Notes on $\mathfrak{p}$-adic Whittaker functions

This contains a collection of notes and letters written by me to Freydoon Shahidi in the course of putting together our joint paper, which appeared as 'On irreducibility of standard modules for generic representations', *Annales scientifiques de l'École Normale Supérieure* **31** (1998), 561–589. It is available on the NUMDAM site:

`http://www.numdam.org/`

I do not know the exact dates the originals were written. The correspondence was initiated because I pointed out to Freydoon that some brief remarks at the end of the paper on Whittaker functions by Joe Shalika and me had gone largely unnoticed and almost completely unexploited.

1. A remark on Whittaker models for the unramified principal series

Suppose G to be a reductive group, unramified in the sense of [Casselman-Shalika]. Let $P = MN$ be a minimal parabolic subgroup. Let Ω_ψ be the Whittaker functional, defined on the subspace of all f with support on $w_\ell N$ by the integral

$$\langle f, \Omega_\psi \rangle = \int_N \psi^{-1}(n) f(w_\ell n) \, dn \ .$$

and extended by exactness.

At the end of [Casselman-Shalika] we show that there exists a canonical map from I_N to $I_{\psi,N}$ for any admissible representation I. Dualizing, we get from Ω_ψ an element of the dual of the Jacquet module of I.

Let $I = \mathrm{Ind}(\chi \mid P, G)$ with χ unramified. Then for χ not fixed by any element of the Weyl group we have that the canonical map takes Ω_ψ to

$$\sum_w \gamma_w(\chi) \, \Lambda_w$$

where the Λ_w are the functionals defining the usual intertwining operators. In particular Λ_1 takes f in I to $f(1)$. Thus for a near 0 we have

$$\langle R_a f, \Omega_\psi \rangle = \sum_w \gamma_w(\chi) \, \delta^{1/2}(w\chi)(a) \langle f, \Lambda_w \rangle \ .$$

The point is that the coefficients are independent of f. Therefore we can calculate the coefficient $\gamma_w(\sigma)$ if we know just one f with $\langle f, \Lambda_w \rangle \neq 0$. But the main result of [Casselman-Shalika] tells us what the coefficients are for $f = \varphi_K$. In particular, we see that $\gamma_1(\chi)$ never vanishes in the positive cone $X^{++}(P)$ of characters where $|\chi(a_\alpha)| < 1$.

Now suppose that f lies in the kernel of the Whittaker map. Then $\langle \pi(g) f, \Omega_\psi \rangle = 0$ for all g in G. This means that for a near enough to 0 (how near will depend on g)

$$\langle \pi(a)\pi(g) v, \Omega_\psi \rangle = \sum_w \gamma_w(\chi) \, \delta^{1/2}(w\chi)(a) \langle \pi(g) f, \Lambda_w \rangle = 0$$

for all g in G. By assumption on the regularity of χ, this implies that each one of the components must vanish identically. Thus $\gamma_1(\chi) \langle \pi(g) f, \Lambda \rangle = \gamma_1(\chi) f(g) = 0$ for all g in G, and $f(g)$ must vanish identically.

2. Brief remarks on the general case

Let now G, $P = MN$ be arbitrary, ψ a generic character of $N_\emptyset$. Let σ be for the moment any irreducible representation of M with a Whittaker model $\langle \sigma(g)v, \Omega_P \rangle$ for ψ_M, the restriction of ψ to $M \cap N_\emptyset$. Let Ω be the Whittaker functional on $\mathrm{Ind}(\sigma \mid P, G)$ determined by Ω_P. The result at the end of [Casselman-Shalika] tells that for σ in the positive cone and a near 1 in A we have the equality

$$\langle \pi(a)f, \Omega_\psi \rangle = \gamma_1(\sigma)\delta_P^{1/2}(a)\sigma(a)\langle f(1), \Omega_P \rangle + \cdots$$

where the other terms are A-finite and do not involve $\sigma(a)$. The same reasoning as before now tells us that it suffices to show that $\gamma_1(\sigma)$ never vanishes (under the assumption on σ).

There is a very simple formula for $\gamma_1(\sigma)$, which I discovered by analogy with the formula for the asymptotic behaviour of matrix coefficients. Formally we have

$$\gamma_1(\sigma) = \langle \Omega_\psi, \Lambda_{\mathrm{opp}} \rangle$$

where Λ_{opp} is the functional determining by Frobenius reciprocity the intertwining operator

$$T\colon \mathrm{Ind}(\sigma \mid P, G) \mapsto \mathrm{Ind}(\sigma \mid P^{\mathrm{opp}}, G)$$

given formally by

$$\langle f, \Lambda_{\mathrm{opp}} \rangle = \int_{N^{\mathrm{opp}}} f(n)\, dn \ .$$

This integral makes no sense in our circumstances, but we can reinterpret it as the constant $c(\sigma)$ such that

$$T^*\Omega_{\mathrm{opp}} = c(\sigma)\Omega \ .$$

Here Ω_{opp} is the Whittaker function given by integration over N in $\mathrm{Ind}(\sigma \mid P^{\mathrm{opp}}, G)$.

At any rate, in order to see that the Whittaker model of $\mathrm{Ind}(\sigma \mid P, G)$ is an embedding for σ in the positive cone, we must show now that $c(\sigma) \neq 0$ if σ lies in that region. Now the intertwining operator is, as you know, a product of operators where P is essentially maximal, so we may reduce to that case.

But this is where I have not been able to do what you have. If, however, I assume that P is minimal, then the calculations of Keys (and myself) exhibit this constant $c(\sigma)$ as an L-function, which can easily be seen not to vanish.

I have told you the formula for γ_1 without justification. I found what I remember to be a very simple proof of this formula which I am having some trouble recalling. I do recall that I interpreted Ω as a limit of K-finite functions, and applied the formula for matrix coefficients, but there is some limit process I cannot remember. I'll forward that to you when I get it.

Presumably your global-local technique could be inserted into this argument to deal with the case of σ cuspidal. I don't see that it gets us any further along in the case when σ is not cuspidal, but perhaps you will get something out of it I don't.

The technique I have sketched has the apparent advantage over yours that it handles everything from scratch. Otherwise it is certainly not essentially different.

Cheers.

3. Letter

Dear Freydoon,

In this note I will prove what you might call the fundamental formula relating asymptotics of Whittaker functions and intertwining operators.

Suppose that $P = MN$ is a parabolic subgroup of G, ψ a nondegenerate character of $N_\emptyset$, ψ_P one of $M \cap N_\emptyset$. Let (σ, U) be an irreducible representation of M with a Whittaker model with respect to ψ_P determined by the functional Ω_P. Suppose that (σ lies in a region where $(\delta^{1/2}\sigma, U)$ is a summand of the Jacquet module of $\mathrm{Ind}(\sigma \mid P, G)$. Let Ω be a Whittaker functional on $\mathrm{Ind}(\sigma \mid P, G)$ and $\overline{\Omega}$ be the one defined on $\mathrm{Ind}(\sigma \mid \overline{P}, G)$ defined formally by the integral

$$\langle f, \overline{\Omega} \rangle = \int_N \psi^{-1}(n) \langle f(n), \Omega_P \rangle \, dn \ .$$

Let

$$T\colon \int (\sigma \mid \overline{P}, G) \to \mathrm{Ind}(\sigma \mid P, G)$$

be the intertwining operator determined by the U-valued functional

$$\Lambda_N(f) = \int_N f(n) \, dn$$

so that $Tf(1) = \Lambda_N(f)$. Let γ be the constant such that

$$T^* \Omega = \gamma \, \overline{\Omega} \ .$$

Suppose that f in $\mathrm{Ind}(\sigma \mid \overline{P}, G)$ has support on $\overline{P}N$, say on $\overline{P}\omega$. Then

$$\begin{aligned} \langle R_a f, \overline{\Omega} \rangle &= \int_N \psi^{-1}(n) \langle f(na), \Omega_P \rangle \, dn \\ &= \delta^{-1/2}(a)\sigma(a) \int_\omega \psi^{-1}(n) \langle f(a^{-1}na), \Omega_P \rangle \, dn \\ &= \delta^{1/2}\sigma(a) \int_{a^{-1}\omega a} \psi^{-1}(ana^{-1}) \langle f(n), \Omega_P \rangle \\ &\sim \delta^{1/2}(a)\sigma(a) \int_N \langle f(n), \Omega_P \rangle \, dn \\ &= \delta^{1/2}(a)\sigma(a) \langle \Lambda_N(f), \Omega_P \rangle \end{aligned}$$

since $\psi(n)$ is 1 for n near 1. If we then look at arbitrary f in $\mathrm{Ind}(\sigma \mid \overline{P}, G)$ we see that the full expansion has this as the term we want (involving the summand $\delta^{1/2}\sigma$ of the Jacquet module). But this can be expressed as saying that for f in $\mathrm{Ind}(\sigma \mid \overline{P}, G)$

$$\langle f, \gamma^{-1} T^* \Omega \rangle = \gamma^{-1} \langle Tf, \Omega \rangle \sim \delta^{1/2}(a)\sigma(a) \langle Tf(1), \Omega_P \rangle \ .$$

But since T is generically surjective, we see that for all φ in $\mathrm{Ind}(\sigma \mid P, G)$

$$\langle R_a \varphi, \Omega \rangle \sim \delta^{1/2}\sigma(a) \langle \varphi(1), \Omega_P \rangle \ .$$

which is what had to be proven. In all these expressions, we are referring to only the term we want in the asymptotic expression for Whittaker functions, which can be taken as the leading term in the expansion if σ lies in the appropriate region.

In other words, the claim follows formally from the simple calculation of asymptotic term for $\mathrm{Ind}(\sigma \mid \overline{P}, G)$.

Note that this result is not what I claimed in an earlier note, but it has the virtue of being true.

4. Letter

Dear Freydoon,

Let me show you what happens for $\mathrm{SL}_2(F)$. Of course you already know the answer, but the derivation will probably be new to you.

Let σ be a character of $F^\times$, f in $\mathrm{Ind}(\sigma \mid P, G)$, ψ for the moment any character of N (possibly even trivial).

Then at least formally

$$\langle R_a f, \Omega_\psi \rangle = \int_N \psi^{-1}(n) f(wna)\, dn \; .$$

Here

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and we may as well set

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \; .$$

Let

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \; .$$

In these circumstances

$$\begin{aligned} wn &= \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix} \\ &= \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 0 \end{pmatrix} \\ wna &= \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\alpha x^{-1} & 0 \\ 0 & -\alpha^{-1} x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha^2 x^{-1} & 0 \end{pmatrix} \end{aligned}$$

Fix an ideal $\mathfrak{a}$ in $\mathfrak{o}_F$, and define f by the condition that it have support on $P\overline{N}$ and the formula

$$f(ny\nu) = \delta^{1/2}(\eta)\sigma(\eta)$$

if

$$\nu \in \begin{pmatrix} 1 & 0 \\ \mathfrak{a} & 0 \end{pmatrix}, \quad y = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$$

and 0 otherwise.

Assume σ in the region where the Whittaker integral converges. We have

$$\begin{aligned} \langle R_a f, \Omega_\psi \rangle &= \int_{\alpha^2 x^{-1} \in \mathfrak{a}} \psi^{-1}(x)\delta^{1/2}(\alpha x^{-1})\sigma(-\alpha x^{-1})\, dx \\ &= \sigma(-1)\delta^{1/2}(\alpha)\sigma(\alpha) \int_{x^{-1} \in \alpha^{-2}\mathfrak{a}} \psi^{-1}(x)\delta^{-1/2}(x)\sigma^{-1}(x)\, dx \\ &= \sigma(-1)\delta^{1/2}(\alpha)\sigma(\alpha) \int_{|x|^{-1} \le |\alpha|^{-2}/N\mathfrak{a}} \psi^{-1}(x)\delta^{-1/2}(x)\sigma^{-1}(x)\, dx \\ &= \sigma(-1)\delta^{1/2}(\alpha)\sigma(\alpha) \int_{|\alpha|^2 N\mathfrak{a} \le |x|} \psi^{-1}(x)\delta^{-1/2}(x)\sigma^{-1}(x)\, dx \end{aligned}$$

Now suppose that ψ is non-trivial. Then the integral is equal to

$$\int_{|\alpha|^2 N\mathfrak{a} \le |x| \le T} \psi^{-1}(x)\delta^{-1/2}(x)\sigma^{-1}(x)\,dx$$

as long as T is large enough. This expression is defined for all σ, and analytic in σ. If σ lies in the opposite region, it may be expressed as

$$\int_{|x| \le T} \psi^{-1}(x)\delta^{-1/2}(x)\sigma^{-1}(x)\,dx - \int_{|x| < |\alpha|^2 N\mathfrak{a}} \psi^{-1}(x)\delta^{-1/2}(x)\sigma^{-1}(x)\,dx \ .$$

The first integral is Tate's factor $\gamma(\sigma^{-1}, \psi^{-1})$. Since ψ is trivial near the origin, if $|\alpha|$ is near 0 then the second is also

$$\int_{|x| < |\alpha|^2 N\mathfrak{a}} \delta^{-1/2}(x)\sigma^{-1}(x)\,dx$$

which is the same as

$$\int_{|x| \ge |\alpha|^2 N\mathfrak{a}} \delta^{-1/2}(x)\sigma^{-1}(x)\,dx \ .$$

This is a very natural transformation, as is explained in a recent paper of mine with a title something like 'Extended automorphic forms' in the *Math. Annalen*. Of course it can be verified explicitly. And this is equal to

$$\delta^{1/2}(\alpha)\sigma^{-1}(\alpha) \int_N f(wn)\,dn \ .$$

So the final formula is that for a near 1 we have

$$\langle R_a, \Omega_\psi \rangle = \delta^{1/2}(\alpha) \left[\sigma(\alpha)\gamma(\sigma^{-1}, \psi^{-1}) + \sigma^{-1}(\alpha)\langle f, \Lambda_N \rangle \right] \ .$$

Of course in some sense you already know this formula. What I like about the derivation is that it makes clear how Tate's factor appears, without making assumptions on σ first.

In the next letter I will do the same for $SU(3)$. This is what I have spent a lot of time on. Again it will give a result you are familiar with, but dealing with all characters at once, as opposed to the technique of you and Keys.

By the way, I am writing these as rapidly as I can. I don't think there are any serious mistakes, but if the argument atr any point seems incorrect or confusing, please let me know immediately rather than waste your time.

5. Letter

Dear Freydoon,

In this note I shall show what happens for $SU(3)$. Of course once again the calculations will be somewhat familiar to you from the work of Keys and yourself. But I hope that they are somewhat more direct than those you are familiar with. The point once again is that I hope to introduce the Tate factors and the number $\lambda(E/F,\psi)$ in as direct a fashion as possible.

[Note: I realized just after doing this that my factors and Tate's differ slightly. I write

$$\gamma(\chi,\psi)=\int\chi(x)\psi(-x)\,dx$$

where Tate defines it as

$$\int\chi(y)\psi(-x)\,dx/|x|\ .$$

This shift shouldn't cause too much trouble.]

So let now E/F be a quadratic extension, and H the Hermitian matrix

$$\begin{pmatrix}0&0&1\\0&\gamma&0\\1&0&0\end{pmatrix}$$

with γ in F, and G the group $SU(H)$.

Then

$$w=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}$$

$$M=\left\{\begin{pmatrix}\alpha&0&0\\0&\overline{\alpha}/\alpha&0\\0&0&1/\overline{\alpha}\end{pmatrix}\right\}$$

$$N=\left\{\begin{pmatrix}1&-\gamma\overline{x}&y\\0&1&x\\0&0&1\end{pmatrix}\middle|\,\mathrm{Trace}(y)=-\gamma\,\mathrm{Norm}(x)\right\}$$

Now

$$\langle R_a\varphi,\Omega\rangle=\int_N\psi^{-1}(n)\varphi(wna)\,dn\ .$$

If

$$n=\begin{pmatrix}1&-\gamma\overline{x}&y\\0&1&x\\0&0&1\end{pmatrix}$$

$$a=\begin{pmatrix}\alpha&0&0\\0&\overline{\alpha}/\alpha&0\\0&0&1/\overline{\alpha}\end{pmatrix}$$

then

$$
\begin{aligned}
\delta(a) &= |\overline{\alpha}^2/\alpha|_E |\alpha\overline{\alpha}|_F \\
&= |\alpha|_E^2 \\
\delta^{1/2}(a) &= |\alpha|_E \\
wn &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -x \\ 1 & -\gamma\overline{x} & y \end{pmatrix} \\
&= \begin{pmatrix} 1 & \gamma\overline{x}/\overline{y} & 1/y \\ 0 & 1 & -x/y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\overline{y} & 0 & 0 \\ 0 & \overline{y}/y & 0 \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x/\overline{y} & 1 & 0 \\ 1/y & -\gamma\overline{x}/y & 1 \end{pmatrix} \\
wna &= \begin{pmatrix} 1 & \gamma\overline{x}/\overline{y} & 1/y \\ 0 & 1 & -x/y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha/\overline{y} & 0 & 0 \\ 0 & \overline{\alpha y}/\alpha y & 0 \\ 0 & 0 & y/\overline{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ (\alpha^2/\overline{\alpha})x/\overline{y} & 1 & 0 \\ \alpha\overline{\alpha}/y & -\gamma(\overline{\alpha}^2/\alpha)\overline{x}/y & 1 \end{pmatrix}
\end{aligned}
$$

If φ is defined as for SL_2 by the condition that $y \in \mathfrak{a}$, then similar calculations show that for $|\alpha|$ small and T large we have

$$
\begin{aligned}
\langle R_a\varphi, \Omega\rangle &= \int_N \psi^{-1}(n)\varphi(wna)\,dn \\
&= \delta^{1/2}(a)\left(\sigma(a)\Gamma + \sigma^{-1}(a)\langle\varphi, \Lambda_N\rangle\right)
\end{aligned}
$$

where

$$\Gamma = \int_{|y|\le T} \psi^{-1}(x)\,|y|_E^{-1}\sigma^{-1}(\overline{y})\,dn\ .$$

It is this factor Γ that we are interested in. It is trickier to calculate than the easy one for $\mathrm{SL}_2(F)$. Note that there is no real dependence on T, so I can write it formally as

$$\Gamma = \int_N \psi^{-1}(x)\,|y|_E^{-1}\sigma^{-1}(\overline{y})\,dn\ .$$

Formally this can be written as

$$
\begin{aligned}
\Gamma &= \langle\Phi, \Omega_\psi\rangle \\
&= \int_N \psi^{-1}(n)\Phi(wn)\,dn
\end{aligned}
$$

where

$$\Phi(na\overline{n}) = \delta^{1/2}(a)\sigma(a)$$

and it is perhaps useful to know that *this expression for Γ is in fact valid in all circumstances — for any group whatsoever, and for any induced representation*. Note that

$$\Phi = \lim_{a\to 0} R_a\varphi$$

in some sense. I am not going to worry about convergence questions in the rest of this note, but I'm pretty sure everything can be justified — either by considering functions as distributions, or applying the tricks of my *Annalen* article. I think it is important to realize that these formal integrals are perfectly OK to work with directly, because it simplifies life a lot, and certainly makes arguments clearer. In a sense, the rest of this note will recover Keys' lengthy calculations by these formal tricks, and therefore, I hope, showing that they should be in everybody's toolkit.

Let

$$\chi(y) = |y|_E^{-1}\sigma^{-1}(\overline{y})$$

so that, considering the way integration over N works, we want to calculate

$$\int_N \psi^{-1}(x)\,dx \int_{\mathrm{Trace}(y)=-\gamma x\overline{x}} \chi(y)\,dy\ .$$

Let

$$F(c) = \int_{\mathrm{Trace}(y)=c} \chi(y)\,dy$$

for c in F.

We can calculate this by Fourier duality. It is the same as

$$\begin{aligned}
&= \int_F \psi_F(\lambda c)\,d\lambda \int_E \chi(y)\psi_E(-\lambda y)\,dy \\
&= \gamma_E(\chi,\psi_E) \int_F \psi_F(c\lambda)|\lambda|_E^{-1}\chi^{-1}(\lambda)\,d\lambda \\
&= \chi)-1)\gamma_E(\chi,\psi_E)\gamma_F(|\bullet|_F^{-2}\chi^{-1}|F^\times,\psi_F)|c|_F\chi(c)
\end{aligned}$$

which makes our original integral into

$$\begin{aligned}
&= \langle\text{constant}\rangle\chi(-\gamma)|\gamma| \int_E \psi(-x)|x\overline{x}|\chi(x\overline{x})\,dx \\
&= \lambda(E/F,\psi)^{-1}\chi(\gamma)|\gamma|\gamma_E(\chi,\psi)\gamma_F(|\bullet|_F\chi,\psi)\gamma_F(|\bullet|_F\chi\mathrm{sgn}_E,\psi)\gamma_F(|\bullet|_F^{-2}\chi^{-1},\psi) \\
&= \chi(\gamma)\lambda(E/F,\psi)^{-1}\gamma_E(\chi,\psi)\gamma_F(|\bullet|_F(\chi|F^\times)\,\mathrm{sgn}_E,\psi_F)\ .
\end{aligned}$$

which differs somehow from what you and Keys have! Oops! But you can see the idea, I hope.

In the next and, for the moment, last note I just give a direct proof of Hasse's product formula.

6. Letter

Dear Shahidi,

In these notes I will tidy up a few items, and give a self-contained account of Hasse's formula, which I needed in the last notes.

Suppose V to be a vector space over F, Q a non-degenerate anisotropic quadratic form on V, so that $Q(x+y) - Q(x) - Q(y)$ is a non-degenerate bilinear symmetric pairing.

For y in F, let $\mu(y)\,dy$ be the volume of $Q^{-1}(y)\,dy$, so that we have the volume formula

$$\int_V f(Q(x))\,dx = \int_F f(y)\mu(y)\,dy$$

for all suitable functions f on F. The function μ vanishes identically in the neighbourhood of y unless y lies in the image of Q. On the image, it scales to some extent.

Even if $Q(x)$ is isotropic, these definitions can be construed in terms of distributions — the distribution μ is characterized by

$$\langle f, \mu\rangle = \int_V f(Q(x))\,dx\ .$$

I will not need that generalization here.

Lemma. *For any α in $(F^\times)^2$*

$$\mu(\alpha y) = |\alpha|_F^{(n-2)/2}\mu(y)\ .$$

We have by definition

$$\int_V f(Q(x))\,dx = \int_F f(y)\mu(y)\,dy$$

But then

$$\begin{aligned}\int_V f(\alpha^2 Q(x))\,dx &= \int_V f(Q(\alpha x))\,dx \\ &= |\alpha|_F^{-n}\int_V f(Q(u))\,du \quad (u=\alpha x, dx = |\alpha|_F^{-n}\,du) \\ &= \int_F f(\alpha^2 y)\mu(y)\,dy \\ &= |\alpha|_F^{-2}\int_F f(u)\mu(\alpha^{-2}u)\,du \\ \mu(\alpha^{-2}u) &= |\alpha|_F^{-n+2} \\ \mu(\alpha u) &= |\alpha|_F^{(n-2)/2}\end{aligned}$$

Let γ_Q be the Fourier transform of μ. Thus

$$\begin{aligned}\gamma_{Q,\psi}(\lambda) &= \int_F \psi(-\lambda y)\mu(y)\,dy \\ &= \int_V \psi(-\lambda Q(x))\,dx\ .\end{aligned}$$

It follows from the quasi-homogeneity that this integral is conditionally convergent for all $\lambda \neq 0$. Weil's formulation is that

$$\lim_{R\to\infty}\int_{\|v\|\le R}\psi(-\lambda Q(x))\,dx$$

exists for all $\lambda \neq 0$. This is valid for any local field, as indeed is my formulation.

Suppose now that E is a quadratic extension of F, $V = E$, $Q(x) = x\overline{x}$. The same argument as above shows that $\mu(\alpha y) = \mu(y)$ if α lies in the subgroup $\mathrm{Norm}(E^\times)$ of index two in $F^\times$. Therefore

$$\mu_{E/F}(y) = (1 + \mathrm{sgn}_E(y))$$

for all y in $F^\times$, if the measures on E and F are compatible. (Here and everywhere I am probably hopeless in getting things correct to more than some positive constant.)

Then $\gamma_{Q,\psi}$ is the Fourier transform of this sum, which is a δ function plus $\gamma(\mathrm{sgn}_E, \psi)\,|\lambda|^{-1}\mathrm{sgn}_E(\lambda)$, according to Tate's thesis. If $\lambda = 1$, this is also Langlands' factor $\lambda(E/F, \psi)$.

Explicitly:

$$\int_E \psi_F(-\lambda x\overline{x})\,dx = \lambda(E/F,\psi)|\lambda|^{-1}\mathrm{sgn}_E(\lambda)\ .$$

Suppose χ to be a character of $F^\times$. Recall that according to Tate

$$\gamma(\chi\overline{\chi}, \psi_E) = \int_E \psi_E(-x)\chi(x\overline{x})\,dx\ .$$

A formula due to Hasse if not to Gauss relates this to γ-factors for F. We rewrite the integral as

$$\gamma(\chi\overline{\chi}, \psi_E) = \int_F \chi(y)\,dy \int_{\mathrm{Norm}(x)=y} \psi_E(-x)\,dx$$

and represent χ by Fourier duality

$$\chi(y) = \int_F \widehat{\chi}(\lambda)\psi(\lambda y)\,d\lambda$$

to get

$$\begin{aligned}\gamma(\chi\overline{\chi}, \psi) &= \int_{F,F} \widehat{\chi}(\lambda)\,dy\,\psi(\lambda y)\,d\lambda \int_{\mathrm{Norm}(x)=y} \psi_E(-x)\,dx \\ &= \int_{F,E} \widehat{\chi}(\lambda)\psi_F(-x-\overline{x})\psi(\lambda x\overline{x})\,dx\,d\lambda\ .\end{aligned}$$

But

$$\psi(\lambda x\overline{x} - x - \overline{x}) = \psi(-1/\lambda)\psi(\lambda(x - 1/\lambda)(\overline{x} - 1/\lambda))$$

and

$$\widehat{\chi}(\lambda) = \gamma(\chi,\psi)\,|\lambda|^{-1}\chi^{-1}(\lambda)$$

so this becomes

$$\begin{aligned}&= \gamma(\chi,\psi)\int_F |\lambda|^{-1}\chi^{-1}(\lambda)\psi(-1/\lambda)d\lambda \int_E \psi(\lambda x\overline{x})\,dx \\ &= \gamma(\chi,\psi)\lambda(E/F,\psi)\mathrm{sgn}_{E/F}(-1)\int_F \chi(\kappa)\mathrm{sgn}_E(\kappa)\psi(-\kappa)\,d\kappa \quad (\kappa = 1/\lambda) \\ &= \gamma(\chi,\psi)\lambda(E/F,\psi)\mathrm{sgn}_{E/F}(-1)\gamma(\chi\,\mathrm{sgn}_E,\psi)\end{aligned}$$

which is a variant of Hasse's formula.

Actually I am not sure if my $\lambda(E/F, \psi)$ agrees with yours. It seems not to! Maybe you can correct what I have.