Geometrical symmetry and the fine structure of regular polyhedra

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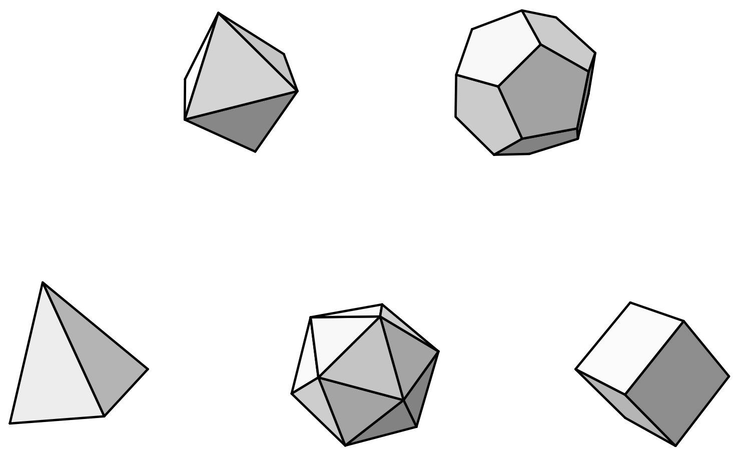
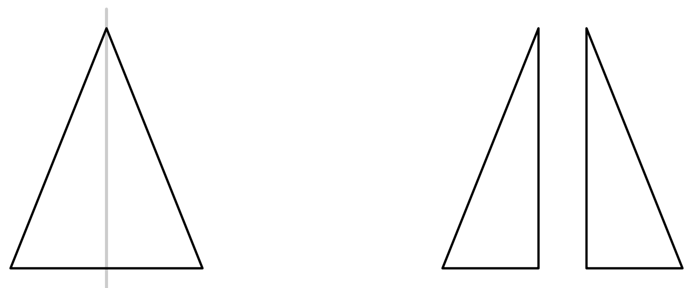
We shall be concerned with geometrical figures with a high degree of symmetry, in both $2D$ and $3D$. In $3D$ the most symmetrical figures are the five Platonic solids, which we shall see how to construct in the last section. There are many ways to do this, and many described in the literature, but the most satisfactory method is one which extends to a wide variety of regular figures of all kinds. This depends on understanding its symmetry transformations, which make up what is called a **Coxeter group**, generated by reflections of a particular kind.



1. Mathematical symmetry

In common English usage, the term *symmetry* seems to have meant at first the property of being *balanced* or *well-proportioned*. This original meaning continues in a slightly more technical sense in the phrases *bilateral symmetry* and *mirror symmetry* applied to a figure which looks the same as its image in a mirror. For example, the triangle on the left has mirror symmetry while the one on the right does not.

In effect, the reason we say the figure on the left has mirror symmetry is that we can slice it with a line to divide it into two halves which are congruent to one another, but with orientation reversed, as if reflected in a mirror. The line is called an **axis of symmetry** of the triangle.

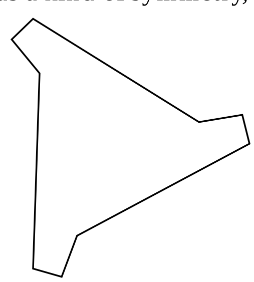


In mathematics, we distinguish between degrees of symmetry. For example, the equilateral triangle shown below has more symmetry than the triangles above.

To be precise, it has three axes of symmetry.

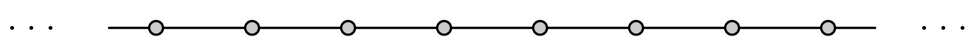
Exercise. *Is it true that every triangle with three axes of symmetry is equilateral? Explain.*

The kinds of symmetry a finite plane figure can have is very limited. *Exactly what kinds of symmetry can a figure have?* First of all, there are several elementary kinds of symmetry. We have already seen mirror or **reflection** symmetry. But the figure in this picture also has a kind of symmetry, which we shall call **rotational**:



That is to say, we can rotate this figure by 120° and we just get the same figure again. As the equilateral triangle demonstrates, a finite figure can have both reflection and rotation symmetry.

A figure can also have **translation** symmetry.



This requires that the figure be infinite in extent.

It is in fact possible to classify all the possible kinds of symmetry a plane figure can have, according to the set of all of its symmetries. *Here by a* **symmetry** *of a figure we mean any rigid transformation which takes the figure into itself.* I remind you that every rigid $2D$ transformation of a figure is one which doesn't distort it in any way—i.e. which preserves relative lengths and angles. Any rigid transformation can be obtained as the composition of a rotation or reflection with a translation.

From now on we shall restrict ourselves to figures which are of bounded size. This makes our discussion much simpler (and perhaps less interesting). In order to avoid technicalities we shall assume that the figure is the union of the interior of some finite region of the plane and its boundary.

There are two basic and self-evident principles we work with:

- **Composition principle.** *The composition of two symmetries of a figure, one applied after the other, is also a symmetry of the figure.*
- **Inversion principle.** *The inverse of any symmetry of a figure is also a symmetry of the figure.*

Any set of transformations of an object satisfying these conditions is called a **group**. We are examining the possible **symmetry groups** of geometrical figures.

These principles allow us to narrow down the possibilities quite a bit. Suppose that we are given a bounded figure with more than one symmetry. It is not too difficult to see that if it has an infinite number of symmetries then it has to be a circle. Suppose, then, that it has only a finite number of symmetries $s_1, s_2, \ldots, s_n$ (including the trivial one which doesn't transform anything). If P is any point of the figure, consider the vector average

$$\overline{P} = \frac{s_1(P) + s_2(P) + \cdots + s_N(P)}{n} .$$

which is also the centre of gravity of the n points. If we apply s_i to it, we get

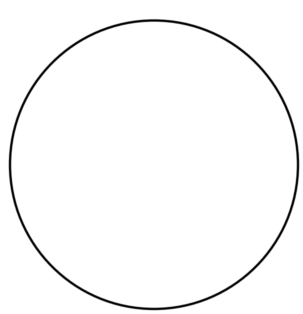
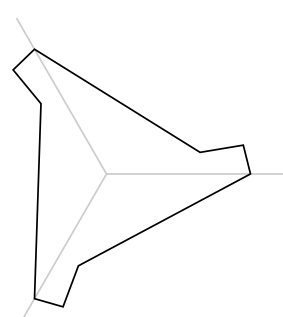
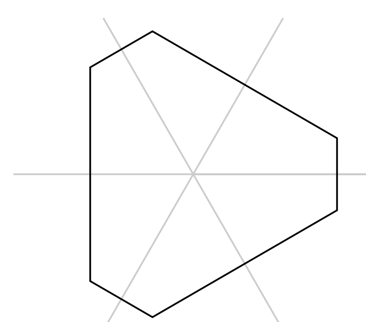
$$s_i\overline{P} = \frac{s_i s_1(P) + s_i s_2(P) + \cdots + s_i s_N(P)}{n} .$$

By the composition principle, each of the products is also a symmetry of the figure. These products are all different, for if $s_i s_j = s_i s_k$ then we can apply s_i^{-1} to both sides and see that $s_j = s_k$. But if the products are all different then since there are n of them they must range over the whole set of symmetries. The sum for $s_i\overline{P}$ is therefore over the same set as that defining $\overline{P}$. In other words

$$s_i\overline{P} = \overline{P} .$$

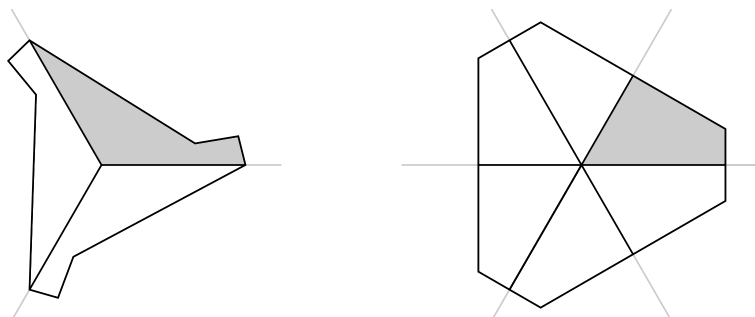
This tells us that *if a figure possesses only a finite number of symmetries then there is one point somewhere in the plane fixed by all of its symmetries.* I will not go into details here, but it is not difficult to deduce from this result that bounded figures, as far as their symmetry is concerned, fall into three classes: (1) total circular symmetry; (2) a finite degree of rotation symmetry; (3) a finite degree of mixed rotation and reflection symmetry. In the second category are those figures with no noticeable symmetry—for the only symmetry is the rotation by $0°$— and in the last category are those figures whose only non-trivial symmetry is a single reflection.

Here are pictures of examples from each of the three types:

The only figure with circular symmetry is . . . well . . . a circle. The rest of the figures will be polygons of various shapes. In the case of rotation symmetry, the figure will have n-fold rotation symmetry for some integer n, in which case it can be rotated into itself through $360^\circ/n$. If it has mixed symmetry, then it will be invariant under n rotations and n reflections for some integer n, a total of $2n$. The rotations are all generated by a single rotation through $360^\circ/n$, and the reflections will be through n lines evenly spaced at $360^\circ/n$ apart. As already mentioned, a figure can have no symmetry whatsoever, as a special case of rotation symmetry with $n = 1$.

Suppose a figure has N symmetry transformations in all (and is not a circle). It can always be partitioned into N distinct regions, each of which is congruent to the others. If we move the figure so that one of its axes of symmetry—if it possesses any—is the x-axis, then these regions are just the intersections of the figure with one of the sections $0 \le \theta < 360^\circ/N$. I shall call them the symmetry **chambers** of the figure.



There are exactly as many chambers in this partition as there are symmetries of the figure. Suppose we fix one of the pieces in the partition. Call it, say, C. If s is a symmetry transformation of the figure then s will transform C to one of the other pieces of the partition. In this way we specify an exact association between pieces of the partition and symmetries of the figure—that is to say, *as s ranges over all symmetries of the figure $s(C)$ ranges over all the pieces of the partition*.

This is one way of using geometry to classify the symmetries. Another is to describe the symmetries more directly in geometrical terms: to a figure of mixed symmetry with $2n$ symmetries n of them are rotations by multiples of $360^\circ/n$ and the other n are reflections in various axes of symmetry of the figure. These two ways of describing the symmetries are rather different in flavour. They are perhaps complementary. Both descriptions are valuable.

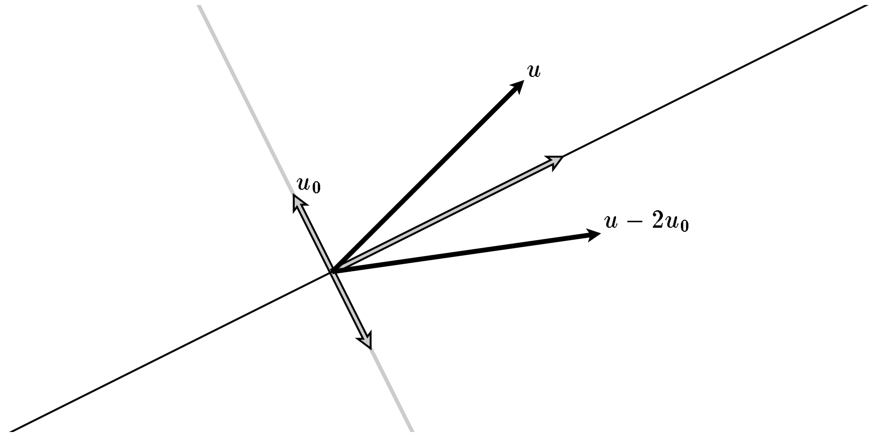
2. Reflections

The key to our construction of the Platonic solids is understanding certain reflection symmetries of these figures. In this section we shall look in detail at reflections.

A reflection is the mathematical way to transform something into its mirror image. In $2D$ a reflection is associated to a line, and in $3D$ it is associated to a plane. In both cases, we shall be concerned here only with reflections in lines or planes that pass through the origin. Such reflections are linear transformations.

In either $2D$ or $3D$ the object through which things are reflected is described by a single equation $f = 0$ where in $2D$ the linear function f has the form $f(x, y) = Ax + By$ and in $3D$ it has the form $f(x, y, z) = Ax + By + Cz$. In either case the function f specifies a vector α which is perpendicular to the reflection line or surface, since the equation can be read $\alpha \bullet v = 0$ where in $2D$ the vector α is (A, B) and in $3D$ it is (A, B, C).

Given α, how can we specify the reflection precisely?



Suppose we are given u and the vector α perpendicular to the line we are reflecting in. Let

$$u_0 = \left(\frac{u \bullet \alpha}{\alpha \bullet \alpha}\right) \alpha$$

be the projection of u along α. The figure shows that the reflection of u in the line perpendicular to α is

$$u - 2u_0 \ .$$

This formula works in any number of dimensions:

- *The reflection of u in $\alpha \bullet v = 0$ is*
$$r_\alpha u = u - 2\left(\frac{u \bullet \alpha}{\alpha \bullet \alpha}\right) \alpha \ .$$

It is often convenient to arrange that $\|\alpha\| = 1$. Suppose now that the line we reflecting in has angle θ with respect to the x-axis. The vector $(\cos\theta, \sin\theta)$ lies along this line, and to obtain α we rotate this by $90°$. We get

$$\alpha = [\,-\sin\theta \ \ \cos\theta\,] \ .$$

The matrix corresponding to the reflection is the one whose columns are the images of $(1,0)$ and $(0,1)$ with respect to the reflection. We get these columns to be

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2(-\sin\theta) \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 1 - 2\sin^2\theta \\ 2\sin\theta\cos\theta \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2(\cos\theta) \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 2\sin\theta\cos\theta \\ 1 - 2\cos^2\theta \end{bmatrix} .$$

By using the trigonometrical formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \ . \end{aligned}$$

we see that

- *The matrix for reflection in the line at angle θ with respect to the x-axis is*

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} .$$

As a check on this calculation, note that this matrix has determinant -1. Note also that θ and $\theta + 180°$ give the same matrix, as they should, since these two angles determine the same line of reflection. Finally, this matrix can be computed by a direct geometric argument.

Exercise. *Write a procedure* **`reflect`** *with one argument θ, which changes the CTM by reflecting in the line at angle θ with respect to the x-axis.*

Exercise. *Let v_0 be a point in $2D$. Find a formula for a rotation of angle θ around v_0. Check by showing that v_0 is fixed by your transformation.*

Exercise. *Show that every rigid motion in $2D$ which is not a translation is a rotation around some point. The main problem is to show that there exists some point fixed by the transformation. You should be able to calculate what it is.*

Exercise. *Given a line $Ax + By + C = 0$ write down a formula for reflection in this line.*

Exercise. *A* **sliding reflection** *in $2D$ reflects in some line called the axis of the transformation, and then slides along parallel to this axis. Given a line $Ax + By + C = 0$ and a shift distance a write down a formula for the combination of reflection in the line followed by a shift of a parallel to it.*

Exercise. *Show that every rigid transformation in $2D$ which is not a rigid motion is either a reflection through some line or a sliding reflection.*

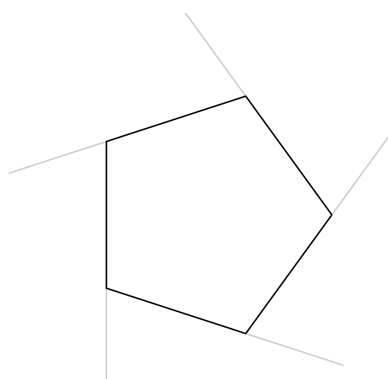
Exercise. *Find a formula for rotation around an axis $v_0 + t\tau$ in $3D$.*

Exercise. *A helical motion in $3D$ is a motion which twists and shifts around an axis all at the same time. That is to say it rotates around an axis and then shifts parallel to it. Show that every rigid motion in $3D$ which is not a translation or a rotation is a helical motion.*

Exercise. *Write down the parametrization for a helix in $3D$ winding around the z-axis and shifting a distance a in every coil.*

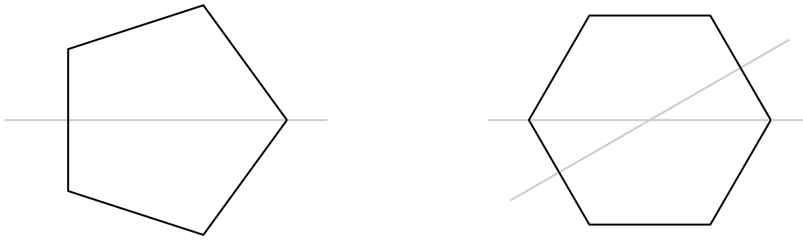
3. Regular polygons

We will in this section restrict our attention to **regular polygons**, that is to say polygons with as much symmetry as possible. All vertices will be at a fixed distance from the origin, and the lengths of all sides are the same.

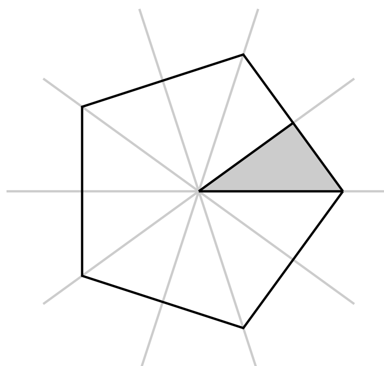


As we travel around the outside of the polygon of n sides, we make n turns for a total of $360°$. The internal angle at each corner is therefore $180° - 360°/n$.

If the figure has n sides, it has $2n$ symmetries in all—n rotations and n reflections. The geometry of the reflections is a bit different for even and odd values of n.

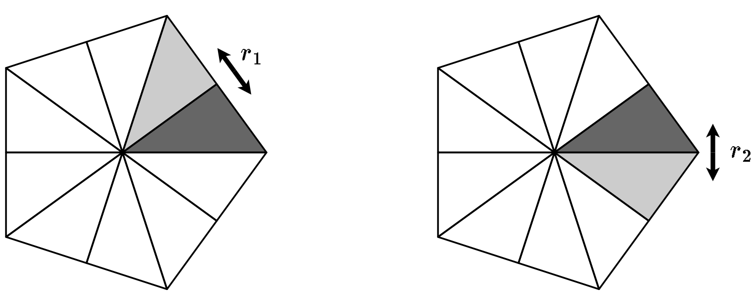


When n is odd, all reflections are through a line cutting through a vertex and the middle of a side, while if n is even there are two kinds of reflections, in lines through vertices and in lines bisecting sides. In either case, the partition of the polygon into chambers determined by its symmetry is into triangles with one vertex at the centre, one at a vertex of the polygon, and one in the middle of a side. These chambers have the properties (1) any two are equivalent with respect to symmetries of the figure, and (2) no symmetry of the figure is also a symmetry of a chamber.

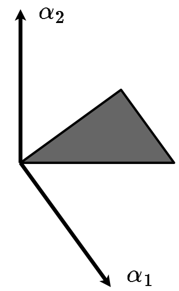


We can see easily from this why there are exactly $2n$ symmetries, 2 each for each of the n sides, since that's how the chambers are distributed.

The main reason why we are looking at regular polygons in $2D$ is because these polygons are the faces of the Platonic solids (regular polyhedra) in $3D$. There is one feature of the symmetries of a regular polygon which will be crucial in constructing the regular polyhedra. Fix a single chamber. The property I want to single out here is that *each of the lines bordering that chamber* (its sides or edges) *is an axis of symmetry for the figure*. To be able to discuss what is going on more precisely, label those sides. The side meeting the side of the polygon will be indexed by 1, the one meeting the vertex of the polygon will be labeled by 2. Similarly, let r_1 and r_2 be the reflections in those sides. The effect of r_1 is therefore to interchange the two chambers on the left, and the effect of r_2 is to interchange the two on the right.



Let α_1 and α_2 be the unit vectors perpendicular to the lines of reflection of r_1 and r_2. We choose the direction of α_i so that the chamber is on the same side of the line through the origin perpendicular to α_i as α_i is.



In our examples

$$\alpha_1 = (\sin\theta, -\cos\theta), \quad \alpha_2 = (0, 1)$$

where $\theta = \theta_n = 360^\circ/2n$. These explicit vectors depend on the particular orientation of the polygon, but the important properties are that both α_1 and α_2 have length 1 and that the angle between the two is $180^\circ - \theta_n$. More precisely:

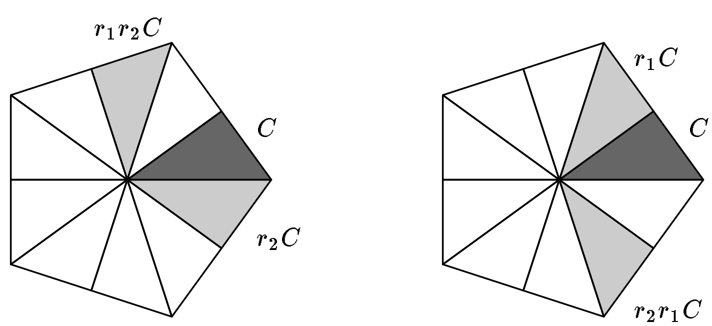
$$\alpha_1 \bullet \alpha_1 = 1$$
$$\alpha_2 \bullet \alpha_2 = 1$$
$$\alpha_1 \bullet \alpha_2 = -\cos(180^\circ/n)$$

Here is a table for small values of n:

n	$-\cos(180^\circ/n)$
2	0
3	$-1/2$
4	$-\sqrt{2}/2$
5	$-1/4 - \sqrt{5}/4$
6	$-\sqrt{3}/2$

Exercise. *In all these cases* $\cos(180^\circ/n)$ *involves a single radical, such as* $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$. *Are there any others?*

What happens if we apply several reflections alternately? If we apply $r_1 r_2$ to C (first r_2, then r_1) we get the figure on the left, and if we apply $r_2 r_1$ (first r_1, then r_2) we get that on the right.

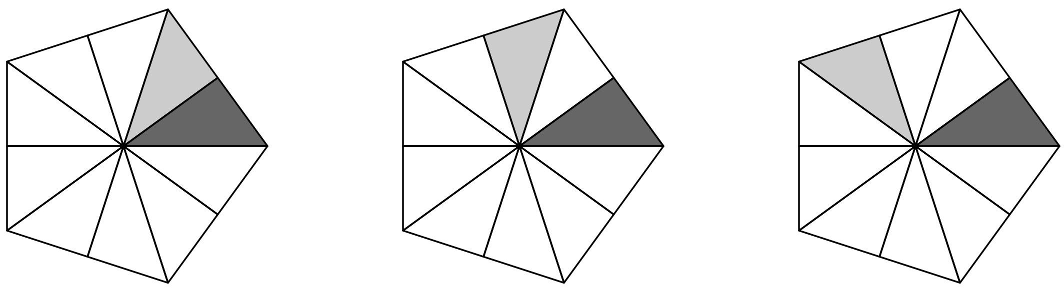


That is to say that $r_1 r_2$ amounts to rotation counter-clockwise by $360^\circ/n$, while $r_2 r_1$ amounts to a rotation by the same angle in the opposite direction. Let ρ be rotation by $360^\circ/n$. Then the rotation symmetries of the n-sided regular polygon are the transformation $1, \rho, \ldots, \rho^{n-1}$. All the compositions $r_2 \rho^m$ with $0 \le m < n$ have negative determinant, and in fact run through all reflection symmetries. This implies, among other things, that every one of its symmetries can be expressed as a product in some order of r_1 and r_2. We shall more about this in the next section.

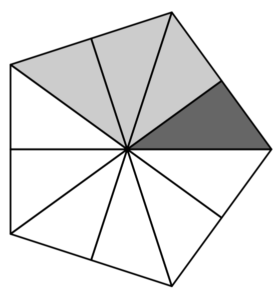
Exercise. *If r is a reflection through the x-axis and ρ a rotation through θ, describe the line of reflection for $r\rho$.*

4. Listing the symmetries of a regular polygon

The triangles in the figures above don't seem to be added in a geometrically readable fashion. Now, however, let's look at the sequences of chambers $r_1(C)$, $r_1r_2(C)$, $r_1r_2r_1(C)$:

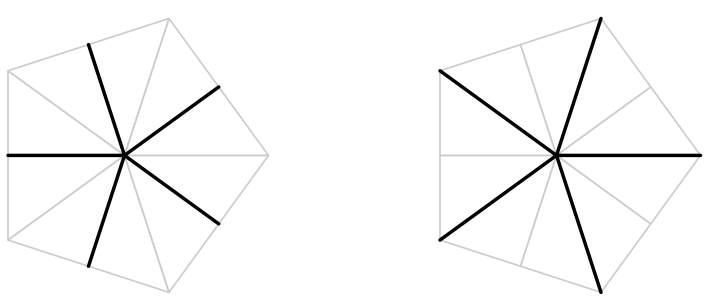


Here, the chambers $r_1(C)$, $r_1r_2(C)$, $r_1r_2r_1(C)$ form a geometrically connected **chain** of chambers, pictured all at once in this figure:



Exercise. *Write a* PostScript *program to draw the picture above. Incorporate routines which change the CTM by a sequence of reflections r_1 and r_2, where the sequence is given as an array of entries* 1 *and* 2.

Where are we heading? Suppose we are given a chain of chambers $C_0 = C$, C_1, C_2, …, C_n from C to a final chamber C_n. *How can we write down a string of reflections whose product takes C to C_n?* The answer is very simple, but not quite intuitive. First of all, label each radial segment in the polygon by either 1 or 2—the ones going to sides by 1, the ones going out to the vertices of the polygon by 2. This is consistent with our initial labeling. For example, on the left I have marked all the segments labeled 1 and on the right those labeled 2.



The main property of this labeling is that *the labels are preserved by the symmetries of the figure*: any symmetry will take a side of type 1 into another of type 1, and one of type 2 to another of type 2. The classification of sides into types is intrinsic to the geometry of the polygon.

- *Suppose we are given a chain of chambers $C_0 = C, C_1, C_2, \ldots, C_n$ from C to a final chamber C_n. Suppose the successive segments separating these are labeled $i_1, i_2, \ldots i_n$. If for each m we set*

$$r_m = r_{i_1} r_{i_2} \ldots r_{i_m}$$

then $r_m(C) = C_m$ for all m.

The point is that the geometrical chain of chambers relates directly to multiplication by the simple reflections r_1 and r_2. The unusual feature is that *we build the chain in the order opposite to that in which the reflections are applied*. The reason this is important is that it gives us a way to label all symmetries of the figure in a way that relates directly to geometry. I shall elaborate on this in a moment.

The proof of the assertion is elementary. Let $r = r_i$ with i either 1 or 2. The chambers C and $r(C)$ are separated by a side of type i. When we apply s to this pair, the pair $s(C)$ and $sr(C)$ we get is again separated by a side of type i, since the type of a side does not change when it is transformed by a symmetry.

Consider, for example, $r_2 r_1 r_2$. We start with the pair C, $r_2(C)$ separated by a side of type 1. Then we want to know that r_2 and $r_2 r_1(C)$ are separated by a side of type 1. But this is the pair we get by applying r_2 to the pair C, $r_1(C)$. Etc.

One consequence is another proof of this basic fact:

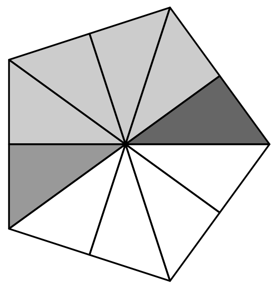
- *Every symmetry of a regular polygon can be expressed as a product of r_1 and r_2.*

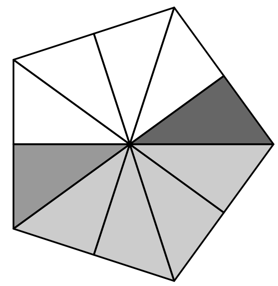
Explicitly, if s is a symmetry of the polygon, connect C and $s(C)$ by a chain of chambers. This chain corresponds to an expression for s in terms of r_1 and r_2.

We can in fact find a well defined and **unique** expression for every symmetry s, if we follow these rules: (1) We connect $s(C)$ and C by a chain of shortest length and write s in terms of that chain. (2) There is one symmetry s for which there are two chains of minimal length connecting $s(C)$ to C. For example if $N = 5$ then

$$r_1 r_2 r_1 r_2 r_1(C) = r_2 r_1 r_2 r_1 r_2(C)$$

as the picture shows.





In this case we choose as the expression for s the one which is least in what is called **inverse dictionary order**, namely $r_1r_2r_1r_2r_1$. This is because $r_2r_1r_2r_1r_2$ would come after $r_1r_2r_1r_2r_1$ in a dictionary where words were read backwards. In $2D$ this ordering is not important, but in $3D$ it helps us avoid chaos.

We can now make a list of all symmetries of a regular polygon. The order in which the elements are listed is to be in inverse dictionary order. For $N = 5$ our list is this (where we use the 'empty' expression $\emptyset$ for the trivial symmetry):

$\emptyset$
r_1
r_2r_1
$r_1r_2r_1$
$r_2r_1r_2r_1$
$r_1r_2r_1r_2r_1$
r_2
r_1r_2
$r_2r_1r_2$
$r_1r_2r_1r_2$

Exercise. *Let $n = 6$. Draw the chain for $r_1r_2r_1r_2$.*

Exercise. *List all the symmetries of the equilateral triangle, following the scheme above.*

Exercise. *List all the symmetries of the square, following the scheme above.*

5. Regular polyhedra

A **polyhedron** in three dimensions is any figure which is a union of plane polygonal figures forming its boundary. These polygonal surfaces are called the **faces** of the polyhedron. The sides of the faces are called the **edges** of the polyhedron, and their corners are called its **vertices**.

A **regular polyhedron** in $3D$ is a polyhedron all of whose faces, edges, and vertices look the same. It is enough to require that *every face is a regular polygon and that any two faces must be congruent.*

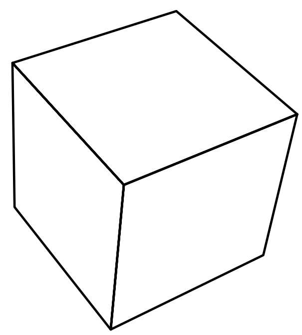
It was known very early to Greek mathematicians that there are only five regular polyhedra. As far as I know, the earliest surviving discussion from the classical Greek period is in Euclid's Book XIII, but the main facts were certainly known earlier. Several Greek books written shortly after Euclid's refer to earlier treatises apparently then still in existence. Three of the regular polyhedra (the tetrahedron, cube, octahedron) are relatively simple, but the remaining two (dodecahedron, icosahedron) are considerably more complicated. It is apparently not known exactly how they were first discovered. (Refer to the introduction by Heath to Book XIII in his edition of *Euclid*.)

We shall concern ourselves with two questions: (1) *Why are there exactly five regular polyhedra?* (2) *How can one construct them?* We interpret the second question as meaning: *Specify explicitly all of the vertices of the polyhedra.*

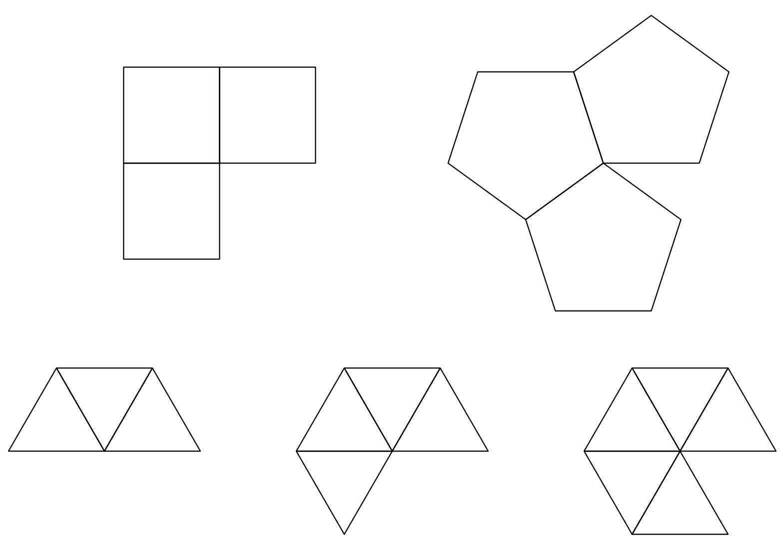
Exercise. *Answering the second question for the cube is trivial. The octahedron is not much more complicated, since it can be derived from the cube. The tetrahedron is not quite so simple, but still reasonably elementary. List all the vertices of a tetrahedron centred at the origin if one vertex is a* $(0,0,1)$ *and a second is of the form* $(x,0,z)$ *with* $x > 0$*.*

The first question will be dealt with in this section, the second in the next.

Suppose we have a regular polyhedron. Its faces will all be congruent regular polygons, say of m sides. Each of its vertices will look the same; let n be the number of faces surrounding each of them. For the cube, for example, $n = 3$ and $m = 4$.



Pick one of its vertices. Cut away from the polyhedron all of its faces except the ones touching this vertex. On each of these faces' sides the angle between two neighbouring edges will be $180^\circ - 360^\circ/m$. I now claim that as I cycle around the vertex adding up the angles I meet on the faces, I have to get a total of less than 360°. Roughly speaking, this claim amounts to the assertion that if we cut out the faces of a regular polyhedron around a given vertex and then flatten them, we get one of the following figures:



To be precise, the claim implies that

$$180 - 360/m < 360/n, \quad 180 < 360\left(\frac{1}{m} + \frac{1}{n}\right), \quad \frac{1}{2} < \frac{1}{m} + \frac{1}{n} .$$

The integers m and n must be at least 3, since our figure is assumed to be genuinely three-dimensional. The possibilities are hence quite limited, as the following table shows with more precision than the figure above.

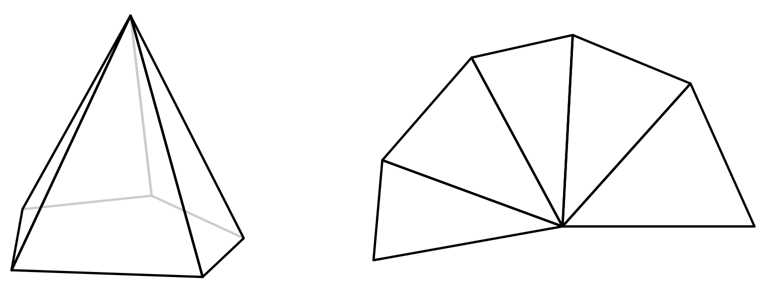
	m	n	$1/m + 1/n$
•	3	3	2/3
•		4	7/12
•		5	8/15
		6	1/2
•	4	3	7/12
		4	1/2
•	5	3	8/15
		4	9/20
	6	3	1/2

In particular, we cannot have m or n greater than 5. Thus we get five possibilities in all. There is some subtle logic here—*this argument asserts that there at most* 5 *possibilities, but it does not guarantee that each possibility is actually realized by a regular polyhedron.* It is the construction in the next section that will do that.

The claim I have made is a special case of a much more general result about arbitrary convex polyhedra.

- *For any convex polyhedron, the sum of angles on the faces around any of its vertices is less than* 360°*.*

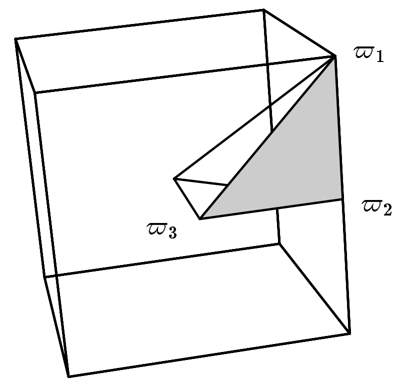
The term *convex* means, roughly, that the polyhedron bends outward at all vertices and edges. As before, what the result means is that if we are given something like the cone on the left, then if we cut it and flatten it out we get the figure on the right:



This result is almost intuitively true. However, as far as I can see, this intuition is based only on special cases which do not cover all possibilities. This result is, at any rate, proven in Euclid (Book XI, Propositions 20 and 21), and it is used exactly as we are using it in the classification of regular polyhedra. It is also possible to describe in geometric terms the **defect** of the vertex (360° less the sum of the vertex angles), but that is another story.

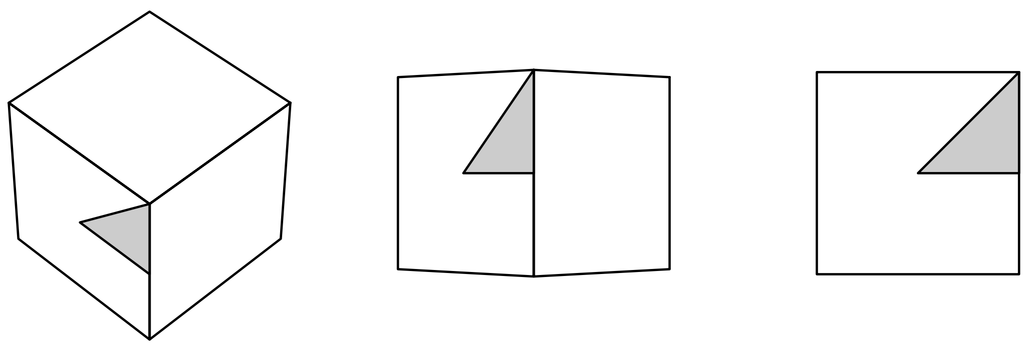
6. Construction

Suppose we are given a regular polyhedron. We can partition its surface into chambers, just as we partitioned a regular polyhedron into chambers. In fact, each chamber of the polyhedron will be a chamber of one of the regular polygons making up the polyhedron's faces. Fix one of the chambers, call it C. We can extend it into the third dimension by joining its vertices to the origin O. Let ϖ_1 be the vertex at a corner of the polyhedron, ϖ_2 in the middle of an edge, ϖ_3 in the centre of a face.

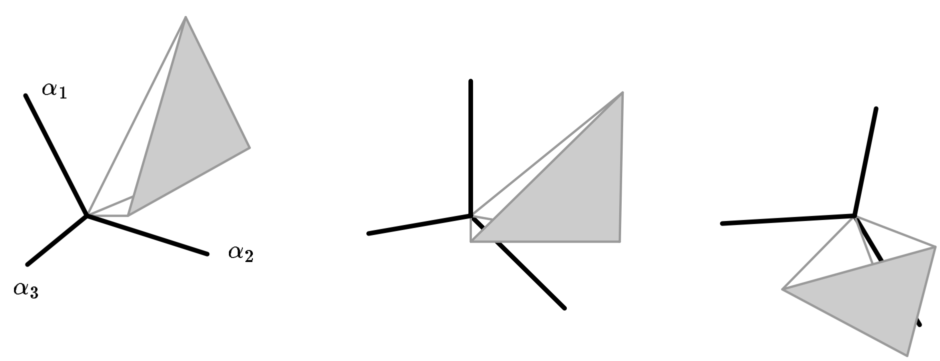


Each one of the ϖ_i gives rise to an integer n_i, half the number of chambers surrounding the point ϖ_i. For example, for the cube we have

$$n_1 = 3, \quad n_2 = 2, \quad n_3 = 4 .$$



I now want to introduce three vectors $\alpha_1, \alpha_2, \alpha_3$. Each of them will have length 1. The vector α_1 be perpendicular to the face $O\varpi_2\varpi_3$, and it will be on the same side of this plane as the chamber C. Similarly, α_2 be perpendicular to the face $O\varpi_1\varpi_3$, and on the same side of this face as C. And α_3 be perpendicular to the face $O\varpi_1\varpi_2$, on the same side as C.



More precisely, in all cases we have the rules

$$\|\alpha_i\| = 1$$
$$\alpha_i \bullet \varpi_j = 0 \text{ if } j \neq i$$
$$\alpha_i \bullet \varpi_i > 0$$

which determine the α_i completely. We can also see what the angles between the various α_i must be. For example, α_1 and α_2 are both perpendicular to ϖ_3. Therefore if we project them onto a plane perpendicular to ϖ_3 they preserve their length. But if we project the polyhedron onto that plane we just get a regular polygon with n_3 sides and ϖ_3 as centre. We can apply what we said in a previous section about the α in that case:

$$\alpha_1 \bullet \alpha_2 = \cos(180^\circ - 180^\circ/n_3) \ .$$

Similarly, if we project the polyhedron onto a plane perpendicular to ϖ_1, the configuration of ϖ_1 and its neighbouring vertices projects to a regular polygon of n_1 sides, and hence all in all

$$\alpha_2 \bullet \alpha_3 = \cos(180^\circ - 180^\circ/n_1) \ .$$

It is also simple to see that

$$\alpha_1 \bullet \alpha_3 = 0$$

All in all

$$\begin{aligned} \alpha_1 \bullet \alpha_2 &= -\cos(180^\circ/n_3) \\ \alpha_1 \bullet \alpha_3 &= \quad 0 \\ \alpha_2 \bullet \alpha_3 &= -\cos(180^\circ/n_1) \end{aligned}$$

The real point of all this work is that

- *These equations, a choice of scale and orientation of the polyhedron, and a choice of the n_i allow us to calculate the α_i explicitly.*
- *From the α_i we can calculate the ϖ_i.*
- *From these and some reasoning about the symmetries of the polyhedron we can construct all the faces.*

We shall deal with these in order. We will take the radius of the polyhedron to be 1.

• Fix the vector α_1 to be at $(1, 0, 0)$. We may assume the vector α_2 to be in the (x, y) plane with $y > 0$, and the vector α_3 to lie in the region $z > 0$. This only amounts to an alignment of the polyhedron.

We start with

$$\alpha_1 = (1, 0, 0) \ .$$

The angle between α_1 and α_2 is equal to $180^\circ - 180^\circ/n_3$. We also know that $\|\alpha_2\| = 1$. Therefore

$$\alpha_2 = \big(-\cos(180^\circ/n_3), \sin(180^\circ/n_3), 0\big) = (x_2, y_2, 0) \ .$$

Note that $y_2 > 0$.

Let

$$\alpha_3 = (x_3, y_3, z_3) \ .$$

Recall that $n_2 = 2$ in all cases. Thus

$$\begin{aligned} \alpha_1 \bullet \alpha_3 &= -\cos(180^\circ/n_2) \\ &= x_3 \\ \alpha_2 \bullet \alpha_3 &= -\cos(180^\circ/n_1) \\ &= x_2 x_3 + y_2 y_3 \end{aligned}$$

so that

$$\begin{aligned} x_3 &= 0 \\ y_3 &= -\cos(180^\circ/n_1)/y_2 \\ z_3 &= \sqrt{1 - x_3^2 - y_3^2}\,. \end{aligned}$$

• How to locate the vectors ϖ_i? Let η_i be three vectors such that

$$\alpha_i \bullet \eta_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then on the one hand, since ϖ_i satisfies the same conditions of perpendicularity, and $\varpi_i \bullet \alpha_i > 0$, ϖ_i is a positive multiple of η_i. On the other we can find the η_i in a very simple way. The equations for the η_i assert that they are the columns of the matrix N such that

$${}^t\!A\,N = I$$

so that

$$N = {}^t\!A^{-1}\,.$$

To find ϖ_i we just have to scale η_i correctly. For $i = 1$ this is simple, because $\|\varpi_1\| = 1$. For $i = 2$ or 3 let $\theta_{1,i}$ be the angle between η_1 and η_i, which is the same as the angle between ϖ_1 and ϖ_i. Then the length of ϖ_i is $\cos\theta_{1,i}$.

• We now know the points ϖ_1, ϖ_2, and ϖ_3 on a single face containing the chamber C. Its centre is ϖ_3, and ϖ_1 is one of its vertices. Let r_i be reflection through the plane perpendicular to α_i. Thus r_1 and r_2 both leave ϖ_3 fixed and fix as well the face containing it. We get the other vertices on this face by applying the symmetries generated by r_1 and r_2, which we know how to list because of our the results in $2D$. The other faces of the polygon are what we get by applying various symmetries of the polyhedron to this one face. Therefore we have to learn something about the symmetries of the various polyhedra.

The reflection r_3 will reflect this face into some other one. It will also reflect the chamber C into some other chamber on this face. It is true here as in $2D$ that the number of symmetries is the same as the number of chambers. The total number of chambers is the product of the number of chambers on one face and the number of faces. We have the following table:

Type	Number of faces	Type of face	Number of symmetries
tetrahedron	4	triangle	24
cube	6	square	48
octahedron	8	triangle	48
dodecahedron	12	pentagon	120
icosahedron	20	triangle	120

Symmetries correspond to chambers. If we are given a chamber then we can construct a path from it to the chamber C, and corresponding to this path is an expression as a product of the r_i. The r_i are listed from left to right in the order in which edges are crossed, and r_i is inserted in an edge of type i is crossed. In $3D$ we label the edges much as we did for $2D$—the edge through which α_i reflects is labeled by i. For each face this gives a different indexing from the one we assigned earlier.

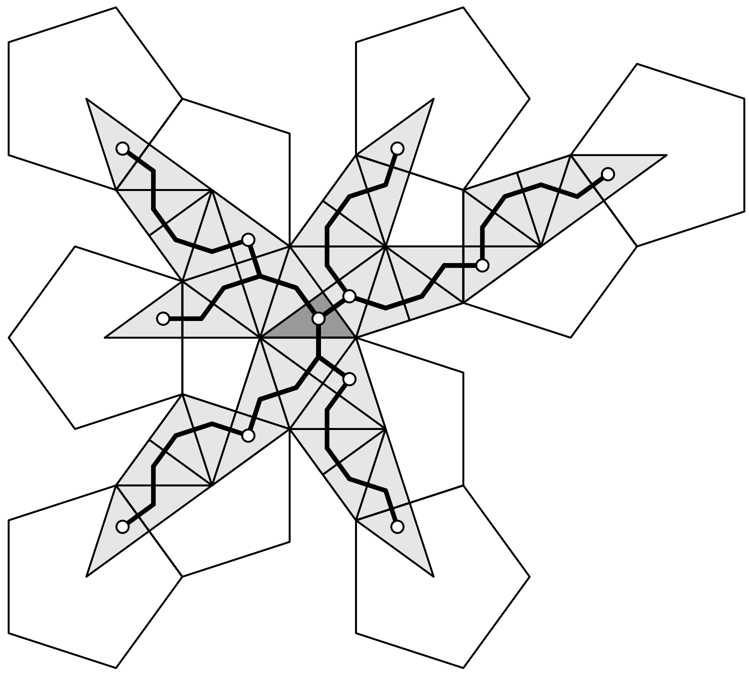
To each symmetry we can assign a unique expression as a product of the r_i according to these rules: (1) We choose the shortest expression if it is unique. This corresponds to a shortest path among the chambers. (2) If there are several shortest paths, we choose the one least in the inverse dictionary order. For each face there will exist a shortest one of these expressions taking our original face into it.

We get these lists generating all the faces. Of course all these expressions have to end in r_3.

Type	Symmetries defining faces
tetrahedron	$\emptyset$
	r_3
	r_2r_3
	$r_1r_2r_3$
cube	$\emptyset$
	r_3
	r_2r_3
	$r_1r_2r_3$
	$r_2r_1r_2r_3$
	$r_3r_2r_1r_2r_3$
octahedron	$\emptyset$
	r_3
	r_2r_3
	$r_1r_2r_3$
	$r_3r_1r_2r_3$
	$r_2r_3r_1r_2r_3$
	$r_3r_2r_3r_1r_2r_3$
	$r_3r_2r_3$
dodecahedron	$\emptyset$
	r_3
	r_2r_3
	$r_1r_2r_3$
	$r_2r_1r_2r_3$
	$r_1r_2r_1r_2r_3$
	$r_3r_1r_2r_1r_2r_3$
	$r_2r_3r_1r_2r_1r_2r_3$
	$r_1r_2r_3r_1r_2r_1r_2r_3$
	$r_2r_1r_2r_3r_1r_2r_1r_2r_3$
	$r_3r_2r_1r_2r_3r_1r_2r_1r_2r_3$
	$r_3r_2r_1r_2r_3$
icosahedron	$\emptyset$
	r_3
	r_2r_3
	$r_1r_2r_3$
	$r_3r_1r_2r_3$
	$r_2r_3r_1r_2r_3$
	$r_1r_2r_3r_1r_2r_3$
	$r_3r_1r_2r_3r_1r_2r_3$
	$r_2r_3r_1r_2r_3r_1r_2r_3$
	$r_1r_2r_3r_1r_2r_3r_1r_2r_3$
	$r_3r_1r_2r_3r_1r_2r_3r_1r_2r_3$
	$r_2r_3r_1r_2r_3r_1r_2r_3r_1r_2r_3$
	$r_3r_2r_3r_1r_2r_3r_1r_2r_3r_1r_2r_3$
	$r_3r_2r_3r_1r_2r_3r_1r_2r_3$
	$r_3r_2r_3r_1r_2r_3$
	$r_2r_3r_2r_3r_1r_2r_3$
	$r_1r_2r_3r_2r_3r_1r_2r_3$
	$r_3r_2r_3$
	$r_2r_3r_2r_3$
	$r_1r_2r_3r_2r_3$

In all cases, if we are given this list of expressions we can recreate the full list of symmetries in a simple fashion—we just tack onto these expressions the strings representing the symmetries of the original face.

In the following figure, these paths are shown in the case of the dodecahedron.



7. Final remarks

One can construct regular polyhedra in higher dimensions as well, although it is nearly impossible to picture them. As Coxeter's book *Regular Polytopes* explains, the most interesting things happen in dimension 4. One can also classify regular figures in affine and non-Euclidean geometry. The secret to understanding the construction of every regular figure in every case is again its symmetry group. It is always generated by reflections with certain products equal to two-dimensional rotations of certain kinds.

8. References

1. Jim Blinn, 'The Three-Dimensional Kaleidoscope', Chapter 9 in *Jim Blinn's Corner*. Morgan Kauffmann, San Francisco, 1996. From the author's column in the September, 1988 issue of *Computer graphics and applications*. The Platonic solids are mentioned in many places in the literature of computer graphics, but this is the only place I have seen the symmetry group dealt with. What I call a *chamber* is called by Blinn a *seed triangle*. Quite reasonably, since it grows to be a tree.

2. Harold S. M. Coxeter, *Regular polytopes*. Dover, 1973. Difficult to read, but nonetheless a classic.

3. *The thirteen books of Euclid's Elements*, edited with commentary by T. L. Heath. Second edition, Dover, 1956. The comments after Propositions XI.20–21, and before Book XIII are especially relevant.

4. Branko Grünbaum, 'Geometry strikes again', Mathematics Magazine, **58** (1988), 13–18. Fine examples of poor pictures in the mathematical literature. The MAA has cleaned up its logo since then, but pictures in its journals continue to be of dubious quality. It is arguable that they need a special graphics editor.