Mathematics 308—Fall 1996

Curves

So far we have seen two ways to draw paths in PostScript—line segments and arcs of a circle. PostScript allows a third method which is much more versatile. Of course one way to draw complicated curves is by drawing a sequence of line segments, but this usually requires a very large number of segments to be at all acceptable, and is also not very scalable. It is better, if possible, to produce smooth curves–at least as smooth as the physical device at hand will allow. PostScript does this with **Bezier cubic curves**.

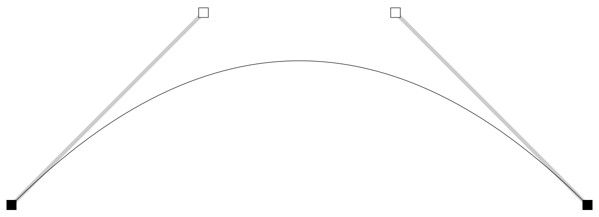
This is the last major component of PostScript you will have to learn. In the rest of this course we shall learn how to manipulate tools we have already been introduced to.

1. Bezier curves

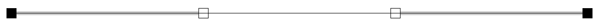
In PostScript, to add a curved path to a path already begun, you put in a command sequence like

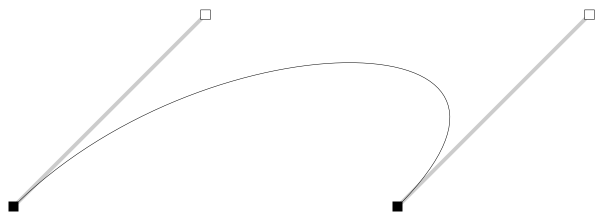
```
1 1
2 1
3 0
curveto
```

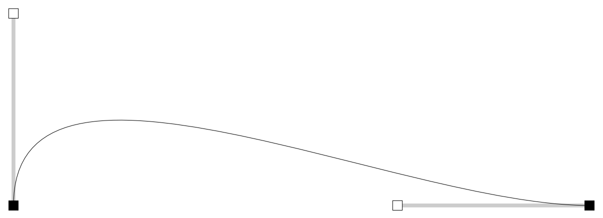
This makes a curve starting at the current point P_0, ending at $P_3 = (3, 0)$, and in between following a path controlled by the intermediate points $P_1 = (1, 1)$ and $P_2 = (2, 1)$. If we want to start at $P_0 = (0, 0)$ we would therefore preface this by a command `0 0 moveto`. What we get is this (where the four relevant points are marked):

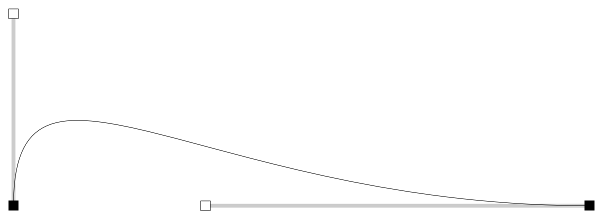


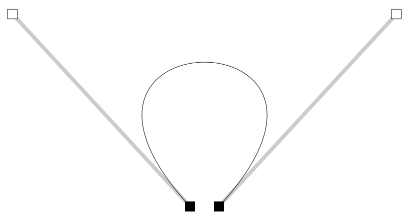
In these notes I shall call P_0 and P_3 the **end points** and P_1 and P_2 the **control points** of the curve. In order to know how to draw curves effectively, we have to understand how the control points determine the curve. We shall see later the exact mathematics of what is going on, but right now I shall simply exhibit several examples.

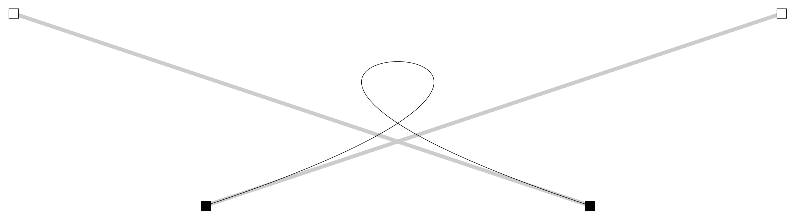


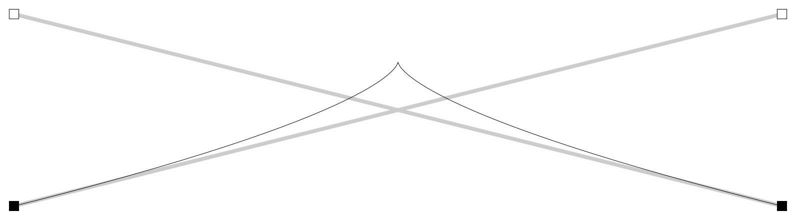






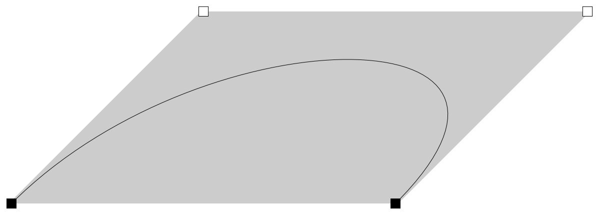






You should be able to see from these examples that the use of control points to specify curves is rather intuitive.

The following facts may give some feeling for how things go: (1) The path starts at P_0 and ends at P_3. (2) When the curve starts out from P_0 it is heading straight for P_1. (3) Similarly, when it arrives at P_3 it is coming from the direction of P_2. (4) If we wrap up the four points P_i in a quadrilateral box, then the whole curve is contained inside that box.



(4) The longer the line from P_0 to P_1, the tighter the curve sticks to that line when it starts out from P_0. Similarly for P_2 and P_3.

Curves drawn by using control points in this way are called **Bezier curves** after a twentieth century French automobile designer who first used them extensively in computer graphics, even though their use in mathematics under the name of **cubic interpolation curves** is much older.

One natural feature of Bezier curves described by control points is that they are stable under arbitrary affine transformations—that is to say that the affine transformation of a Bezier curve is the Bezier curve defined by the affine transformations of its control points.

Exercise 1.1. *Write a PostScript procedure* `pixelcurve` *with arguments* 4 *arrays* P_0, P_1, P_2, P_3 *of size* 2, *with the effect of drawing the corresponding Bezier curve, including also black pixels of width* 0.05” *at each of these points.*

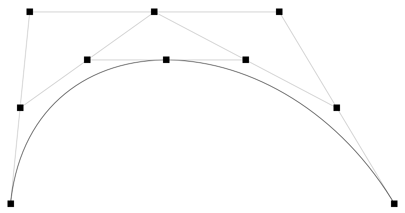
2. How the computer draws Bezier curves

These properties may become clearer if I explain how the computer actually goes about drawing the curve. First of all, it ‘thinks of’ any path as a succession of small points (pixels) on the particular device it is dealing with. This is somewhat easier to see on a computer screen, certainly if you use a magnifying glass, but remains true even of the highest resolution printers. So in order to draw something it just has to decide which pixels to color. It does this by an elegant recursive procedure, something akin to the following way to draw a straight line segment: (1) Color the pixels at each end. (2) Color the pixel at the middle. (3) This divides the segment into two halves. Apply steps (2) and (3) again to each of them. And so on, until the segments you are looking at are so small that they cannot be distinguished from pixels.

The analogous construction for Bezier curves goes like this:

Start with a Bezier curve, say with control points P_0, $P_{1/3}$, $P_{2/3}$, P_1, and perform the following construction. Set

$$\begin{aligned}
P_{1/6} &= \text{the median between } P_0 \text{ and } P_{1/3} \\
P_{5/6} &= \text{the median between } P_{2/3} \text{ and } P_1 \\
Q &= \text{the median between } P_{1/3} \text{ and } P_{2/3} \\
P_{2/6} &= \text{the median between } P_{1/6} \text{ and } Q \\
P_{4/6} &= \text{the median between } Q \text{ and } P_{5/6} \\
P_{1/2} &= \text{the median between } P_{2/6} \text{ and } P_{4/6}
\end{aligned}$$



Then the Bezier curve can be split into two halves, each of which is itself a Bezier cubic. The points P_0, $P_{1/6}$, $P_{2/6}$, and $P_{1/2}$ are the control points for the first half (and in particular $P_{1/2}$ lies on the curve); similarly, $P_{1/2}$, $P_{4/6}$, $P_{5/6}$ are control points for the second half. If we keep subdividing in this way we get a sequence of midpoints for the smaller segments (actually a kind of branched list), and to draw the curve we just plot these points after the curve has been subdivided far enough. This sort of subdivision can be done very rapidly by a computer, and in fact drawing the pixels to go on a straight line is not a great deal faster.

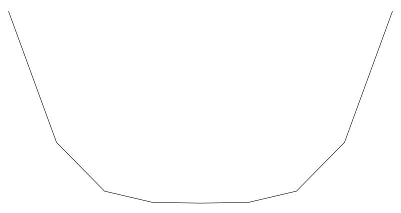
At any rate, all the properties (1) - (4) of Bezier curves can be seen easily from this construction.

Exercise 2.1. *Label the points P_* and Q yourself in the figure above, picking the left end as P_0 and the right as P_1.*

Exercise 2.2. *The point $P_{1/6}$ is $(1/2)P_0 + (1/2)P_{1/3}$. Find similar expressions for all the constructed points in terms of the original four.*

3. The mathematics of Bezier curves

I shall begin by looking at a reasonable problem in mathematical graphics. Suppose that you would like to draw the graph of $y = x^4$ in the range $x = -1$ to $x = 1$. One way to do this would be to draw it as a sequence of lots of straight lines. Here I used 8 segments.



The overall shape is not too bad, but the breaks are quite visible. You can certainly improve the quality of the curve by using more segments, but then the number of segments required to satisfy the eye changes with the scale used in representing the curve. The trouble is that the human eye can perceive that the directions (or first derivatives) vary discontinuously in this scheme. This is not at all something to be taken for granted—the more we learn about eyes in nature the more we learn that most features like this are ‘hard-wired’. (In contrast, the human eye apparently has trouble perceiving discontinuities in the second derivative.) So we would like to require that the tangent direction of the curve vary continuously as well. We can do this if we approximate the curve by **cubic segments** or in other words by piecewise cubic parametrizations.

A line segment is determined by its endpoints. A cubic segment is determined by specifying both position and velocity at the ends. In other words, any parametrization of a path associates to every value of the parameter t a point $(x(t), y(t))$ and a velocity $(x'(t), y'(t))$. To approximate a path by cubic segments we pick a certain number of values of the parameter, and then draw a cubic path between any two successive values by specifying that it agree in both position and velocity at the ends of the path. Mathematically, the technical basis for this construction is this result:

- *Given t_0, t_1, y_0, y_1, v_0, v_1, there exists a unique cubic polynomial $P(t)$ such that*

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= v_0 \\ y(t_1) &= y_1 \\ y'(t_1) &= v_1 \ . \end{aligned}$$

If

$$P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

then the conditions on $P(t)$ set up four equations in the four unknowns a_i which turn out to have a unique solution (assuming of course that $t_0 \neq t_1$. Here are the equations:

$$\begin{array}{ccccccccc} a_0 & + & a_1 t_0 & + & a_2 t_0^2 & + & a_3 t_0^3 & = & y_0 \\ & & a_1 & + & 2a_2 t_0 & + & 3a_3 t_0^2 & = & v_0 \\ a_0 & + & a_1 t_1 & + & a_2 t_1^2 & + & a_3 t_1^3 & = & y_1 \\ & & a_1 & + & 2a_2 t_1 & + & 3a_3 t_1^2 & = & v_1 \end{array}$$

The coefficient matrix is

$$\begin{array}{cccc} 1 & t_0 & t_0^2 & t_0^3 \\ & 1 & 2t_0 & 3t_0^2 \\ 1 & t_1 & t_1^2 & t_1^3 \\ & 1 & 2t_1 & 3t_1^2 \end{array}$$

One thing that simplifies the mathematics quite a bit is to **normalize** the parameter variable t, so that instead of going from t_0 to t_1 it goes from 0 to 1. We can do this by defining a new parameter variable

$$s = \frac{t - t_0}{t_1 - t_0} \ .$$

Note that s takes values 0 and 1 at the ends $t = t_0$ and $t = t_1$. Changing the parameter varible in this way doesn't affect the curve traversed. It simplifies the assertion above.

- *Given y_0, y_1, v_0, v_1, there exists a unique cubic polynomial $P(t)$ such that*

$$\begin{aligned} y(0) &= y_0 \\ y'(0) &= v_0 \\ y(1) &= y_1 \\ y'(1) &= v_1 \ . \end{aligned}$$

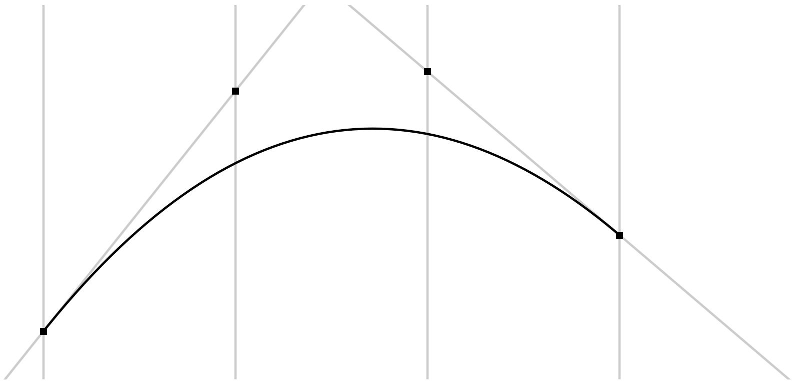
The coefficient matrix is now

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}$$

I leave it to you as an exercise to check now by direct row reduction that the determinant is not zero. One can actually solve this system of equations explicitly, but I will do something else to write down an explicit formula for the cubic we are looking for, because it is closely related to the way PostScript handles this business.

Return to unnormalized parameters.

Given the data t_0 etc. as given above I define $y_{1/3}$ to be the y coordinate of the point where the line at (t_0, y_0) with slope v_0 crosses the vertical line one third of the way from t_0 to t_1. Similarly I define $y_{2/3}$ to be the y coordinate of the point where the line at (t_1, y_1) with slope v_1 crosses the vertical line two thirds of the way across.



If we let Δt be the width of the span from left to right, then elementary geometry tells us that

$$\begin{aligned} \Delta t &= t_1 - t_0 \\ y_{1/3} &= y_0 + \frac{\Delta t}{3} v_0 \\ y_{2/3} &= y_1 - \frac{\Delta t}{3} v_1 \ . \end{aligned}$$

These are in a very rough fashion the **interpolated values** of y we might guess at for the values $1/3$ and $2/3$ of the way across the interval, on the basis of the data given at the endpoints.

Return to the normalized parameter s. Now $\Delta s = 1$, which makes formulas simpler.

- *In terms of the variable s, the cubic polynomial we are looking for is now*

$$P(s) = (1-s)^3 y_0 + 3s(1-s)^2 y_{1/3} + 3s^2(1-s) y_{2/3} + s^3 y_1$$

We can verify immediately that

$$\begin{aligned} P(0) &= y_0 \\ P(1) &= y_1 \ . \end{aligned}$$

We can also calculate that

$$\begin{aligned} P'(s) &= -6(1-s)^2 y_0 + 3(1-s)^2 y_{1/3} - 6s(1-s) y_{1/3} + 6s(1-s) y_{2/3} - 3s^2 y_{2/3} + 3s^2 y_1 \\ &= 3\big((1-s)^2(y_{1/3} - y_0) + 2s(1-s)(y_{2/3} - y_{1/3}) + s^2(y_1 - y_{2/3})\big) \\ &= 3\big((1-s)^2 z_0 + 2s(1-s) z_1 + s^2 z_2\big) \end{aligned}$$

where

$$\begin{aligned} z_0 &= y_{1/3} - y_0 \\ z_1 &= y_{2/3} - y_{1/3} \\ z_2 &= y_1 - y_{2/3} \ . \end{aligned}$$

Thus

$$\begin{aligned} P'(0) &= 3(y_{1/3} - y_0) \\ &= v_0 \\ P'(1) &= 3(y_1 - y_{2/3}) \\ &= v_3 \end{aligned}$$

Let

$$\begin{aligned} P_0 &= (t_0, y_0) \\ P_{1/3} &= (t_{1/3}, y_{1/3}) \\ P_{2/3} &= (t_{2/3}, y_{2/3}) \\ P_1 &= (t_1, y_1) \end{aligned}$$

The essential mathematics of Bezier curves is contained in the following statement:

- *The graph of the cubic $y = P(t)$ determined by these formulas is exactly the Bezier curve determined by the four points P_0, $P_{1/3}$, $P_{2/3}$, P_1.*

This result explains one of the facts about Bezier curves we noticed earlier. It is one way in which Bezier curves are analogous to line segments. A line segment between two points is made up of the points in the plane which are weighted combinations of its endpoints. The point t of the way from P_0 to P_1 will be (in vector notation)

$$(1-t)P_0 + tP_1 \ .$$

For example the mid-point of the segment is

$$\frac{P_0 + P_1}{2}$$

and the point one third of the way is

$$\frac{2P_0 + P_1}{3}$$

which is to say that it is a weighted combination of the endpoints with P_0 given twice as much weight as P_1.

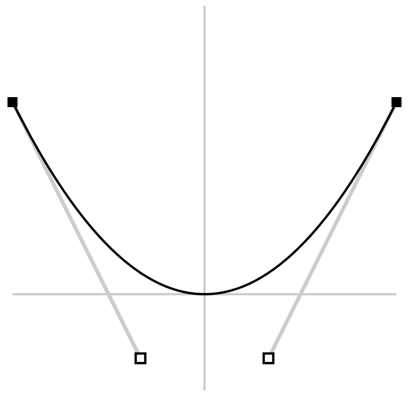
Every Bezier segment is therefore contained within the quadrilateral whose outside corners are the control points of the segment. This is because the coefficients t^3 etc. that appear in the formula all add up to 1 and are non-negative, so that each point on the segment is a kind of weighted average of the four control points of the segment. The reason the coefficients all add up to 1 is because of the binomial formula of degree 3, which says that

$$(a+b)^3 = a^3 + 3\,a^2b + 2ab^2 + b^3 \ .$$

Here we set $a = 1 - t$, $b = t$.

This allows us to use Bezier curves (equivalently, the PostScript command **`curveto`**) to draw an approximation to the graph of any function $y = f(x)$ between x values x_0 and x_f by choosing a sequence $x_0, x_1, \ldots, x_n = x_f$ of x values reasonably close together and using the values of $f(x_m)$ and $f'(x_m)$ to determine the necessary control points in between. This construction guarantees that these cubic segments will have matched slopes at the endpoints.

Let's see how we would draw the graph of the parabola, for example, between $x = -1$ and $x = 1$.



We have the endpoints

$$\begin{aligned} P_0 &= (-1, 1) \\ P_1 &= (\ \ 1, 1)\ . \end{aligned}$$

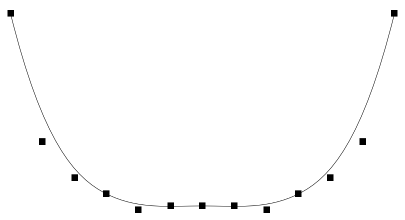
The interval is $\Delta t = 2$. The values of $x_{1/3}$ and $x_{2/3}$ are $-1/3$ and $1/3$. The slope at the left is -2, that at the right is 2. Therefore

$$\begin{aligned} y_{1/3} &= 1 + (1/3)(2)(-2) \\ &= -1/3 \\ y_{2/3} &= 1 - (1/3)(2)(2) \\ &= -1/3 \\ P_{1/3} &= (-1/3, -1/3) \\ P_{2/3} &= (\ \ 1/3, -1/3)\ . \end{aligned}$$

For another example, let's draw the graph of $y = x^4$ for $x = -1$ to $x = 1$. The derivative is $4x^3$. Divide it up into (say) 4 segments $[-1.0, -0.5]$, $[-0.5, 0.0]$, $[0.0, 0.5]$, $[0.5, 1.0]$. We have this table, with the control points interpolated.

x	y	x'	y'
−1.0000	1.0000	1.0	−4.0
−0.8333	0.3333		
−0.6667	0.1458		
−0.5000	0.0625	1.0	−0.5
−0.3333	0.0208		
−0.1667	0.0000		
0.0000	0.0000	1.0	0.0
0.1667	0.0000		
0.3333	0.0208		
0.5000	0.0625	1.0	0.5
0.6667	0.1458		
0.8333	0.3333		
1.0000	1.0000	1.0	4.0

Here is the curve you get, with control points marked. It is almost indistinguishable from the one you get with 16 segments, so the approximation is pretty good. It is perhaps only when you see where the control points lie that you notice the slight rise in the middle.



Exercise 3.1. *Write the simplest procedure you can with these properties: • it has two arguments x_0 and x_1 and • it draws the graph of $y = x^2$ between x_0 and x_1 with a single Bezier curve.*

Exercise 3.2. *Do for $y = x^5$ between $x = -1$ and $x = 1$ in the same way I did $y = x^4$.*

Exercise 3.3. *The purpose of this exercise is to prove the assertion about Bezier curves and cubic polynomials. Let*

$$P(s) = (1-s)^3 y_0 + 3s(1-s)^2 y_{1/3} + 3s^2(1-s) y_{2/3} + s^3 y_1$$

The point is to verify that this formula agrees with the geometrical process described earlier. Let $P_{1/6}$ etc. be the points defined in section 2. (1) In an exercise to that section you (should have) found that

$$P_{1/2} = \frac{P_0 + 2P_1 + 2P_2 + P_3}{6} .$$

Verify that $P(1/2) = P_{1/2}$. (2) The rest of that construction relied on the claim that the curve from P_0 to $P_{1/2}$ was itself a Bezier curve with control points $P_{1/6}$ and $P_{2/6}$. Not the first half of the parametrized curve has a normalized parametrization $s \mapsto P(s/2)$ as s goes from 0 to 1. Verify that

$$P(s/2) = (1-s)^3 P_0 + 3s(1-s)^2 P_{1/6} + 3s^2(1-s)P_{2/6} + s^3 P_{1/2}$$

by using the expressions for $P_{1/6}$ and $P_{2/6}$ in terms of P_0, P_1, P_2, P_3.

4. Drawing an hyperbola

A parabola is too simple to give you an idea of how useful Bezier curves are. This is because the Bezier curve is guaranteed to match exactly the graph of any polynomial of degree 3 or less, so no approximation is involved. In this section we will develop a procedure **`hyperbola`** with these properties: • it has three arguments—x_0, x_1, N. • It draws the graph of the upper branch of an hyperbola

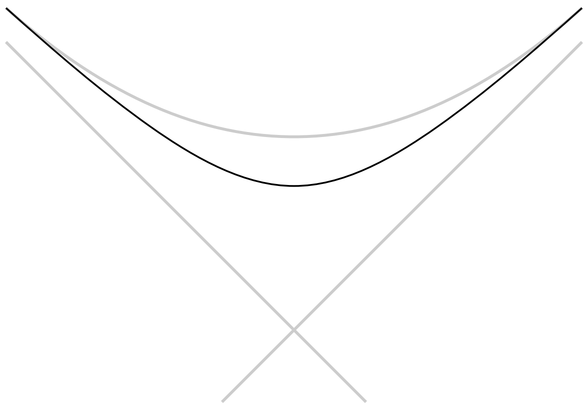
$$y^2 - x^2 = 1, \quad y = \sqrt{1 + x^2} .$$

by using N Bezier segments in between.

As is usually the best idea, the procedure builds the path without drawing it. Thus the sequence

```
newpath
-2 2 4 hyperbola
stroke
```

will draw



I have added the asymptotes to convince you it really is an hyperbola, and I have also shown in light grey what you get with **`-2 2 1 hyperbola`** so you can see that several Bezier segments are necessary.

What goes into the procedure? We can immediately write down the skeleton

```
/hyperbola {
16 dict begin
/N exch def
/x1 exch def
/x0 exch def

...

end
} def
```

and we must now fill in the real computation. First we set a step size $h = (x_1 - x_0)/N$:

```
/h x1 x0 sub N div def
```

Then we introduce variables x and y which are going to change as the procedure runs, and move to the first point on the graph. It will also help to keep a variable s to hold the current value of the slope. Note that if

$$y = f(x) = \sqrt{1+x^2} = (1+x^2)^{1/2}$$

then

$$s = f'(x) = (1/2)(2x)(1+x^2)^{-1/2} = \frac{x}{\sqrt{1+x^2}} = \frac{x}{y} .$$

```
/x x0 def
/y 1 x x mul add sqrt def
/s x y div def
x y moveto
```

Now we must build N Bezier segments, using a **`repeat`** loop.

```
N {
   x 0.33333 h mul add
   y h 0.33333 mul s mul add
   /x x h add def
   /y 1 x x mul add sqrt def
   /s x y div def
   x h 0.33333 mul sub
   y h 0.33333 mul s mul sub
   x y
   curveto
} repeat
```

and that's it.

We could make this program somewhat more readable and more flexible if we add to it a couple of procedures which calculate $f(x)$ and $f'(x)$, instead of doing it in line. Each of the procedures should have a single argument x. They are short enough that we do not need to use a variable inside them. Explicitly:

```
% sqrt(1 + x^2)

/f {
   dup mul 1 add sqrt
} def

% x/sqrt(1 + x^2)
```

```
/df {
    dup dup mul 1 add sqrt div
} def
```

The new loop would be

```
N {
    x 0.33333 h mul add
    y h 0.33333 mul s mul add
    /x x h add def
    /y x f def
    /s x df def
    x h 0.33333 mul sub
    y h 0.33333 mul s mul sub
    x y
    curveto
} repeat
```

I would prefer to have a single procedure that calculates $f(x)$ and $f'(x)$ all in one package. For one thing, it would be more efficient since we wouldn't have to calculate square roots more often than necessary. For another, I prefer to have things that go together ... well, go together. The calculation of $f(x)$ and $f'(x)$ are related, and if you modify one to draw some different graph, then you will have to modify the other. For this reason they should be dealt with in one visible unit. I can do this by using a single procedure with one argument x and as output an array of two numbers $[y\ s]$. But now a variable to hold the value of x is useful. It might be a good idea here to exhibit all procedures we have written:

```
% [sqrt(1 + x^2), x/sqrt(1 + x^2)]

/f {
    2 dict begin
    /x exch def
    /y 1 x x mul add sqrt def
[
    y
    x y div
]
end
} def

% x0 x1 N

/hyperbola {
16 dict begin
/N exch def
/x1 exch def
/x0 exch def

% h = (x1 - x0)/N
/h x1 x0 sub N div def

/x x0 def
/F x f def
/y F 0 get def
/s F 1 get def
x y moveto

N {
    x 0.33333 h mul add
    y h 0.33333 mul s mul add
    /x x h add def
    /F x f def
    /y F 0 get def
    /s F 1 get def
    x h 0.33333 mul sub
    y h 0.33333 mul s mul sub
    x y
    curveto
} repeat

end
} def
```

It is true that using a dictionary and an array has made the program somewhat less efficient than it was to start. On the good side, the program is now perhaps a bit more readable—or perhaps not, depending probably on your own taste. It has one great virtue, however—it is a great deal more flexible. *If we want to draw some other graph, we need only to rewrite the single procedure* **f***, making sure that it, too, has a single argument x and returns an array of two values $[y\ s]$.*

Exercise 4.1. *Modify the procedure* **f** *so you can use essentially the same stuff to graph the function $y = x^3 - x$ between -1 and 1.*

Exercise 4.2. *Modify the procedure* **f** *so you can use essentially the same stuff to graph the function $y = \sin x$ between -1 and 1. Be careful about degrees and radians. Use it to draw the graph of $\sin x$ between $x = 0$ and $x = \pi$ with 1, 2, 4 segments, all on one plot. Make them successively lighter so you can distinguish them.*

5. Parametrized curves

Not all curves in the plane are the graphs of functions. But almost any curve we can imagine is the union of a number of smooth segments, where each segment is a **parametrized path** in the plane.

I recall that a parametrized path is a pair of functions $(x(t), y(t))$ defined for the parameter t in some given range, and that as t varies over the range we get a one dimensional curve of points in the plane. It is useful to keep in mind here a mildly important distinction between the **curve** and the **path**. The curve is a geometrical object—the thing you see—while the (parametrized) path is a particular way of describing points on the curve. If the curve is the unit circle around the origin, for example, we can think of it as the union of two graphs

$$y = \pm\sqrt{1 - x^2}$$

as x varies from -1 to 1, and we can also think of it as the curve traversed by the points

$$(\cos t, \sin t)$$

as t ranges from 0 to 2π.

The best way to think of a parametrized path is to think of the number t as representing time, and to picture a point moving along the path as t increases. The position at time t is $(x(t), y(t))$. At time $t + h$ we are at position $(x(t + h), y(t + h))$, and the velocity at time t is therefore

$$\lim_{h \to 0} \big((x(t + h) - x(t))/h, (y(t + h) - y(t))/h\big) = (x'(t), y'(t)) \ .$$

To draw a parametrized curve we apply to each coordinate the procedure above. Suppose we are given endpoints (x_0, y_0) and (x_1, y_1), as well as velocity vectors (x'_0, y'_0) and (x'_0, y'_1). Then we get cubic polynomials $P_x(t)$ and $P_y(t)$ determined by the endpoint data. The path $(P_x(t), P_y(t))$ will be a cubic approximation to the true curve. It can be drawn in PostScript in a way similar to that described above for graphs.

We associate to each of the coordinates x and y some interpolated points

$$\begin{bmatrix} x_{1/3} \\ y_{1/3} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{\Delta t}{3} \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix}, \quad \begin{bmatrix} x_{2/3} \\ y_{2/3} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \frac{\Delta t}{3} \begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix}$$

These are the control points of the cubic segment.

And now the way you draw this cubic segment in PostScript is by the sequence

```
newpath
x[0] y[0] moveto
x[1/3] y[1/3]
x[2/3] y[2/3]
x[1] y[1]
curveto
```

where I have written **`x[0]`** for x_0 etc.

Let's look again at the problem posed at the beginning of this section, and see how we would draw a parametrized path by using Bezier curves. We must first divide it up into a certain number of segments. Calculate the position and velocity at the endpoints of each of the segments. Move to the first point and add one Bezier curve for each segment. (The command **`moveto`** is not necessary in the intermediate curves, because drawing a Bezier curve in PostScript advances the current point to the end-point.) Then stroke it or fill it or clip it (if in the latter cases it is closed).

One curious fact is that although PostScript has a special command for drawing arcs of circles, behind the scenes it draws an arc as a sequence of Bezier curves! Internally, every path in PostScript is stored as a sequence of commands **`moveto`**, **`lineto`**, **`curveto`**, and **`closepath`**, along with accompanying data specified in device coordinates. A path is built without delay as you build it with commands like these in your program—the conversion from user to device coordinates takes placs at the earliest moment possible. For example, if you type **`0 0 moveto`** the point **`0 0`** is converted when the command **`moveto`** is encountered, since it is just there that PostScript knows that **`0 0`** are the coordinates of a point. A path disappears when you start up a new one with **`newpath`**. (There are some magical tricks PostScript can perform by working with the internal representation of a path. If you want to know more about it, you can begin by looking up the PostScript command **`pathforall`**.)

6. Drawing graphs automatically

In the next section I'll explain a technique for drawing parametrized curves which will become a standard trick in your bag of tools. There are several new ingredients in it, and it may help if I explain one of them by improving the procedure for drawing hyperbolas along the same lines. We have already made that procedure reasonably flexible by isolating how the actual function $f(x)$ is used to draw the graph. What I will do now is show how *you can change functions by using the graphing function as an argument to a general procedure for making graphs.* The *only* part of the procedure **`hyperbola`** which must be changed is the very first part. Since the procedure no longer draws only hyperbolas, its name must be changed. And it has an extra argument, the **name** of f. This must be a procedure into which you put x and out of which you get an array $[f(x)\ f'(x)]$. So we must read this fourth argument, which in fact I make the first of four, and we must convert that name into a procedure so we can call it. The few lines of this procedure where it differs from the older one are here:

```
/mkgraph {
16 dict begin
/N exch def
/x1 exch def
/x0 exch def
/f exch cvx def
```

Very simple. Here is a complete program which uses this to draw the graph of $y = x^4$.

```
% /f x0 x1 N

/mkgraph {
16 dict begin
/N exch def
/x1 exch def
/x0 exch def
/f exch cvx def

% h = (x1 - x0)/N
/h x1 x0 sub N div def

/x x0 def
/F x f def
/y F 0 get def
/s F 1 get def

x y moveto

N {
    x 0.33333 h mul add
    y h 0.33333 mul s mul add
    /x x h add def
    /F x f def
    /y F 0 get def
    /s F 1 get def
    x h 0.33333 mul sub
y h 0.33333 mul s mul sub
x y
curveto
} repeat

end
} def

% [x^4 4x^3]

/quartic {
    2 dict begin
    /x exch def
[
x x mul x mul x mul
x x mul x mul 4 mul
]
end
} def

% ----------------------------------------

72 72 scale
4.25 5.5 translate
0.012 setlinewidth

newpath
/quartic -1 1 8 mkgraph
stroke
```

In the next section I shall introduce a procedure which is rather similar to this one. It differs in these aspects: (1) It deals with parametrized curves instead of graphs. (2) It allows you to use a single procedure to draw any one of a large family of curves, for example all of the graphs $y = cx^4$ where c is a constant you can specify when you draw the curve. (3) It adds the new path to the path that already exists, if there is one.

Exercise 6.1. *Write a PostScript procedure with the same arguments as* **`mkgraph`** *but which simply draws a polygon among the successive points. (This can be used to debug your calculus.)*

Exercise 6.2. *Write a PostScript procedure that will graph a polynomial between x_0 and x_1 with N Bezier segments. There are a number of things you have to think about: (1) For evaluating a polynomial in a program it is easiest to use an expression like $5x^3 + 2x + 3x + 4 = ((5x + 2)x + 3)x + 4$. (2) You will have to add an argument to this procedure to pass the polynomial coefficients as an array. Recall that* **`length`** *returns the size of an array.*

7. Drawing parametrized paths automatically

If you are given a parametrized path and you want to draw it by using Bezier curves, you must calculate position and velocity at several points of the path. This is tedious and prone to error, and you will quickly ask if there is some way to get PostScript to do the work. This is certainly possible, if you can write a PostScript routine which calculates position and velocity for the parametrization. One tricky point is that we don't want to rewrite the drawing routine for every path we draw, but would like instead to put in the **parametrization** itself as an argument passed to the routine. The parametrization should be a procedure which has a single argument, namely a value of t, and returns data giving position and velocity for that value of t. We shall in fact do this, and make the name of the routine which calculates position and velocity one of the arguments. How PostScript handles this is somewhat technical, but you won't have to understand underlying details to understand the routine. Another tricky point is that we would like to make a routine for drawing a family of curves—we don't want to have to make separate routines for $y = x^2$, $y = 2x^2$, etc. We would like to be able at least to write one routine that can draw $y = cx^2$ for any specified constant c. This will be accomplished by passing to the routine an array of **parameters** to pick a specific curve from a family. The terminology is a bit clumsy: we have the parameters determining which path from a family is to be drawn and the variable t which parametrizes the curve. The last, to avoid confusion, I shall call the **parametrizing variable**.

I will first simply lay out the main routine we shall use from now on for drawing parametrized paths. It is a bit complicated. In the next two sections I shall explain how to use it and how it works.

```
% stack:  [parameters] /f t0 t1 N

/mkpath {
16 dict begin
/N exch def
/t1 exch def
/t0 exch def
/f exch cvx def
/pars exch def

/h t1 t0 sub N div def
/h3 h 0.333333 mul def

/currentloc pars t0
f def
pars t0 f 0 get
aload pop

thereisacurrentpoint
{
   lineto
}
{
   moveto
}
ifelse

N {                          % x y = currentpoint
   currentloc 0 get 0 get      % x0 dx0
   currentloc 1 get 0 get
   h3 mul
   add

   currentloc 0 get 1 get
   currentloc 1 get 1 get
   h3 mul
   add

   /t0 t0 h add def
   /currentloc pars t0 f def
   currentloc 0 get 0 get
   currentloc 1 get 0 get
   h3 mul sub
   currentloc 0 get 1 get
   currentloc 1 get 1 get
   h3 mul sub
   currentloc 0 get 0 get
   currentloc 0 get 1 get
   curveto
} repeat
end
} def
```

We shall assume that this code is in a file called `mkpath.inc` (for *inclusion*).

How to use it

The input to the procedure consists of five items on the stack.

- *First comes an array* **`[ ... ]`** *of parameters, which I shall say something more about in a moment. It can be just the empty array* **`[]`***, but it must be there.*
- *Then follows the name of a routine specifying the parametrization in PostScript. A name in PostScript starts with the symbol* **`/`***. As I have already said, this routine has two arguments. The first is an array of things which the routine can use to do its calculation. The second is the variable t. Its output (left on the stack) will be a 2×2 matrix written according to my conventions as an array of two $2D$ arrays.*
- *The third is the initial value of t for the path.*
- *The fourth is the final value of t.*
- *Fifth is the number N of Bezier segments to be drawn.*

The most important, and complicated, item here is the parametrization routine.

For example, suppose we want to draw circles of varying radii, centred at the origin. The parametrization of such a circle is

$$t \mapsto P(t) = (R\cos t, R\sin t)$$

and the velocity vector of this parametrization is

$$t \mapsto P'(t) = (-R\sin t, R\cos t)$$

if t is in radians. The variable t is the parameter which traverse the path, while R is a parameter specifying which of several possible circles are to be drawn. Thus the input to the circle drawing routine will be a pair **`[R] t`** and output will be **`[[xt yt][dxt dyt]]`**.

Here is a more explicit block of PostScript code (assuming π defined elsewhere). Note that PostScript uses degrees instead of radians, but that the formula for the velocity vector assumes radians, and we must convert from radians to degrees.

```
/circle {
4 dict begin
/t exch def
/pars exch def
/R pars 0 get def
/t t 180 mul pi div def
[
[ t cos R mul t sin R mul ]
[ t sin neg R mul t cos R mul ]
]
end
} def

(mkpath.inc) run

newpath
[2] /circle 0 2 pi mul 8 mkpath
closepath
stroke
```

This example shows among other things that the routine **`mkpath`** does not actually draw a path, but builds it just as the PostScript **`curveto`** does, and allows the programmer to decide what to do with it—to stroke it or fill it, for example, or even clip to it. In fact, the routine **`mkpath`** does something a bit more complicated; *it adds to any path already drawn, by beginning with a straight line from the last point drawn to to the start of the path segment about to be built.* This is the way the PostScript **`arc`** behaves, and it seems a good model to follow. At any rate, it is for this reason that the routine tests whether or not there already exists a current point. The overall effect is to allow you to build up a series of paths so as to make a rather complicated one. You might even draw several successive segments of the same path, for example, using a different value of the integer N (the fifth argument) to allow for parts of the path being straighter or more curvy.

Another possibly interesting point is how to pass a procedure to another procedure. What we do in fact is to pass the *name* of the procedure (begun with **/**) to our routine, and the routine converts the name to a procedure with the command **`cvx`**.

The array of parameters can be of any size you want. It can even be empty if in fact you just want to draw one of a kind. But you will probably find that most paths you want to draw are just part of a larger family.

There is one other point. It is very common to get the formula for the velocity vector wrong, and usually you will draw rather wild curves instead of the correct one. There is a simple way to see if this is the case: you can use a routine that might be called **`mkpolypath`** which simply draws a polygonal path instead of one made up of Bezier segments. It has exactly the same usage as **`mkpath`**, but ignores the velocity vector in building the path, and can hence be used to see if the rest of your routine is working.

Exercise 7.1. *Write a procedure* **`mkpolypath`** *which has the same arguments as* **`mkpath`** *but draws a polygon instead.*

Exercise 7.2. *Write down a parametrization of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ .$$

Write a procedure which will draw it. For example you would write **`3 4 drawellipse`**.

Exercise 7.3. *Draw the image of the* 12×12 *grid centred at* $(0,0)$*, under the map* $(x,y) \mapsto (x^2 - y^2, 2xy)$*. The spacing between grid lines is* 0.25*.*

How it works

The basic idea is simply to automate the procedure you used earlier to draw the graph of $y = x^4$. The point is that if we are given a parametrization of a path we can draw an approximation by Bezier curves using the velocity vectors associated to the parametrization in order to construct control points. The routine is fairly straightforward, except that it calls a procedure **`thereisacurrentpoint`** to tell whether the path being drawn is the beginning of a new path or the continuation of an old one. You don't have to know the details of the procedure called — it operates in a very simple manner at a somewhat low level of PostScript.

This routine all by itself is very useful, and is capable of making interesting pictures. In combination with a programmed version of the chain rule from second year calculus it is capable of producing spectacular ones. We shall see later how the basic routine **`mkpath`** can be incorporated in others, for example to draw three dimensional parametrized paths in perspective.