, in the case at hand we can prove everything directly. Again let $R = U(\mathfrak{n})$, and let I be the ideal generated by $\mathfrak{n}$. If ν is a generator of $\mathfrak{n}$, the ring R is just a polynomial algebra in ν. The group $\mathrm{Tor}_1^R(I^n, A)$ is the subspace of A annihilated by I^n. The completion of R with respect to I is still a principal ideal domain, and the torsion in V_1 is finite dimensional. In particular, there exists some $k \ge 0$ annihilating all its torsion. The canonical projection from $\mathrm{Tor}_1^R(R/I^n, V_1)$ to $\mathrm{Tor}_1^R(R/I^{n-k}, V_1)$ may be identified with multiplication by ν^k, which annihilates all torsion. This is a special case of what I called the variant of Artin-Rees.

One immediate corollary of the Lemma is this:

14.3. Corollary. *The subspace* $\mathfrak{n}^k V_1$ *is closed in* V_1 *with finite codimension.*

In order to fully understand the Bruhat filtration of the Jacquet module, we must figure out what $(V_w)_{[\mathfrak{n}]}$ is. The group N is isomorphic to the additive group $\mathbb{R}$. Its Schwartz space $\mathcal{S}(N)$ is defined by this identification. An application of the same calculations we made for Langlands' Theorem, using the Iwasawa decomposition, tells us:

14.4. Proposition. *The restriction of* V_w *to* N *is isomorphic to the Schwartz space* $\mathcal{S}(N)$.

This is a slight generalization of Schwartz' identification of $\mathcal{S}(\mathbb{R})$ with the space of those smooth functions on the projective line that vanish of infinite order at ∞ (which is in fact a special case).

14.5. Proposition. *A function* f *in* V_w *lies in* $\mathfrak{n}^k V_w$ *if and only if*

$$\int_N P(n) f(n)\, dn = 0$$

for every polynomial P *of degree* $< k$.

This identifies the $\mathfrak{n}$-torsion in the dual of V_w. It is a simple exercise in calculus.

14.6. Corollary. *The space* $\mathfrak{n}^k V_w$ *is closed in* V_w*, and every quotient* $V_w/\mathfrak{n}^k V_w$ *is free of rank one over* $U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n})$.

This gives us:

14.7. Proposition. *Each space* $\mathfrak{n}^k V$ *is closed in* V *and has finite codimension.*

Proof. If $U \to V$ is a continuous map of Fréchet spaces with image of finite codimension, then its image is closed. This is because if F is a finite-dimensional complement, the associated map from $U \oplus F$ to V is continuous and surjective, so that the induced map from $\big(U/\ker(f)\big) \oplus F$ is an isomorphism of topological vector spaces, by the open mapping theorem. The long exact sequence

$$\ldots \to V_w/\mathfrak{n}^k V_w \to V/\mathfrak{n}^k V \to V_1/\mathfrak{n}^k V_1 \to 0$$

tells us that $\mathfrak{n}^k V$ has finite codimension, but it is also the image of a map from $\otimes^{(k)}\mathfrak{n} \otimes V$ to V. qed

Now, let's try to understand what we have on hand. Dual to the Bruhat filtration of $\mathrm{Ind}(\chi)$ by double cosets PwN is that of its contragredient $\mathrm{Ind}(\chi^{-1})$ by cosets $Pw\overline{N}$. Let the corresponding exact sequence be

$$0 \to \widetilde{V}_w \to \widetilde{V} \to \widetilde{V}_1 \to 0\,,$$

giving rise to

$$0 \to (\widetilde{V}_w)_{[\overline{\mathfrak{n}}]} \to \widetilde{V}_{[\overline{\mathfrak{n}}]} \to (\widetilde{V}_1)_{[\overline{\mathfrak{n}}]} \to 0\,.$$

These two coset decompositions are transversal to one other–the coset $P\overline{N}$ is open in G and contains $P = PN$, while PwN is open in G and contains $Pw\overline{N}$. Every smooth function on PwN therefore determines a Taylor series along $Pw = Pw\overline{N}$, and in particular every polynomial $P(n)$ on PwN determines one. As we have seen these are all annihilated by some power of $\mathfrak{n}$, so it should not be too surprising to see:

14.8. Proposition. *As a module over* $\mathfrak{g}$*, the* $\mathfrak{n}$*-completion of* V_w *is isomorphic to the* $\mathfrak{n}$*-completion of the continuous dual* $(\widetilde{V}_w)_{[\overline{\mathfrak{n}}]}$.

Similarly for $\widetilde{V}_1$ and V_1.

So now we find ourselves in exactly the same situation we saw in the p-adic case—Bruhat filtrations of $\mathrm{Ind}(\chi)$ and $\mathrm{Ind}(\chi^{-1})$ with corresponding terms in the associated graded spaces dual to one another. In the next section we shall see that this duality matches with the asymptotic expansion of matrix coefficients.

Generically, the Bruhat filtration of Jacquet modules will split, but for isolated values of χ it will not.

There is one final remark to make. Of course the K-finite principal series $V_{(K)}$ embeds into the smooth one V, inducing a map of their Jacquet modules. It is a consequence of the 'automatic continuity' theorem in [Wallach:1983] that this is an isomorphism. Thus, although there is no Bruhat filtration of $V_{(K)}$, there is one of its Jacquet module. It's a curious fact, and presumably a fundamental one.

15. The explicit formula.

Let me recall where we are in the discussion. We want to calculate $\langle \varphi, R_a f \rangle$ for φ in $\mathrm{Ind}^\infty(\chi^{-1})$, f in $\mathrm{Ind}^\infty(\chi)$. As with p-adic groups, since the asymptotic expansion of matrix coefficients factors through Jacquet modules, but here I do not see how to use that to simplify calculations. The problem is that we must look at what happens for smooth functions.

I'll look here at a principal series representation of $\mathrm{SL}_2(\mathbb{R})$. Let

$$a = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} .$$

Set $\chi(a) = |t|^s$ as in the discussion of Langlands' formula.

We express f in $\mathrm{Ind}(\chi)$ and φ in $\mathrm{Ind}(\chi^{-1})$ as sums of functions with support on the open Bruhat double cosets:

$$f = f_w + f_1, \quad \varphi = \varphi_w + \varphi_1$$

where $*_w$ has support on PwN, $*_1$ on $P\overline{N}$. We will see what happens to each term in

$$\langle \varphi, R_a f \rangle = \langle \varphi_1, R_a f_1 \rangle + \langle \varphi_1, R_a f_w \rangle + \langle \varphi_w, R_a f_1 \rangle + \langle \varphi_w, R_a f_w \rangle$$

as $t \to 0$.

I'll look first at $\langle \varphi_w, R_a f_w \rangle$ because it offers an interesting comparison with the verification of Langlands' formula. To simplify notation I'll set $\varphi = \varphi_w$, $f = f_w$. We get here as before that

$$\langle \varphi, R_a f \rangle = \delta^{1/2} \chi^{-1}(a) \int_N f(wn) \varphi(w \cdot ana^{-1}) \, dn \,,$$

We must look at the integral, also as before. But here things are somewhat simpler analytically—the functions $f(wn)$ and $\varphi(wn)$ are both of compact support as functions of n. Set

$$n = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$$

and change both the variable n and the functions $f(wn)$ and $\varphi(w \cdot ana^{-1})$ of n to functions of x. So we are now considering the integral

$$\int_{\mathbb{R}} f(x) \varphi(t^2 x) \, dx$$

where f and φ are both of compact support on $\mathbb{R}$. You can see immediately and roughly what is going to happen—as $t \to 0$ $\varphi(t^2x)$ will be more or less determined by its behaviour near 0, or in other words by its Taylor series at 0. We get formally

$$\int_{\mathbb{R}} f(x)\varphi(t^2x)\,dx = \int_{\mathbb{R}} f(x) \sum_{m\ge 0} t^{2m}\,\frac{x^m}{m!}\,\varphi^{(m)}(0)\,dx = \sum_{m\ge 0} t^{2m}\,\frac{\varphi^{(m)}(0)}{m!}\int_{\mathbb{R}} x^m f(x)\,dx\,.$$

Because f and φ have compact support, it is not hard to justify this as an asymptotic expansion.

We can find a more enlightening interpretation of this. Let

$$\overline{\nu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

be a generator of $\overline{\mathfrak{n}}$. In terms of our choice of coordinates in $\overline{N}$ the functional

$$\widetilde{\Omega}_{w,m}\colon\ \mathrm{Ind}(\chi^{-1}) \to \mathbb{C},\ \varphi \longmapsto R_{\overline{\nu}^m} f(w)$$

is the same as that taking it to $\varphi^{(m)}(0)$. It is annihilated by $\overline{\mathfrak{n}}^m$. Let

$$\Omega_{w,m}\colon\ \mathrm{Ind}(\chi) \to \mathbb{C},\ f \mapsto \int_N x^m f(wn)\,dn\,,$$

which is a meromorphic function of χ. In these terms, the formula becomes

$$\langle \varphi, R_a f\rangle \sim \delta^{1/2}(a)\chi^{-1}(a)\Big(\sum_{m\ge 0} \widetilde{\Omega}_{w,m}(\varphi)\,\Omega_{w,m}(f)\Big)\,.$$

Actually, this whole expansion can be deduced from the algebra of Verma modules, if one knows just the leading term. The expansion expresses, essentially, the unique pairing between generic terms in Jacquet module of the principal series. After all, generic Verma modules are irreducible, so the pairing is unique up to scalar multiplication.

The term $\langle \varphi_1, R_a f_1\rangle$ has a similar asymptotic expression in terms of the Jacquet modules of $\widehat{V}_1$ and $\widehat{V}$, and the cross terms vanish asymptotically. Define functionals $\Omega_{1,m}$ $\widetilde{\Omega}_{1,m}$ as I did $\Omega_{w,m}$ and $\widetilde{\Omega}_{w,m}$ In the end we get as asymptotic expansion a sum of two infinite series

$$\langle \varphi, R_a f\rangle \sim \delta^{1/2}(a)\chi^{-1}(a)\Big(\sum_{m\ge 0} \widetilde{\Omega}_{w,m}(\varphi)\,\Omega_{w,m}(f)\Big) + \delta^{1/2}(a)\chi(a)\Big(\sum_{m\ge 0} \widetilde{\Omega}_{1,m}(\varphi)\Omega_{1,m}(f)\Big)\,.$$

There is one series for each component in the Bruhat filtration. This is therefore the analogue for real groups of Macdonald's formula for spherical functions on $\mathrm{SL}_2(\mathbb{Q}_p)$. Of course, Macdonald's formula is an exact formula, but here we are given an asymptotic expansion. But in fact, K-finite matrix coefficients satisfy an analytic ordinary differential equation, and the formula for them becomes one involving convergent series valid everywhere except for $a = I$, where the differential equation has a (regular) singularity.

Macdonald's formula, incidentally, is proven along similar lines in [Casselman:2009].

16. Whittaker functions. Whittaker functions for real groups also satisfy a differential equation with regular singularities along the walls $\alpha = 0$ for simple roots α (although irregular at infinity in other directions). Suppose ψ to be a non-degenerate character of $\mathfrak{n}$. If V is a finitely generated $(\mathfrak{g}, K)$-module, then its **Kostant module** (following [Kostant:1979]) is the space $\widehat{V}^{[\mathfrak{n},\psi]}$ of $\mathfrak{n}_\psi$-torsion in the continuous dual of its canonical G-representation—that is to say continuous linear functionals annihilated by some power of the $U(\mathfrak{n})$-ideal generated by the $x - \psi(x)$ for x in $\mathfrak{n}$. We have a map from $\widehat{V}^{[\mathfrak{n},\psi]} \otimes V$, taking $W \otimes v$ to the series expansion of $\langle W, \pi(a)v \rangle$ at $a = 0$. How can we fit this into a scheme such as we have seen above?

I have only a rough idea of what to propose. In the p-adic case we have a canonical map from V_N to $V_{\psi,N}$. In the real case, both the Jacquet module and the Kostant module are very different as modules over $U(\mathfrak{n})$, but have similar structures as modules over $\overline{\mathfrak{n}}$. A Kostant module is finitely generated over $\overline{\mathfrak{n}}$, and if $U = \widehat{V}^{[\mathfrak{n},\psi]}$ we get a map from the Jacquet module of V to the $\overline{\mathfrak{n}}$-completion. An explicit formula for the expansion of Whittaker functions on the smooth principal series would then follow from a calculation of the scalars defined by intertwining operators on Whittaker models. This would not be all that different, conceptually, from what happens for p-adic groups.

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